LOCAL LIPSCHITZ CONTINUITY OF THE DIAMETRIC COMPLETION MAPPING

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Abstract. The diametric completion mapping associates with every closed bounded set C in a normed linear space the set $\gamma(C)$ of its completions, that is, of the diametrically complete sets containing C and having the same diameter. We prove local Lipschitz continuity of this set-valued mapping, with respect to two possible arguments: either as a function on the space of closed, bounded and convex sets, while the norm is fixed, or as a function on the space of equivalent norms, while the set C is fixed. In the first case, our result is valid in spaces with Jung constant less than 2, whereas the result in the second case is only proved for finite dimensional spaces. In this setting, we further show: (i) the maximal volume completion is a continuous selection for γ if the space is strictly convex, (ii) $\gamma(C)$ is convex for all C if and only if the space has the property, studied by Eggleston, that every diametrically maximal set is of constant width.

The existence of nontrivial curves of constant width was already known to Euler [7]. They have since become an active subject of study, appearing in many different contexts and having surprising applications in different areas of mathematics and engineering. The notion of diametrically maximal sets appeared later, at the beginning of last century, as a generalization of constant width sets [12]. A set in a metric space is *diametrically maximal* if the addition of any point to the set increases the diameter. Other names such as *complete* (which will also be used in this note), diametrically complete or entire sets have been used in the literature to refer to this concept. In this paper, we study diametrically maximal sets in normed linear spaces. It is well known that every bounded set C can be completed, in the sense that there is always a complete set D containing C such that $\operatorname{diam} C = \operatorname{diam} D$ (see, for example, the survey [5]). Such a set D is called a *completion* of C. In general, a set can have infinitely many completions. We denote by $\mathcal{H}$ the family of all nonempty, closed, bounded and convex sets in our space, endowed with the Hausdorff metric. The *diametric completion mapping* $\gamma : \mathcal{H} \to 2^{\mathcal{H}}$ associates with each $C \in \mathcal{H}$ the family of all its completions. The question with which we are mainly concerned in this paper can be phrased in a simple manner: is γ a continuous function?

Every complete set satisfies the spherical intersection property and therefore it is an intersection of closed balls [5]. For this reason, γ is somehow related to the ball hull mapping β, which associates with every bounded set the intersection of all balls containing it. It is known that β need not be continuous, even in three dimensional spaces [15]. Hence, having in mind that $\gamma = \gamma \circ \beta$, one might expect a similar behavior of γ. To our surprise, this is not so and we show that γ is locally Lipschitz continuous in the wide class of normed spaces with Jung constant less than 2. This class includes the spaces with

2010 *Mathematics Subject Classification:* 46B20, 52A05, 52A21

The first author was partially supported by the DGICYT project MTM 2009-07848.

normal structure [8] and, in particular, the finite dimensional spaces. This result leads us to study the continuity of two other mappings in the vein of the previous one, which we call the *wide spherical hull* η and the *tight spherical hull* θ. In two-dimensional spaces, these mappings are known as 'circular intersection' and 'circular hull', respectively. Both notions have been considered in several papers, often with different names (see [2], [3] and references therein) in connection with sets of constant width, overconvexity and some variational problems. For recent results on this subject, we refer to [4], [11] and [13]. We prove that η and θ are always locally Lipschitz continuous in an open dense set of $\mathcal{H}$. Finally, we consider the continuity of γ also with respect to the space of all equivalent norms. Also in this case we prove the local Lipschitz continuity, but only in the context of finite dimensional spaces. In this setting, we further show that the maximal volume completion is a continuous selection for γ when the space is strictly convex.

Sets of constant width are always diametrically maximal, and the converse is true in Euclidean spaces and Minkowski planes. The corresponding result for general Minkowski spaces is false, even in three dimensional spaces with smooth and strictly convex norm [6]. Following Eggleston [6], we say that the normed space $(X, \|\cdot\|)$ has property (A) if every diametrically maximal set in X has constant width. In Section 3, we give a characterization of finite dimensional spaces with property (A) in terms of the convexity of $\gamma(C)$ for every $C \in \mathcal{H}$. This result illustrates why the basic properties of γ are relevant for the study of convex bodies of constant width. We prove also that the set of norms on X having property (A) is closed, improving a previous result of Yost [19], who has shown that the set of norms failing property (A) is a G_δ set.

1. Local Lipschitz continuity of γ

The setting for the following is a normed real vector space $(X, \|\cdot\|)$. We denote its unit ball by B, and $B(x, r) = x + rB$ is the ball with center $x \in X$ and radius $r > 0$. For a subset $C \subset X$, we let $\operatorname{diam}_{\|\cdot\|} C$ denote the diameter of C (when there is no danger of confusion we shall write $\operatorname{diam} C$ or simply d_C). Given a point $x \in X$, the radius of C with respect to x is defined as

$$r(x, C) = \sup\{\|x - y\| : y \in C\}.$$

If C is a bounded set, then C is included in the ball with center x and radius $r(x, C)$, for every $x \in X$. When $x \in C$, it is clear that $r(x, C) \leq \operatorname{diam} C$, and the problem of covering a bounded set C by a ball with radius smaller than $\operatorname{diam} C$ is a classical subject in convex geometry and functional analysis. In this latter context, a normed space X has *normal structure* if for each nonempty, bounded, convex subset $C \subset X$ there is a point $x \in C$ such that $r(x, C) < d_C$. Different generalizations of this property can be found in the literature [8]. The Jung constant $J(X)$ is defined by

$$J(X) = \sup\{2r(C) : C \text{ convex}, \operatorname{diam} C = 1\},$$

where $r(C) = \inf\{r(x, C) : x \in X\}$ is the radius of Chebyshev of C [1]. Notice that $J(X) < 2$ implies that every bounded set C can be included in a ball with radius smaller than $\operatorname{diam} C$. However, in this case, the center of the ball need not be in C. It is well known that $J(X) < 2$ does not imply normal structure: the space ℓ_∞ with the usual sup norm is an example. Indeed, $J(\ell_\infty) = 1$ (every convex set C with $\operatorname{diam} C = 1$ can be included in a ball of the same diameter), but ℓ_∞ does not have normal structure (for instance, take for C the closed convex hull of the basic unit vectors).

Recall that, for any metric space (X, δ), the *Hausdorff metric* ρ on the space $\mathcal{C}(X)$ of nonempty, closed, bounded subsets of X is defined by

$$\rho(C, D) := \max\{\sup_{x\in C} \inf_{y\in D} \delta(x, y), \sup_{x\in D} \inf_{y\in C} \delta(x, y)\}$$

for $C, D \in \mathcal{C}(X)$. For $(\mathbb{R}^n, \|\cdot\|)$, we denote by ρ the Hausdorff metric on $\mathcal{C}(\mathbb{R}^n)$ that is induced by the norm; this can also be represented by

$$\rho(C, D) \;=\; \inf\{\varepsilon \geq 0 : \; C \subset D + \varepsilon B,\; D \subset C + \varepsilon B\}.$$

By Δ we denote the Hausdorff metric on $\mathcal{C}(\mathcal{H})$ that is induced by the metric ρ. Continuity in the following refers to these metrics.

Theorem 1. *The diametric completion mapping γ is locally Lipschitz continuous in spaces with Jung constant less than 2.*

Proof. Let X satisfy $J(X) < 2$. If we let $J(X)/2 < \tau < 1$, then for each set $C \in \mathcal{H}$ there is a point $x \in X$ with

$$r(x, C) \leq \tau d_C.$$

Consider two sets $C, C' \in \mathcal{H}$ and the real numbers $D_{CC'} = \max\{\operatorname{diam} C, \operatorname{diam} C'\}$ and $d_{CC'} = \min\{\operatorname{diam} C, \operatorname{diam} C'\}$. We shall show that, if C, C' satisfy the condition

$$\rho(C, C') \;\leq\; \beta\, d_{CC'} \tag{1}$$

with

$$\beta = \frac{1-\tau}{10\tau + 2} \tag{2}$$

then

$$\Delta(\gamma(C), \gamma(C')) \;\leq\; \frac{D_{CC'}}{\beta\, d_{CC'}}\; \rho(C, C'). \tag{3}$$

Notice that, when $D_{CC'} = 0$, the above constant (which has no meaning since $d_{CC'} = 0$ as well) can be replaced by 1 since, in this case, C and C' are singletons, hence diametrically maximal, and thus $\gamma(C) = \{C\}$, $\gamma(C') = \{C'\}$. Therefore, we may assume that $D_{CC'} > 0$ and, using (1) together with

$$d_{CC'} \geq D_{CC'} - 2\rho(C, C')$$

we obtain $d_{CC'} \geq D_{CC'} - 2\beta d_{CC'}$ and therefore

$$d_{CC'} \geq \frac{D_{CC'}}{1+2\beta} > 0 .$$

Finally, if we use the above estimate in (3), we have

$$\Delta(\gamma(C), \gamma(C')) \leq \frac{1+2\beta}{\beta} \rho(C, C')$$

for $C, C' \in \mathcal{H}$ satisfying (1), an inequality that together with the definition (2) of β proves the local Lipschitz continuity of γ.

To prove (3), we shall need some estimates involving some constants that will be defined right now. If we put

$$\varepsilon = \frac{\rho(C, C')}{\beta d_{CC'}},$$

then $\varepsilon \leq 1$ and, without loss of generality, we may assume that $\varepsilon > 0$. Define

$$\mu = \frac{\varepsilon + \tau}{\tau}, \qquad \lambda = \frac{\tau}{\varepsilon + \tau}, \qquad \alpha = \lambda + (1 - \lambda)\tau,$$

so that $\mu > 1$, $\lambda < 1$ and $\alpha < 1$. The proof consists in considering an arbitrary $K \in \gamma(C)$ and finding $K' \in \gamma(C')$ satisfying (5) below, to apply then an argument of symmetry to get (3). Let $K \in \gamma(C)$ be a completion of C. Since $\operatorname{diam} K = d_C$ and $J(X) < 2$, there exists a point w such that $r(w, K) < d_C$. Incidentally, having in mind that K is diametrically maximal, w must be an interior point of K, necessarily. Therefore, after applying a translation to C, we may assume that

$$K \subset \tau d_C B$$

and that 0 is a point of K. We claim that there is $K' \in \gamma(C')$, a completion of C', such that

$$\lambda K \subset K' \subset \mu K. \tag{4}$$

Before proving the above claim, which is the cornerstone of the proof, let us see why these inclusions imply (3). On the one hand, from $K' \subset \mu K$ we deduce that

$$K' \subset K + (\mu - 1)K \subset K + (\mu - 1)\tau d_C B = K + \varepsilon d_C B.$$

On the other hand, similarly, from $\lambda K \subset K'$ and $\lambda^{-1} = \mu$, we obtain $K \subset K' + \varepsilon d_C B$, hence $\rho(K, K') \leq \varepsilon d_C$ and therefore

$$\rho(K, K') \leq \varepsilon D_{CC'}. \tag{5}$$

Since the assumptions are symmetric in C and C', also to any $K' \in \gamma(C')$ there exists $K \in \gamma(C)$ satisfying (5). By the definition of the Hausdorff metric Δ, this means that

$$\Delta(\gamma(C), \gamma(C')) \leq \varepsilon D_{C,C'} = \frac{D_{CC'}}{\beta\, d_{CC'}} \rho(C, C'),$$

as desired.

Now, to prove the claim, let $x \in \lambda K$ and $z \in C$. Then $\lambda z \in \lambda K$, hence $\|x - \lambda z\| \le \lambda d_C$ and therefore

$$\|x - z\| \le \|x - \lambda z\| + \|\lambda z - z\| \le \lambda d_C + (1-\lambda)\tau d_C = \alpha d_C. \tag{6}$$

Consider now

$$\delta = \beta \varepsilon d_C = \frac{d_C}{d_{CC'}}\, \rho(C, C') \tag{7}$$

and notice that

$$\rho(C, C') \le \delta. \tag{8}$$

Therefore, the diameters of C and C' satisfy $d_C \le d_{C'} + 2\delta$ and $d_{C'} \le d_C + 2\delta$. Also, since $\beta\varepsilon \le \beta < 1/4$, we have $d_C \le 2d_{C'}$. Using these facts, we get the inequalities

$$\frac{1-\lambda}{2\lambda}\, d_{C'} \ge \frac{1-\lambda}{4\lambda}\, d_C = \frac{1}{4\tau}\, \varepsilon\, d_C \ge \beta\varepsilon d_C = \delta,$$

which can be used to obtain the following estimate of the diameter of λK:

$$\operatorname{diam} \lambda K = \lambda d_C \le \lambda(d_{C'} + 2\delta) \le d_{C'}. \tag{9}$$

To obtain a similar estimate for the diameter of the set $M := \lambda K \cup C'$, let $z' \in C'$ and $x \in \lambda K$. By (8), there exists $z \in C$ with $\|z' - z\| \le \delta$. Using (6) and the inequalities

$$\frac{1-\alpha}{2\alpha+1}\, d_{C'} \ge \frac{1-\alpha}{4\alpha+2}\, d_C = \frac{1-\tau}{2[(2\tau+1)\varepsilon + 3\tau]}\, \varepsilon d_C \ge \beta\varepsilon d_C = \delta,$$

we get

$$\|x - z'\| \le \|x - z\| + \delta \le \alpha d_C + \delta \le \alpha(d_{C'} + 2\delta) + \delta \le d_{C'}. \tag{10}$$

Now (9) and (10) show that the diameter of $M = \lambda K \cup C'$ is exactly $d_{C'}$. Therefore, if K' is any completion of M, it is also a completion of C'. By construction, we have $\lambda K \subset K'$ and the first inclusion of (4) is proved.

To prove now that $K' \subset \mu K$, once we know that $0 \in K$ and $\lambda K \subset K'$, it is enough to show that no point from the boundary of μK is in K'. First of all, we need the following estimate:

$$\begin{aligned} \frac{\mu - (\mu-1)\tau - 1}{2(\mu - (\mu-1)\tau) + 1}\, d_{C'} &\ge \frac{\mu - (\mu-1)\tau - 1}{4(\mu - (\mu-1)\tau) + 2}\, d_C \\ &= \frac{1-\tau}{4(1-\tau)\varepsilon + 6\tau}\, \varepsilon\, d_C \\ &> \beta\, \varepsilon\, d_C = \delta, \end{aligned} \tag{11}$$

where the last inequality follows from $\tau > J(X)/2$ and hence $\tau > 1/2$. If y is an arbitrary point in the boundary of μK, then $(1/\mu)y$ is in the boundary of K. Since K is a completion of C, this fact implies that for each $n \in \mathbb{N}$ there is $z_n \in C$ with $\|(1/\mu)y - z_n\| \ge d_C - 1/n$, hence $\|y - \mu z_n\| \ge \mu d_C - \mu(1/n)$. By (8), there exists $z_n' \in C'$ such that $\|z_n - z_n'\| \le \delta$.

This gives

$$\begin{aligned}\mu d_C - \frac{\mu}{n} &\leq \|y - \mu z_n\| \\ &\leq \|y - z_n\| + \|z_n - \mu z_n\| \\ &\leq \|y - z_n'\| + \delta + (\mu - 1)\tau d_C\end{aligned}$$

and hence

$$\begin{aligned}\|y - z_n'\| &\geq \mu d_C - (\mu - 1)\tau d_C - \delta - \frac{\mu}{n} \\ &\geq (\mu - (\mu - 1)\tau)(d_{C'} - 2\delta) - \delta - \frac{\mu}{n} \\ &> d_{C'},\end{aligned}$$

where the last inequality is true for $n \in \mathbb{N}$ big enough, in virtue of (11). Thus, $y \notin C'$ and hence $y \notin K'$. This completes the proof of $K' \subset \mu K$ and hence of (4). □

Let us say that a normed space X satisfies property (G) if every bounded set $C \subset X$ can be included in a ball with radius smaller than $\operatorname{diam} C$. This property is a particular case of the notion of relative normal structure [10]. It is natural to ask whether the condition $J(X) < 2$ (which is a uniform version of property (G)) can be replaced in Theorem 1 by property (G).

If $C \subset X$ is a bounded subset of a normed space X, the *wide spherical hull* $\eta(C)$ and the *tight spherical hull* $\theta(C)$ of C are defined as

$$\eta(C) = \bigcap_{x \in C} B(x, \operatorname{diam} C)$$

and

$$\theta(C) = \bigcap_{x \in \eta(C)} B(x, \operatorname{diam} C),$$

respectively. These two mappings, which associate with each set an intersection of closed balls containing it, have been studied in connection with different questions in convexity, variational and functional analysis. However, some of their basic properties and possible applications to set covering, approximation and optimization problems have been hardly explored.

Corollary 2. *The wide spherical hull mapping η is locally Lipschitz continuous in spaces with Jung constant smaller than* 2.

The proof of this result is just a direct application of Theorem 1, together with the following characterization of $\eta(C)$ given in [13]: $\eta(C) = \bigcup\{D : D \in \gamma(C)\}$. In the general case, a similar result is available for convex bodies with nonempty interior. Given $C \in \mathcal{H}$, denote by $B(C, r) \subset \mathcal{H}$ the closed ball with center C and radius r. Recall that the inner radius $\operatorname{inr}_{\|\cdot\|}(C)$ is the supremum of the radii of all balls contained in C. If there is no risk of confusion, we simply denote $\operatorname{inr}_{\|\cdot\|}(C) = r_C$.

Proposition 3. *Let $C \in \mathcal{H}$ be a set with inner radius $r_C > 0$. Then η and θ are locally Lipschitz continuous mappings in $B(C, r_C/2)$.*

Proof. We shall show that there exists a positive constant λ_C such that

$$\rho(\eta(D), \eta(D')) \leq \lambda_C\, \rho(D, D') \tag{12}$$

for every $D, D' \in B(C, r_C/2)$ with $\rho(D, D') < r_C/6$. Analogously, there exists a positive constant μ_C satisfying

$$\rho(\theta(D), \theta(D')) \leq \mu_C\, \rho(D, D') \tag{13}$$

for every $D, D' \in B(C, r_C/2)$ with $\rho(D, D') < r_C/(4 + 2\lambda_C)$. Given a positive number $\varepsilon < r_C$, denote by $C \sim \varepsilon B$ the inner parallel body

$$C \sim \varepsilon B = \{x \in C : x + \varepsilon B \subset C\}.$$

It is not difficult to show that

$$\rho(C, C \sim \varepsilon B) \leq \frac{\operatorname{diam} C}{r_C}\, \varepsilon \quad \text{for } 0 \leq \varepsilon < r_C. \tag{14}$$

Consider now $D \in B(C, r_C/2)$. Then, since $r_{\eta(D)} \geq r_D \geq r_C/2$ and

$$\operatorname{diam} \eta(D) \leq 2 \operatorname{diam} D \leq 2(\operatorname{diam} C + r_C),$$

we obtain (applying (14) to $\eta(D)$ instead of C)

$$\rho(\eta(D), \eta(D) \sim \varepsilon B) \leq \frac{\operatorname{diam} \eta(D)}{r_{\eta(D)}}\, \varepsilon \leq \frac{4(\operatorname{diam} C + r_C)}{r_C}\, \varepsilon \tag{15}$$

for every $\varepsilon < r_C/2$. Now consider $D, D' \in B(C, r_C/2)$ satisfying $\rho(D, D') = \varepsilon$, with $0 \leq \varepsilon < r_C/6$. If $x \in \eta(D) \sim 3\varepsilon B$, then $\|x - y\| \leq \operatorname{diam} D - 3\varepsilon$ for every $y \in D$. If $y' \in D'$, there is $y \in D$ with $\|y - y'\| \leq \varepsilon$ and so

$$\|x - y'\| \leq \|x - y\| + \|y - y'\| \leq \operatorname{diam} D - 3\varepsilon + \varepsilon \leq \operatorname{diam} D',$$

hence $x \in \eta(D')$. Therefore, we get $\eta(D) \sim 3\varepsilon B \subset \eta(D')$ and, using an argument of symmetry, also $\eta(D') \sim 3\varepsilon B \subset \eta(D)$. Now, if we define

$$\lambda_C := 12(\operatorname{diam} C + r_C)/r_C,$$

we have

$$\begin{aligned} \eta(D) &\subset (\eta(D) - 3\varepsilon B) + \lambda_C\, \varepsilon B \\ &\subset \eta(D') + \lambda_C\, \varepsilon B \end{aligned}$$

and, analogously, $\eta(D') \subset \eta(D) + \lambda_C\, \varepsilon B$, hence (12) is proved.

For the second part of the proof, we use a similar argument: consider now $D, D' \in B(C, r_C/2)$ satisfying $\rho(D, D') = \varepsilon \leq r_C/(4+2\lambda_C)$ and let $x \in \theta(D) \sim (2+\lambda_C)\varepsilon B$. Then, for every $y \in \eta(D)$ we have $\|x-y\| \leq \operatorname{diam} D - (2+\lambda_C)\varepsilon$. If $y' \in \eta(D')$, (12) implies that there is $y \in \eta(D)$ with $\|y-y'\| \leq \lambda_C\,\varepsilon$, and so

$$\|x-y'\| \leq \|x-y\| + \|y-y'\| \leq \operatorname{diam} D - (2+\lambda_C)\varepsilon + \lambda_C\,\varepsilon \leq \operatorname{diam} D',$$

hence $x \in \theta(D')$. Therefore, we have $\theta(D) \sim (2+\lambda_C)\varepsilon B \subset \theta(D')$ and, analogously, $\theta(D') \sim (2+\lambda_C)\varepsilon B \subset \theta(D)$. Then

$$\begin{aligned}
\theta(D) &\subset (\theta(D) \sim (2+\lambda_C)\,\varepsilon B) + \frac{\operatorname{diam}\theta(D)}{r_C/2}(2+\lambda_C)\,\varepsilon B \\
&\subset (\theta(D) \sim (2+\lambda_C)\,\varepsilon B) + \frac{4(\operatorname{diam} C + r_C)}{r_C}(2+\lambda_C)\,\varepsilon B \\
&\subset \theta(D') + \frac{\lambda_C}{3}(2+\lambda_C)\,\varepsilon B
\end{aligned}$$

and, analogously, $\theta(D') \subset \theta(D) + (\lambda_C/3)(2+\lambda_C)\varepsilon B$. As a consequence, if we define $\mu_C := (\lambda_C/3)(2+\lambda_C)$, then (13) is proved. □

The continuity of the diametric completion mapping γ can also be considered from the point of view of the space of equivalent norms. Indeed, if $\|\cdot\|$ and $|\cdot|$ are two different equivalent norms, they induce different mappings $\gamma_{\|\cdot\|}(\cdot)$ and $\gamma_{|\cdot|}(\cdot)$. It is quite natural to ask whether, for given $C \in \mathcal{H}$, the mapping $\|\cdot\| \to \gamma_{\|\cdot\|}(C)$ is continuous or even whether it has a Lipschitz behavior. Given $(X, \|\cdot\|)$ and $0 \leq \varepsilon < 1$, we say that $|\cdot|$ is an ε-equivalent norm on X if

$$(1-\varepsilon)\|x\| \leq |x| \leq (1+\varepsilon)\|x\| \tag{16}$$

for every $x \in X$. If we denote by $B_{\|\cdot\|}$ and $B_{|\cdot|}$ the corresponding unit balls, the previous inequalities can be written as follows:

$$(1-\varepsilon)B_{|\cdot|} \subset B_{\|\cdot\|} \subset (1+\varepsilon)B_{|\cdot|}.$$

Since we are dealing with more than one norm, we will specify in the notation which is the norm we are dealing with. For instance, the self-Jung constant $J^s_{|\cdot|}(X)$ relative to $|\cdot|$, which will be used in the proof of the next result, is defined as

$$J^s_{|\cdot|}(X) = \sup\{2r_C(C) : C \text{ convex}, \operatorname{diam}_{|\cdot|} C = 1\},$$

where $r_C(C) = \inf\{r(x, C) : x \in C\}$ (however, ρ and dist will always refer to the norm $\|\cdot\|$). It is known that, if $\dim X \leq n$, then

$$J^s_{|\cdot|}(X) \leq \frac{2n}{n+1} \tag{17}$$

for every norm $|\cdot|$ on X (see [1]).

Proposition 4. *Let* $(X, \|\cdot\|)$ *be a finite dimensional normed space and let* $|\cdot|$ *be an* ε*-equivalent norm. Then, for every* $C \in \mathcal{H}$*, there is a constant* $M_{\|\cdot\|}(C)$ *such that*

$$\Delta_{\|\cdot\|}(\gamma_{\|\cdot\|}(C), \gamma_{|\cdot|}(C)) \leq \frac{2\varepsilon}{1-\varepsilon} M_{\|\cdot\|}(C).$$

Proof. Consider an arbitrary set $C \in \mathcal{H}$. We begin by observing that $\gamma_{\|\cdot\|}(C)$ is compact: the boundedness is clear, the closedness is easy to prove [14], and then compactness follows from the Blaschke selection theorem. Define

$$r_{\|\cdot\|}(C) = \min\{\mathrm{inr}_{\|\cdot\|}(K) : K \in \gamma_{\|\cdot\|}(C)\}.$$

Since $\mathrm{inr}_{\|\cdot\|}(K) > 0$ for every diametrically maximal set in a finite dimensional space and $\gamma_{\|\cdot\|}(C)$ is compact, it is clear that $r_{\|\cdot\|}(C) > 0$.

Let $D \in \gamma_{\|\cdot\|}(C)$ be an arbitrary $\|\cdot\|$-completion of C. We need to find a $|\cdot|$-completion $D' \in \gamma_{|\cdot|}(C)$ and a constant $M_{\|\cdot\|}(C)$ (not depending on D) satisfying

$$\rho(D, D') \leq \frac{2\varepsilon}{1-\varepsilon} M_{\|\cdot\|}(C). \tag{18}$$

To this end, using (17) we can fix an arbitrary positive number τ satisfying

$$J^s_{|\cdot|}(X)/2 < \tau < 1 \tag{19}$$

and define $C' = \lambda D \cup C$, where

$$\lambda = \frac{\mathrm{diam}_{|\cdot|} C - \tau \,\mathrm{diam}_{|\cdot|} C}{\mathrm{diam}_{|\cdot|} D - \tau \,\mathrm{diam}_{|\cdot|} C} \leq 1. \tag{20}$$

We shall show first that $\mathrm{diam}_{|\cdot|} C' = \mathrm{diam}_{|\cdot|} C$. On one hand,

$$\begin{aligned} \mathrm{diam}_{|\cdot|} \lambda D &= \lambda \,\mathrm{diam}_{|\cdot|} D \\ &= \frac{\mathrm{diam}_{|\cdot|} D - \tau \,\mathrm{diam}_{|\cdot|} D}{\mathrm{diam}_{|\cdot|} D - \tau \,\mathrm{diam}_{|\cdot|} C} \mathrm{diam}_{|\cdot|} C \leq \mathrm{diam}_{|\cdot|} C. \end{aligned}$$

On the other hand we may assume, after a suitable translation, that $0 \in C$ and $C \subset \tau(\mathrm{diam}_{|\cdot|} C) B_{|\cdot|}$. Now, if $u \in \lambda D$ and $c \in C$, there is $w \in D$ such that $u = \lambda w$ and so

$$\begin{aligned} |u - c| &\leq |u - \lambda c| + |\lambda c - c| = |\lambda w - \lambda c| + |\lambda c - c| \\ &\leq \lambda \,\mathrm{diam}_{|\cdot|} D + (1-\lambda)\tau \,\mathrm{diam}_{|\cdot|} C \\ &= \lambda\tau \,\mathrm{diam}_{|\cdot|} C + (1-\tau)\,\mathrm{diam}_{|\cdot|} C + (1-\lambda)\tau \,\mathrm{diam}_{|\cdot|} C \\ &= \mathrm{diam}_{|\cdot|} C. \end{aligned}$$

Consider now $D' \in \gamma_{|\cdot|}(C')$. Since $C \subset C'$ and $\mathrm{diam}_{|\cdot|} C = \mathrm{diam}_{|\cdot|} C'$, it is clear that $D' \in \gamma_{|\cdot|}(C)$. In order to estimate $\rho(D, D')$ we can write

$$\rho(D, D') \leq \rho(D, \lambda D) + \rho(\lambda D, D') \tag{21}$$

and try to obtain an estimate for the right-hand side of the inequality. In the case of the first summand, we have

$$\begin{aligned}
\rho(D,\lambda D) &\leq (1-\lambda)\operatorname{diam}_{\|\cdot\|} D \\
&= \frac{\operatorname{diam}_{|\cdot|} D - \operatorname{diam}_{|\cdot|} C}{\operatorname{diam}_{|\cdot|} D - \tau \operatorname{diam}_{|\cdot|} C} \operatorname{diam}_{\|\cdot\|} D \\
&\leq \frac{\frac{1+\varepsilon}{1-\varepsilon}\operatorname{diam}_{|\cdot|} C - \operatorname{diam}_{|\cdot|} C}{\operatorname{diam}_{|\cdot|} D - \tau \operatorname{diam}_{|\cdot|} C} \operatorname{diam}_{\|\cdot\|} C \\
&= \frac{2\varepsilon}{1-\varepsilon} \frac{\operatorname{diam}_{|\cdot|} C}{\operatorname{diam}_{|\cdot|} D - \tau \operatorname{diam}_{|\cdot|} C} \operatorname{diam}_{\|\cdot\|} C \\
&\leq \frac{2\varepsilon}{1-\varepsilon} \frac{\operatorname{diam}_{\|\cdot\|} C}{1-\tau}. \qquad (22)
\end{aligned}$$

To estimate $\rho(\lambda D, D')$, first notice that $\lambda D \subset D'$. Consider $x \in D' \setminus \lambda D$ and $y \in \lambda D$ such that $y + \lambda r_{\|\cdot\|}(C) B_{\|\cdot\|} \subset \lambda D$. Let z be a point in the segment $[x, y]$ which is in the boundary of λD. Using that λD is $\|\cdot\|$-diametrically maximal and also that $r_{\|\cdot\|}(\cdot, \lambda D)$ is a convex function, we obtain

$$\begin{aligned}
r_{\|\cdot\|}(x,\lambda D) &\geq \operatorname{diam}_{\|\cdot\|} \lambda D + \frac{r_{\|\cdot\|}(z,\lambda D) - r_{\|\cdot\|}(y,\lambda D)}{\|z-y\|} \|x-z\| \\
&\geq \operatorname{diam}_{\|\cdot\|} \lambda D + \frac{\lambda\, r_{\|\cdot\|}(C)}{\operatorname{diam}_{\|\cdot\|} \lambda D - \lambda\, r_{\|\cdot\|}(C)} \|x-z\| \\
&\geq \lambda \operatorname{diam}_{\|\cdot\|} D + \frac{r_{\|\cdot\|}(C)}{\operatorname{diam}_{\|\cdot\|} D - r_{\|\cdot\|}(C)} \operatorname{dist}(x,\lambda D).
\end{aligned}$$

Using

$$\begin{aligned}
r_{\|\cdot\|}(x,\lambda D) &\leq \frac{1}{1-\varepsilon} r_{|\cdot|}(x,\lambda D) \leq \frac{1}{1-\varepsilon} \operatorname{diam}_{|\cdot|} C' \\
&= \frac{1}{1-\varepsilon} \operatorname{diam}_{|\cdot|} C \leq \frac{1+\varepsilon}{1-\varepsilon} \operatorname{diam}_{\|\cdot\|} C,
\end{aligned}$$

we finally get

$$\begin{aligned}
\operatorname{dist}(x,\lambda D) &\leq \frac{\operatorname{diam}_{\|\cdot\|} C - r_{\|\cdot\|}(C)}{r_{\|\cdot\|}(C)} \left((1-\lambda)\operatorname{diam}_{\|\cdot\|} C + \frac{2\varepsilon}{1-\varepsilon} \operatorname{diam}_{\|\cdot\|} C \right) \\
&\leq \frac{2\varepsilon}{1-\varepsilon} \frac{\operatorname{diam}_{\|\cdot\|} C - r_{\|\cdot\|}(C)}{r_{\|\cdot\|}(C)} \left(\frac{\operatorname{diam}_{\|\cdot\|} C}{1-\tau} + \operatorname{diam}_{\|\cdot\|} C \right), \qquad (23)
\end{aligned}$$

where in the last inequality we have used a previous estimate obtained in (22). Since this holds for all $x \in D'$, and since $\lambda D \subset D'$, we deduce that the inequality holds also if $\operatorname{dist}(x, \lambda D)$ is replaced by $\rho(\lambda D, D')$. As a consequence of this fact and (22) we have

$$\rho(D,\lambda D) + \rho(\lambda D, D') \leq \frac{2\varepsilon \operatorname{diam}_{\|\cdot\|} C}{1-\varepsilon} \left(\frac{1}{1-\tau} + \frac{\operatorname{diam}_{\|\cdot\|} C - r_{\|\cdot\|}(C)}{r_{\|\cdot\|}(C)} \left(\frac{1}{1-\tau} + 1 \right) \right).$$

Hence, if we define

$$M_{\|\cdot\|}(C) := \frac{\operatorname{diam}_{\|\cdot\|} C}{1-\tau}\left(1 + \frac{\operatorname{diam}_{\|\cdot\|} C - r_{\|\cdot\|}(C)}{r_{\|\cdot\|}(C)}\,(2-\tau)\right) \tag{24}$$

we finally obtain (18), which finishes the proof. □

Notice that the constant defined in (24) depends on the dimension of the space since we cannot avoid that τ was close to 1 if the dimension is high enough (19).

If X is a normed space of infinite dimension and $J^s_{\|\cdot\|}(X) < 1$, a somehow similar result as the one stated in Proposition 4 holds for any convex set $C \in \mathcal{H}$ with nonempty interior and for any $\varepsilon > 0$ satisfying

$$\frac{1+\varepsilon}{1-\varepsilon}\, J^s_{\|\cdot\|}(X) < 1\,.$$

The above condition is needed because, other than in the finite dimensional case, we have no uniform estimate of $J^s_{|\cdot|}(X)$, and we must use instead that $J^s_{|\cdot|}(X) \le \frac{1+\varepsilon}{1-\varepsilon}\, J^s_{\|\cdot\|}(X)$ when $\|\cdot\|$ and $|\cdot|$ are ε-equivalent. Notice that $r_{\|\cdot\|}(C)$ can be replaced directly by $\operatorname{inr}_{\|\cdot\|} C$ in this case. The proof of this result follows the lines of the proof of Proposition 4 with minor modifications.

2. A continuous selection for γ.

Continuous selections of multivalued mappings have various applications in geometry of Banach spaces, convex sets, fixed point theory, approximation theory, and other fields. It is a natural question to ask whether the diametric completion mapping γ admits a continuous selection and if such a selection can be defined through a completion enjoying special properties, in analogy with the problem of finding selectors for convex sets using a particular type of center such as the classical Steiner point [17], [18]. We show that this is the case in finite dimensional, strictly convex spaces. For this, we employ a result of Groemer [9]. He called a closed set $K \in \mathcal{H}$ a *tight cover* of $C \in \mathcal{H}$ if $C \subset K$ and $\operatorname{diam} K = \operatorname{diam} C$. He showed that among the tight covers of C there is one of maximal volume, that this is a completion of C, and that it is uniquely determined if the norm is strictly convex. In the latter case, we denote this maximal volume completion of C by $\zeta(C)$.

Theorem 5. *In a finite dimensional space X with strictly convex norm, the maximal volume completion ζ is a continuous selection for γ.*

Proof. We need a preparatory result to prove that $\zeta : \mathcal{H} \to \mathcal{H}$ is continuous. Consider C, the family of nonempty, compact subsets of $\mathcal{H}$. For $\mathcal{C} \in \mathsf{C}$ we define

$$\varphi(\mathcal{C}) := \max\{V(K) : K \in \mathcal{C}\},$$

where V is a volume on X (induced by some Euclidean metric, for example). The maximum exists since V is continuous on $\mathcal{H}$, and $\mathcal{C}$ is compact. We show that φ is continuous. Suppose that $\mathcal{C}_i, \mathcal{C} \in \mathsf{C}$ and that $\mathcal{C}_i \to \mathcal{C}$ with respect to the metric Δ. Then:

(a) each $K \in \mathcal{C}$ is the limit of a sequence $(K_i)_{i\in\mathbb{N}}$ with $K_i \in \mathcal{C}_i$,

(b) the limit of any convergent subsequence $(K_{i_j})_{j\in\mathbb{N}}$ with $K_{i_j} \in \mathcal{C}_{i_j}$ belongs to $\mathcal{C}$.

There exists $K \in \mathcal{C}$ with $\varphi(\mathcal{C}) = V(K)$. By (a), there is a sequence $(K_i)_{i\in\mathbb{N}}$ with $K_i \in \mathcal{C}_i$ and $K = \lim_{i\to\infty} K_i$. Then $V(K_i) \le \varphi(\mathcal{C}_i)$ and hence

$$\varphi(\mathcal{C}) = V(K) = \lim_{i\to\infty} V(K_i) \le \liminf_{i\to\infty} \varphi(\mathcal{C}_i) \le \limsup_{i\to\infty} \varphi(\mathcal{C}_i).$$

There is a subsequence $(\mathcal{C}_{i_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} \varphi(\mathcal{C}_{i_j}) = \limsup_{i\to\infty} \varphi(\mathcal{C}_i)$ and there exists $K_{i_j} \in \mathcal{C}_{i_j}$ with $\varphi(\mathcal{C}_{i_j}) = V(K_{i_j})$. A subsequence of $(K_{i_j})_{j\to\infty}$, say the sequence itself (after changing the notation), converges to some convex body K_0. By (b), $K_0 \in \mathcal{C}$, hence

$$\limsup_{i\to\infty} \varphi(\mathcal{C}_i) = \lim_{j\to\infty} \varphi(\mathcal{C}_{i_j}) = \lim_{j\to\infty} V(K_{i_j}) = V(K_0) \le \varphi(\mathcal{C}).$$

This completes the proof of $\lim_{i\to\infty} \varphi(\mathcal{C}_i) = \varphi(\mathcal{C})$. Thus, φ is continuous.

By Theorem 1, the mapping γ is continuous, hence the mapping $\varphi \circ \gamma$ is continuous. Let $C_i, C \in \mathcal{H}$ be such that $C_i \to C$. The sequence $(\zeta(C_i))_{i\in\mathbb{N}}$ of convex bodies is bounded and hence, by the Blaschke selection theorem, it has a subsequence $(\zeta(C_{i_j}))_{j\in\mathbb{N}}$ converging to some convex body K. Since $\zeta(C_{i_j})$ is a tight cover of C_{i_j}, the body K is a tight cover of C. Moreover,

$$V(K) = \lim_{j\to\infty} V(\zeta(C_{i_j})) = \lim_{j\to\infty} \varphi(\gamma(C_{i_j})) = \varphi(\gamma(C)).$$

Thus, K is a maximal tight cover of C and hence $K = \zeta(C)$. Since every convergent subsequence of the sequence $(\zeta(C_i))_{i\in\mathbb{N}}$ converges to $\zeta(C)$, the sequence itself converges to $\zeta(C)$. Thus ζ is continuous. □

For given $C \in \mathcal{H}$, there are different ways of choosing a completion from $\gamma(C)$ with an extremal property (for example, with minimal Hausdorff distance from C), and it is a natural question whether they can play a similar role as the maximal volume completion in the above result. On the other hand, one may ask whether general results on the existence of continuous selections can be applied in the case of the mapping γ. The main difficulty is that most of them, as, for example, the classical Michael selection theorem, are valid only for convex valued mappings. The diametric completion mapping γ is in general not convex, as we point out in the next section.

3. Normed spaces with property (A)

When is γ convex valued? In the infinite dimensional case, this question has been answered in the context of $C(K)$ spaces [14]. Here we are concerned with the finite dimensional case. The Minkowski sum of two sets of constant width is again a set of constant width, while the corresponding result for diametrically maximal sets is false in general [16]. We relate this question to Eggleston's property (A), requiring that every diametrically maximal set is of constant width.

Proposition 6. *A finite dimensional normed space satisfies property (A) if and only if the set $\gamma(C)$ is convex, for every $C \in \mathcal{H}$.*

Proof. Suppose that the space satisfies property (A) and let $K, M \in \gamma(C)$. Since K, M are of (the same) constant width $\lambda = \operatorname{diam} C$, we have $K - K = \lambda B$ and $M - M = \lambda B$. For $\alpha \in [0,1]$ this gives

$$(\alpha K + (1-\alpha)M) - (\alpha K + (1-\alpha)M) = \lambda B.$$

Thus, $\alpha K + (1-\alpha)M$ is of constant width λ and, therefore, diametrically maximal. Since $C = \alpha C + (1-\alpha)C \subset \alpha K + (1-\alpha)M$, it is clear that $\alpha K + (1-\alpha)M \in \gamma(C)$.

Conversely, suppose that $\gamma(C)$ is always convex. Let K be a diametrically maximal set (of diameter > 0, without loss of generality). Let C be a longest segment contained in K. Then K is a completion of C. The segment C is centrally symmetric, without loss of generality with center 0. Then $-K$ is also a completion of C. Since $\gamma(C)$ is convex, the set $(K-K)/2$ is also a completion of C, and it is centrally symmetric. It is not difficult to prove that, when a set is both diametrically maximal and centrally symmetric, it must be a homothet of the unit ball. Therefore, K is of constant width. ☐

Proposition 7. *In a finite dimensional space X, the set of all norms satisfying property (A) is closed.*

Proof. Let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be a sequence of norms on X with unit balls $(B_n)_{n\in\mathbb{N}}$, converging to a norm $\|\cdot\|$ with unit ball B for $n \to \infty$. Suppose that every norm $\|\cdot\|_n$ satisfies property (A), and let K be a convex body which is diametrically maximal with respect to $\|\cdot\|$. Let $\rho > 0$ be given and consider the parallel body $K_\rho := K + \rho B$. We can choose a number $\alpha > 0$ such that

$$\operatorname{diam}_{\|\cdot\|}(K \cup \{y\}) > \operatorname{diam}_{\|\cdot\|} K + \alpha$$

for every $y \in X \setminus K_\rho$. In fact, suppose this were false. Then there exists, to each $m \in \mathbb{N}$, a point $y_m \in X \setminus K_\rho$ with

$$\operatorname{diam}_{\|\cdot\|}(K \cup \{y_m\}) \leq \operatorname{diam}_{\|\cdot\|} K + \frac{1}{m}.$$

The above inequality implies that $(y_m)_{m\in\mathbb{N}}$ is a bounded sequence and so we can assume (going over to a subsequence) that $(y_m)_{m\in\mathbb{N}}$ converges to some y for $m \to \infty$. Then $y \notin K$ and $\operatorname{diam}_{\|\cdot\|}(K \cup \{y\}) \leq \operatorname{diam}_{\|\cdot\|} K$, a contradiction. Now choose $\lambda < 1$ satisfying

$$\lambda^2 > \frac{\operatorname{diam}_{\|\cdot\|} K}{\operatorname{diam}_{\|\cdot\|} K + \alpha}.$$

There exists $n \in \mathbb{N}$ with

$$\lambda\|x\| \leq \|x\|_n \leq \frac{1}{\lambda}\|x\|$$

for all $x \in X$ and thus

$$\lambda \operatorname{diam}_{\|\cdot\|} K \leq \operatorname{diam}_{\|\cdot\|_n} K \leq \frac{1}{\lambda} \operatorname{diam}_{\|\cdot\|} K.$$

Let K_n be a $\|\cdot\|_n$-completion of K and let $y \in X \setminus K_\rho$. By the choice of α, we have

$$\begin{aligned} \operatorname{diam}_{\|\cdot\|_n}(K \cup \{y\}) &\geq \lambda \operatorname{diam}_{\|\cdot\|}(K \cup \{y\}) \\ &> \lambda(\operatorname{diam}_{\|\cdot\|} K + \alpha) > \frac{1}{\lambda} \operatorname{diam}_{\|\cdot\|} K \\ &\geq \operatorname{diam}_{\|\cdot\|_n} K, \end{aligned}$$

showing that $K_n \subset K_\rho$. Since $\|\cdot\|_n$ has property (A), the set K_n is of constant $\|\cdot\|_n$-width, hence

$$(\operatorname{diam}_{\|\cdot\|_n} K_n)B_n = K_n - K_n \subset K_\rho - K_\rho = K - K + 2\rho B$$

and therefore, using that $\lambda B \subset B_n$, we get

$$\lambda^2 (\operatorname{diam}_{\|\cdot\|} K)\, B \subset (\operatorname{diam}_{\|\cdot\|_n} K_n)B_n \subset K - K + 2\rho B.$$

Since λ can be chosen arbitrary close to 1 and ρ arbitrary close to 0, we conclude that $(\operatorname{diam}_{\|\cdot\|} K)\, B \subset K - K$. The relation $K - K \subset (\operatorname{diam}_{\|\cdot\|} K)\, B$ holds trivially, hence K is of constant width. Thus $\|\cdot\|$ satisfies property (A). □

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