

STABLE DETERMINATION OF CONVEX BODIES FROM SECTIONS

KÁROLY J. BÖRÖCZKY¹ and ROLF SCHNEIDER²

¹ Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13–15,
H-1053 Budapest, Hungary
e-mail: carlos@renyi.hu

² Mathematisches Institut, Albert-Ludwigs-Universität, Eckerstr. 1,
D-79104 Freiburg i. Br., Germany
e-mail: rolf.schneider@math.uni-freiburg.de

Communicated by G. Fejes Tóth

(Received February 21, 2007; accepted October 1, 2007)

Abstract

A star body (with respect to the origin 0) in \mathbb{R}^d ($d \geq 3$) which has 0 as center of symmetry is uniquely determined by the $(d-1)$ -dimensional volumes of its sections with hyperplanes through 0. Without the symmetry assumption, we show that a star body is uniquely determined by the volumes and centroids of its hyperplane sections through 0. For convex bodies, we prove a stability version of this result.

1. Introduction

A star body K in Euclidean space \mathbb{R}^d is a nonempty compact set which is starshaped with respect to the origin 0 and has a continuous positive radial function, defined by

$$\rho_K(v) := \max \{ \lambda \geq 0 : \lambda v \in K \} \quad \text{for } v \in \mathbb{R}^d \setminus \{0\}.$$

If $u \in S^{d-1} := \{x \in \mathbb{R}^d : \langle x, x \rangle = 1\}$ (where $\langle \cdot, \cdot \rangle$ is the scalar product) and $u^\perp := \{x \in \mathbb{R}^d : \langle x, u \rangle = 0\}$, then the $(d-1)$ -dimensional volume of the sec-

2000 *Mathematics Subject Classification*. Primary 52A20, 52A40.

Key words and phrases. Geometric tomography, sections of convex bodies, spherical Radon transform, stability.

This work was supported by the EU-project DiscConvGeo (MTKD-CT-2005-014333). The first author was supported by OTKA grants 068398 and 049301.

tion of K by the $(d-1)$ -subspace orthogonal to u is given by

$$(1) \quad v_{d-1}(K \cap u^\perp) = \frac{1}{d-1} \int_{s_u} \rho_K^{d-1} d\sigma_u.$$

Here $s_u := S^{d-1} \cap u^\perp$, and σ_u denotes the $(d-2)$ -dimensional spherical Lebesgue measure on the great subsphere s_u . It is well known that the spherical Radon transform \mathcal{R} , defined by

$$(\mathcal{R}f)(u) := \int_{s_u} f d\sigma_u$$

for continuous functions f , is injective on the even functions (see, e.g., [4], Proposition 3.4.12). It follows that two centrally symmetric star bodies K, L satisfying

$$v_{d-1}(K \cap u^\perp) = v_{d-1}(L \cap u^\perp) \quad \text{for all } u \in S^{d-1}$$

must be identical. This is a classical result of Geometric Tomography, and we refer to the books of Gardner [1] and Groemer [4] for more information.

Without the symmetry assumption, the quoted result does not hold, not even for convex bodies, and the unique determination of non-symmetric star or convex bodies by section data requires more information. Groemer [5] defined halfplanes $H(u, w) := \{x \in u^\perp : \langle x, w \rangle \geq 0\}$ for $u \in S^{d-1}$, $w \in S^{d-1} \cap u^\perp$, and showed that the assumption

$$v_{d-1}(K \cap H(u, w)) = v_{d-1}(L \cap H(u, w))$$

for two star bodies K, L and all pairs (u, w) of orthogonal unit vectors implies $K = L$. He also proved a more general stability result. Goodey and Weil [2] remarked that a star body K seems to be overdetermined by the section function $(u, w) \mapsto v_{d-1}(K \cap H(u, w))$, and they considered instead certain mean values of such section functions, also for lower-dimensional sections. They established a number of corresponding uniqueness and stability results. Surprisingly, they also found cases where uniqueness fails.

In the present note, we first wish to point out that the classical result mentioned above has a counterpart for non-symmetric star bodies if not only the volumes of the hyperplane sections, but also their centroids are taken into account. The centroid (center of gravity) of the section $K \cap u^\perp$ of the star body K can, if polar coordinates are used, be expressed by

$$c_{d-1}(K \cap u^\perp) = \frac{1}{d} \int_{s_u} \rho_K^d(v) v d\sigma_u(v).$$

If K, L are star bodies satisfying

$$(2) \quad v_{d-1}(K \cap u^\perp) = v_{d-1}(L \cap u^\perp) \quad \text{for } u \in S^{d-1}$$

and

$$(3) \quad c_{d-1}(K \cap u^\perp) = c_{d-1}(L \cap u^\perp) \quad \text{for } u \in S^{d-1},$$

then $K = L$. In fact, since the spherical Radon transform of a continuous function on S^{d-1} uniquely determines the even part of the function, the assumption (2) implies that the even part of $\rho_K^{d-1} - \rho_L^{d-1}$ vanishes, thus

$$(4) \quad \rho_K^{d-1}(v) - \rho_L^{d-1}(v) = -\rho_K^{d-1}(-v) + \rho_L^{d-1}(-v) \quad \text{for } v \in S^{d-1}.$$

Similarly, the assumption (3) yields (if the result on the spherical Radon transform is applied coordinate-wise) that the even part of the function $v \mapsto [\rho_K^d(v) - \rho_L^d(v)]v$ vanishes, and this gives

$$(5) \quad \rho_K^d(v) - \rho_L^d(v) = \rho_K^d(-v) - \rho_L^d(-v) \quad \text{for } v \in S^{d-1}.$$

Suppose now that there exists some $v \in S^{d-1}$ with $\rho_K(v) \neq \rho_L(v)$, say $\rho_K(v) < \rho_L(v)$. Then $\rho_K^{d-1}(v) < \rho_L^{d-1}(v)$, hence (4) gives $\rho_L^{d-1}(-v) < \rho_K^{d-1}(-v)$. This yields $\rho_L^d(-v) < \rho_K^d(-v)$, and now (5) gives $\rho_K^d(v) > \rho_L^d(v)$, a contradiction.

The main purpose of this note is to prove a stability version of this new uniqueness result. For that, however, we will have to restrict ourselves to the case of convex bodies.

2. A stability result

We consider the space \mathcal{K}_0^d of convex bodies in \mathbb{R}^d containing 0 and, especially, the space $\mathcal{K}^d(r, R)$ of all convex bodies K satisfying $rB^d \subset K \subset RB^d$, where B^d denotes the unit ball and $0 < r < R$ are given numbers. On \mathcal{K}_0^d we use the radial L_2 -metric, defined by

$$\rho_2(K, L) := \|\rho_K - \rho_L\| \quad \text{for } K, L \in \mathcal{K}_0^d,$$

where

$$\|f\| := \left(\int_{S^{d-1}} |f|^2 d\sigma \right)^{1/2}$$

is the L_2 -norm of the square-integrable function f on S^{d-1} ; here σ denotes spherical Lebesgue measure. We use this notation also for \mathbb{R}^d -valued functions f , denoting by $|f|$ the Euclidean norm of f .

For convex bodies $K, L \in \mathcal{K}^d(r, R)$, the Hausdorff distance $\delta(K, L)$ can be estimated in terms of the radial L_2 -metric, by

$$\delta(K, L) \leq c_d R^2 r^{-(d+3)/(d+1)} \rho_2(K, L)^{2/(d+1)},$$

with an explicit constant c_d depending only on the dimension d ; see Groemer [3], Lemma 3, or [4], Lemma 2.3.2. Thus, the result (8) below leads also to an estimate of the Hausdorff distance, while the assumptions (6) and (7) are weaker than the corresponding ones with the maximum norm.

For convenience, we write

$$v_{d-1}(K, u) := v_{d-1}(K \cap u^\perp), \quad c_{d-1}(K, u) := c_{d-1}(K \cap u^\perp) \quad \text{for } u \in S^{-1}.$$

Now we can formulate our main result.

THEOREM. *Let $K, L \in \mathcal{K}^d(r, R)$. If, for some $\varepsilon \geq 0$,*

$$(6) \quad \|v_{d-1}(K, \cdot) - v_{d-1}(L, \cdot)\| \leq \varepsilon$$

and

$$(7) \quad \|c_{d-1}(K, \cdot) - c_{d-1}(L, \cdot)\| \leq \varepsilon,$$

then

$$(8) \quad \rho_2(K, L) \leq c(d, r, R, \varepsilon_0) \varepsilon^{2/d},$$

with an explicit constant $c(d, r, R, \varepsilon_0)$ depending only on d, r, R and an upper bound ε_0 for ε .

Similarly as in [6], where an analogous result for mean widths and Steiner points of projections was obtained, the proof rests on a stability result for the spherical Radon transform on even functions. In the following lemma (see Theorem 3.4.14 in Groemer [4]), F^+ and F^- denote the even and the odd part of a function F on S^{d-1} , thus

$$F^+(u) := \frac{1}{2}(F(u) + F(-u)), \quad F^-(u) := \frac{1}{2}(F(u) - F(-u)), \quad u \in S^{d-1}.$$

The constant β_d is defined by $\beta_3 = 2^{-3/4}$ and, for $d \geq 4$, by

$$\beta_d = \begin{cases} (d-1)^{-(d-2)/4} 1 \cdot 3 \cdots (d-3) & \text{if } d \text{ is even,} \\ \frac{1}{\sqrt{2}} (d-1)^{-(d-2)/4} 2 \cdot 4 \cdots (d-3) & \text{if } d \text{ is odd.} \end{cases}$$

By ∇_0 we denote the gradient operator on the sphere S^{d-1} . The constant σ_d is given by $\sigma(S^{d-1})$.

LEMMA. *If F_1 and F_2 are twice continuously differentiable functions on S^{d-1} ($d \geq 3$), then*

$$\|F_1^+ - F_2^+\| \leq h_d(F_1, F_2) \|\mathcal{R}F_1 - \mathcal{R}F_2\|^{2/d}$$

with

$$\begin{aligned} & h_d(F_1, F_2) \\ &= \frac{1}{\sigma_{d-1}} \left(2\sigma_{d-1}^2 \beta_d^{-\frac{4}{d-2}} (\|\nabla_0 F_1\|^2 + \|\nabla_0 F_2\|^2) + \|\mathcal{R}F_1 - \mathcal{R}F_2\|^2 \right)^{\frac{d-2}{2d}}. \end{aligned}$$

PROOF OF THE THEOREM. We assume that the assumptions are satisfied and, moreover, that K and L have twice continuously differentiable radial functions. If the theorem is proved under this assumption, then the general case follows by approximation.

Putting $F_1 := \rho_K^{d-1}$, $F_2 := \rho_L^{d-1}$, we have

$$\mathcal{R}F_1 - \mathcal{R}F_2 = (d-1)[v_{d-1}(K, \cdot) - v_{d-1}(L, \cdot)],$$

hence the Lemma together with (6) gives

$$(9) \quad \|(\rho_K^{d-1})^+ - (\rho_L^{d-1})^+\| \leq h_d(F_1, F_2) ((d-1)\varepsilon)^{2/d} =: \eta_1.$$

Next, we choose $e \in S^{d-1}$ and put $G_1(v) := \rho_K^d(v)\langle v, e \rangle$ and $G_2(v) := \rho_L^d(v)\langle v, e \rangle$ for $v \in S^{d-1}$. Then

$$\begin{aligned} \|\mathcal{R}G_1 - \mathcal{R}G_2\|^2 &= \int_{S^{d-1}} |\mathcal{R}G_1 - \mathcal{R}G_2|^2 d\sigma \\ &= \int_{S^{d-1}} \left| \int_{s_u} (\rho_K^d - \rho_L^d)(v)\langle v, e \rangle d\sigma_u(v) \right|^2 d\sigma(u), \end{aligned}$$

where we can estimate

$$\begin{aligned} \left| \int_{s_u} (\rho_K^d - \rho_L^d)(v) \langle v, e \rangle d\sigma_u(v) \right| &= \left| \left\langle \int_{s_u} (\rho_K^d - \rho_L^d)(v) v d\sigma_u(v), e \right\rangle \right| \\ &\leq \left| \int_{s_u} (\rho_K^d - \rho_L^d)(v) v d\sigma_u(v) \right|. \end{aligned}$$

This yields

$$\begin{aligned} \|\mathcal{R}G_1 - \mathcal{R}G_2\|^2 &\leq \int_{S^{d-1}} \left| \int_{s_u} (\rho_K^d - \rho_L^d)(v) v d\sigma_u(v) \right|^2 d\sigma(u) \\ &= d^2 \int_{S^{d-1}} |c_{d-1}(K, u) - c_{d-1}(L, u)|^2 d\sigma(u) \\ &= d^2 \|c_{d-1}(K, \cdot) - c_{d-1}(L, \cdot)\|^2. \end{aligned}$$

The lemma together with (7) gives

$$\|G_1^+ - G_2^+\| \leq h_d(G_1, G_2)(d\varepsilon)^{2/d}.$$

Since $G_1(v) = \rho_K^d(v) \langle v, e \rangle$, we have $G_1^+(v) = (\rho_K^d)^-(v) \langle v, e \rangle$, and a similar relation holds for G_2 . Thus, we get

$$\begin{aligned} &\int_{S^{d-1}} |(\rho_K^d)^- - (\rho_L^d)^-|^2(v) \langle v, e \rangle^2 d\sigma(v) \\ &= \|G_1^+ - G_2^+\|^2 \leq [h_d(G_1, G_2)(d\varepsilon)^{2/d}]^2. \end{aligned}$$

We insert for e the vectors of an orthonormal basis, then summation gives

$$(10) \quad \|(\rho_K^d)^- - (\rho_L^d)^-\| \leq \sqrt{d} h_d(G_1, G_2)(d\varepsilon)^{2/d} =: \eta_2.$$

Explicitly, the inequalities (9) and (10) can be written as

$$(11) \quad I^+ := \int_{S^{d-1}} |\rho_K^{d-1}(v) - \rho_L^{d-1}(v) + \rho_K^{d-1}(-v) - \rho_L^{d-1}(-v)|^2 d\sigma(v) \leq 4\eta_1^2,$$

$$(12) \quad I^- := \int_{S^{d-1}} |\rho_K^d(v) - \rho_L^d(v) - \rho_K^d(-v) + \rho_L^d(-v)|^2 d\sigma(v) \leq 4\eta_2^2.$$

We make use of the identity

$$(13) \quad \rho_K^m - \rho_L^m = (\rho_K - \rho_L) \sum_{i=0}^{m-1} \rho_K^i \rho_L^{m-i},$$

valid for $m \in \mathbb{N}$. Since $K, L \in \mathcal{K}^d(r, R)$, we have $\rho_K(v), \rho_L(v) \geq r > 0$ for $v \in S^{d-1}$, hence

$$\gamma_m := \sum_{i=0}^{m-1} \rho_K^i \rho_L^{m-i} \geq mr^m \quad \text{on } S^{d-1}.$$

Using (13) in (11), we obtain

$$(14) \quad \int_{S^{d-1}} |\rho_K(v) - \rho_L(v) + \alpha(v)|^2 d\sigma(v) \leq \frac{I^+}{(d-1)^2 r^{2(d-1)}}$$

with

$$(15) \quad \alpha(v) := \frac{\rho_K^{d-1}(-v) - \rho_L^{d-1}(-v)}{\gamma_{d-1}(v)}.$$

Similarly, (12) yields

$$(16) \quad \int_{S^{d-1}} |\rho_K(v) - \rho_L(v) - \beta(v)|^2 d\sigma(v) \leq \frac{I^-}{d^2 r^{2d}}$$

with

$$(17) \quad \beta(v) := \frac{\rho_K^d(-v) - \rho_L^d(-v)}{\gamma_d(v)}.$$

Let

$$S^+ := \{v \in S^{n-1} : (\rho_K(v) - \rho_L(v)) \alpha(v) \geq 0\},$$

$$S^- := \{v \in S^{n-1} : (\rho_K(v) - \rho_L(v)) \beta(v) \leq 0\}.$$

Then we get

$$\begin{aligned} \int_{S^+} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) &\leq \int_{S^+} |\rho_K(v) - \rho_L(v) + \alpha(v)|^2 d\sigma(v) \\ &\leq \frac{I^+}{(d-1)^2 r^{2(d-1)}}, \\ \int_{S^-} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) &\leq \int_{S^-} |\rho_K(v) - \rho_L(v) - \beta(v)|^2 d\sigma(v) \\ &\leq \frac{I^-}{d^2 r^{2d}}. \end{aligned}$$

Now $\alpha(v)\beta(v) \geq 0$ for all $v \in S^{d-1}$. It follows that $S^+ \cup S^- = S^{d-1}$ and hence that

$$\begin{aligned} (18) \quad &\int_{S^{d-1}} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) \\ &\leq \int_{S^+} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) + \int_{S^-} |\rho_K(v) - \rho_L(v)|^2 d\sigma(v) \\ &\leq \frac{4\eta_1^2}{(d-1)^2 r^{2(d-1)}} + \frac{4\eta_2^2}{d^2 r^{2d}}. \end{aligned}$$

It remains to estimate the constants $h_d(F_1, F_2)$ and $h_d(G_1, G_2)$. A special case of the estimate on p. 243 of Groemer [4] gives

$$(19) \quad \|\nabla_0 \rho_K^m\| \leq m \sqrt{(d-1)\sigma_d} \frac{R^{m+1}}{r}$$

for $m > 0$. This yields

$$(20) \quad h_d(F_1, F_2) \leq \frac{(d-1)^{\frac{d-2}{d}}}{\sigma_{d-1}} \left(4(d-1)\sigma_{d-1}^2 \sigma_d \beta_d^{-\frac{4}{d-2}} \frac{R^{2d}}{r^2} + \varepsilon_0^2 \right)^{\frac{d-2}{2d}},$$

where ε_0 is an upper bound for ε .

We write $G_1(v) = \rho(v)\eta(v)$ with $\rho(v) := \rho_K^d(v)$ and $\eta(v) := \langle v, e \rangle$. Then

$$\begin{aligned} |\nabla_0 G_1|^2 &= \rho^2 |\nabla_0 \eta|^2 + 2\rho\eta \langle \nabla_0 \rho, \nabla_0 \eta \rangle + \eta^2 |\nabla_0 \rho|^2 \\ &\leq R^{2d} |\nabla_0 \eta|^2 + 2R^d |\nabla_0 \rho| |\nabla_0 \eta| + |\nabla_0 \rho|^2, \end{aligned}$$

hence

$$\|\nabla_0 G_1\|^2 \leq R^{2d} \|\nabla_0 \eta\|^2 + 2R^d \int_{S^{d-1}} |\nabla_0 \rho| |\nabla_0 \eta| \, d\sigma + \|\nabla_0 \rho\|^2.$$

Here

$$\|\nabla_0 \eta\|^2 \leq (d-1)\sigma_d,$$

as proved in [6], and

$$\int_{S^{d-1}} |\nabla_0 \rho| |\nabla_0 \eta| \, d\sigma \leq \|\nabla_0 \rho\| \|\nabla_0 \eta\|.$$

Using (19) for $m = d$, we get

$$\begin{aligned} (21) \quad &h_d(G_1, G_2) \\ &\leq \frac{1}{\sigma_{d-1}} \left(4(d-1)\sigma_{d-1}^2 \sigma_d \beta_d^{-\frac{4}{d-2}} \left[R^{2d} + 2d \frac{R^{2d+1}}{r} + d^2 \frac{R^{2d+2}}{r^2} \right] + d^2 \varepsilon_0^2 \right)^{\frac{d-2}{d}}. \end{aligned}$$

Now an explicit value of the constant $c(d, r, R, \varepsilon_0)$ in the theorem can be read off from (18), (9), (10), (20), (21). \square

REFERENCES

- [1] GARDNER, R. J., *Geometric Tomography*, Encyclopedia of Mathematics and its Applications, vol. 58. Cambridge University Press, Cambridge 1995. *MR 96j:52006*
- [2] GOODEY, P. and WEIL, W., Average section functions for star-shaped sets, *Adv. Appl. Math.*, **36** (2006), no. 1, 70–84. *MR 2007b:52012*
- [3] GROEMER, H., Stability results for convex bodies and related spherical integral transformations, *Adv. Math.*, **109** (1994), no. 1, 45–74. *MR 95j:52005*
- [4] GROEMER, H., *Geometric Applications of Fourier Series and Spherical Harmonics*, Encyclopedia of Mathematics and its Applications, vol. 61. Cambridge University Press, Cambridge 1996. *MR 97j:52001*

- [5] GROEMER, H., On a spherical integral transformation and sections of star bodies, *Monatsh. Math.*, **126** (1998), no. 2, 117–124. *MR 99k*:52005
- [6] SCHNEIDER, R., Stable determination of convex bodies from projections, *Monatsh. Math.*, **150** (2007), no. 3, 241–247. *MR 2008b*:52010