Parabolic Partial Differential Equations

Vorlesung: Armin Schikorra
Mitschrift: Julian Scheuer

Version: June 23, 2017

Contents

0.1. Overview

Parabolic equations such as

$$\partial_t u - Lu = f$$

and their nonlinear counterparts: Equations such as, see

Elliptic PDE: Describe steady states of an energy system, for example a steady heat distribution in an object.

Parabolic PDE: describe the time evolution towards such a steady state.

Flows: Consider the energy functional

$$\mathcal{E} \colon \mathbb{R}^n \to \mathbb{R}.$$

Crititcal points are also called stationary points

Now let $u(0)$ satisfy $D\mathcal{E}(u(0)) \neq 0$. Set

$$u(1) = u(0) - D\mathcal{E}(u(0)),$$

$$u(k+1) = u(k) - D\mathcal{E}(u(k)).$$

Infinitesimally:

$$u(t+h) = u(t) - hD\mathcal{E}(u(t)),$$

i.e.

$$\frac{u(t+h) - u(t)}{h} = -D\mathcal{E}(u(t)).$$

$h \to 0$ gives:

$$\dot{u} = -D\mathcal{E}(u(t)).$$

This is the flow along $\mathcal{E}$.

Example 0.1.1. On $H^1(\Omega)$ consider the energy

$$\mathcal{E}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2.$$

Then

$$D\mathcal{E}(u) = -\Delta u$$

and the flow

$$\dot{u} = \Delta u$$

is called the *heat equation*.

Aim of this lecture: We want to understand fully nonlinear parabolic PDE, e.g.

- Bellmann-equation

$$\dot{u} - \sup_{\alpha \in A} L_\alpha u + \lambda u = 0.$$

- Mean curvature flow

$$\dot{u} = |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

- Kähler-Ricci-Flow

$$\dot{u} = \log \det(D^2 u).$$

We study existence, uniqueness and regularity by using viscosity solutions and comparison principles (cf. [**IS13**]).

CHAPTER I

The Heat Equation

1.1. Definitions

(Cf. [**Eva98**, Section 2]). The Laplace operator Δ is gives as

$$\Delta u(x_1, \dots, x_n) = \partial_{x_i}\partial_{x_i} u(x_1, \dots, x_n).$$

We will use the so-called *Einstein's summation formula* which says that repeated indices are always summed over, that is

$$\partial_{x_i}\partial_{x_i} u(x_1, \dots, x_n) \equiv \sum_{i=1}^{n} \partial_{x_i}\partial_{x_i} u(x_1, \dots, x_n).$$

Sometimes, we write u_{x_i} for $\partial_{x_i} u$.

We want to study time-dependent problems, where we denote with $t \in (0, \infty)$ the time. Sometimes we write $\mathbb{R}^{n+1}_+$ for $\mathbb{R}^n \times (0, \infty)$.

More precisely, we want to study the heat equation "$\partial_t - \Delta$". For example, we want to understand existence, uniqueness questions for solutions $u = u(t, x_1, \dots, x_n) : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ of

$$(\partial_t - \Delta) u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \tag{1.1.1}$$

The right-hand side is zero, and we call this the homogeneous heat equation.

Also we ask us the same questions about the inhomogeneous heat equation, for $f(x,t) : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$

$$(\partial_t - \Delta) u = f \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Let $\Omega \subset \mathbb{R}^n$ be open. Define

$$\Omega_T = \Omega \times (0, T].$$

The Laplace operator for $u\colon \mathbb{R}^n \to \mathbb{R}$ is

$$\Delta u = \sum_{i=1}^{n} u_{ii} = u_{ii},$$

where we use the Einstein summation.

For a domain $X \subset \mathbb{R}^{n+1}$ let $f \in C^k_l(X)$ if and only if

$$\partial_t^l D^k f$$

are continuous. For general X the derivatives have to be continuously extendable up to the boundary.

1.2. Fundamental solution

Studying solutions of the heat equation, a first step might be to find simple solutions. Clearly, any constant function $u \equiv \mathit{const}$ is a solution to (1.1.1). But that is too easy, and gives us no useful information about (1.1.1). Also, any solution $v : \mathbb{R}^n \to \mathbb{R}$ of $\Delta v = 0$ becomes a solution of (1.1.1), simply set $u(x,t) := v(x)$. Again, this does not give us too much information about the structure of (1.1.1). So we need to find a nontrivial, time-dependent solution of (1.1.1). For this we make the interpretation of (1.1.1) as a ordinary differential equation in t. We all know

$$u_t - \mu u = 0$$

has the solution $u(t) = e^{t\mu}u(0)$ for any $\mu \in \mathbb{R}$. So in some sense, one might think that

$$u(t,x) = e^{t\Delta}u(0,x) \tag{1.2.1}$$

is a solution (but it is not clear what $e^{t\Delta}$ means, and we don't want to get into this here; just note this is actually a thing and this is possible). To make (1.2.1) precise and meaningful for us, we use the Fourier transform.

$$\hat{u}(\xi,t) := \int_{\mathbb{R}^n} e^{-i\langle \xi,x\rangle} u(x,t)\ dx.$$

We have (Exercise 1)

$$\widehat{\Delta u}(\xi,t) = -|\xi|^2 \hat{u}(\xi,t),$$

and thus, after Fouriertransform (1.1.1) becomes

$$\partial_t \hat{u}(\xi,t) + |\xi|^2 \hat{u}(\xi,t) = 0 \quad \forall (\xi,t) \in \mathbb{R}^{n+1}_+. \tag{1.2.2}$$

If we fix $\xi \in \mathbb{R}^n$ and set $v(t) := \hat{u}(\xi,t)$, then this is nothing but

$$v'(t) + |\xi|^2 v(t) = 0,$$

and the (unique is $v(0)$ is chosen) solution to this equation is $v(t) = e^{-t|\xi|^2}v(0)$. That is, (1.2.2) implies

$$\hat{u}(\xi,t) = e^{-t|\xi|^2}\hat{u}(\xi,0).$$

The simplest situation arrises, if we assume that $\hat{u}(\xi,0) = 1$. This is not possible for any *function* $u(x,0)$, but $\hat{u}(\xi,0) = 1$ (at least formally) is the Fourier transform of the Dirac measure $u(\cdot,0) := \delta_0$ defined as $\int_{\mathbb{R}^n} f(x)\delta_0(x)\ dx = f(0)$. For this choice of u we have (see Exercise 1)

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},$$

which we shall call the fundamental solution.

DEFINITION 1.2.1 (Fundamental solution). The function

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & t < 0 \end{cases}$$

is called the *fundamental solution of the heat equation*, or the *heat kernel*.

One can show, see Exercise 2, that $\Phi(x,t)$ is the solution to

$$\begin{cases} (\partial_t - \Delta)\Phi = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \Phi(x,0) = \delta_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2.3}$$

Here δ_0 is the Dirac-measure from above.

Another nice feature is

LEMMA 1.2.2. *For any $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\ dx = 1.$$

PROOF. From Exercise 1 and the above calculations we have

$$\int_{\mathbb{R}^n} \Phi(x,t)\ dx = \hat{\Phi}(0,t) = e^{-t0} = 1.$$

□

More generally, the above Fouriertransform argument implies that *any* solution of (1.1.1) has actually the form

$$u(x,t) = \Phi * g \equiv \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\ dy. \tag{1.2.4}$$

This is true since,

$$\hat{u}(\xi,t) = \hat{\Phi}(\xi,t)\ \hat{u}(\xi,0).$$

Using the convolution formula, see Exercise 1, this implies (at least formally, under convergence assumptions) (1.2.4).

Actually, this is precise.

THEOREM 1.2.3 (Potential solution). *Let $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Define u by (1.2.4). Then*

(1) $u \in C^\infty(\mathbb{R}^{n+1}_+)$,
(2) $(\partial_t - \Delta)u = 0$ *in* $\mathbb{R}^{n+1}_+$
(3) For each $x_0 \in \mathbb{R}^n$,

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0).$$

PROOF. For $t > 0$, $\Phi(z,t)$ is smooth in z and t-direction, so by convolution estimates (derivatives commute with the integral), u is smooth.

Also for $t > 0$, we have by commutation of derivatives and integrals,

$$u_t(x) - \Delta u(x,t) = \int_{\mathbb{R}^n} \left(\Phi_t(x-y,t) - \Delta\Phi(x-y,t)\right) g(y)\, dy.$$

The latter is constantly zero by (1.2.3).

Finally, we need to show the boundary data. Pick $\varepsilon > 0$ and $x_0 \in \mathbb{R}^n$. In view of Lemma 1.2.2, for any $(x,t) \in \mathbb{R}^{n+1}_+$,

$$u(x,t) - g(x_0) = \int_{\mathbb{R}^n} \Phi(x-y,t)\left(g(y) - g(x_0)\right)\, dy. \tag{1.2.5}$$

The idea is now to show that if x is sufficiently close to x_0 and t is sufficiently small, then either $|x-y|$ is small, in which case also $g(y) - g(x_0)$ is small; or $|y - x_0|$ is large, but in this case $\Phi(x-y,t)$ is small for small t.

Let $\delta > 0$ so that

$$|g(y) - g(x_0)| < \varepsilon \quad \text{whenever } |y - x_0| < 2\delta,$$

and moreover so that

$$\int_{\mathbb{R}^n\backslash B(0,\frac{1}{\delta})} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4}}\, dz < \varepsilon.$$

The latter is possible, since we can estimate

$$\int_{\mathbb{R}^n\backslash B(0,\frac{1}{\delta})} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4}}\, dz \precsim \int_{\mathbb{R}^n\backslash B(0,\frac{1}{\delta})} |z|^{-2n} \precsim \delta^n.$$

Now we claim that for a uniform constant $C > 0$

$$|u(x,t) - g(x_0)| \leq C\, \varepsilon \quad \text{whenever } |x - x_0| < \delta \text{ and } |t| < \delta^4. \tag{1.2.6}$$

We split the integral in (1.2.5),

$$|u(x,t) - g(x_0)| \leq \int_{B(x,\delta)} \Phi(x-y,t)\left(g(y) - g(x_0)\right)\, dy + \int_{\mathbb{R}^n\backslash B(x,\delta)} \Phi(x-y,t)\left(g(y) - g(x_0)\right)\, dy$$

For the first integral observe $y \in B(x,\delta)$ and $|x - x_0| < \delta$ implies $|y - x_0| < 2\delta$, and thus

$$\int_{B(x,\delta)} \Phi(x-y,t)\left(g(y) - g(x_0)\right) < \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t) = \varepsilon,$$

the last equality in view of Lemma 1.2.2.

As for the second integral,

$$\int_{\mathbb{R}^n\backslash B(x,\delta)} \Phi(x-y,t)\left(g(y) - g(x_0)\right)\, dy \leq 2\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n\backslash B(0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4t}}\, dz$$

By substitution

$$\int_{\mathbb{R}^n\setminus B(0,\delta)} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4t}}\, dz = \int_{\mathbb{R}^n\setminus B(0,\frac{\delta}{\sqrt{t}})} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4}}\, dz \leq \int_{\mathbb{R}^n\setminus B(0,\frac{1}{\delta})} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4}}\, dz < \varepsilon.$$

(1.2.6) is proven. □

In the next step we would like to find a potential representation for solutions of the inhomogeneous equation (for now starting from $u = 0$)

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) & \text{in } \mathbb{R}^{n+1}_+ \\ u(\cdot,0) \equiv 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2.7}$$

Taking the Fourier transform, setting $v(t) := \hat{u}(\xi,t)$ and $g(t) := \hat{f}(\xi,t)$

$$v'(t) + |\xi|^2 v(t) = g(t). \tag{1.2.8}$$

How do we solve this kind of ODE? We use a trick from ODE-theory, called Duhamel's principle.

For any fixed $s > 0$ we solve the homogeneous equation (with variable $t \in (s,\infty)$).

$$\begin{aligned} w_s'(t) + |\xi|^2 w_s(t) = 0, &\quad t > s \\ w_s(s) = g(s). & \end{aligned} \tag{1.2.9}$$

If we now set

$$v(t) := \int_0^t w_s(t)\, ds,$$

we compute that $v(0) = 0$ and

$$v'(t) = w_s(t) + \int_0^t w_s'(t)\, ds \overset{(1.2.9)}{=} g(t) - |\xi|^2 \int_0^t w_s(t)\, ds = g(t) - |\xi|^2 v(t),$$

that is, v solves (1.2.8). On the other hand, we have a formula for w_s:

$$w_s(t) = e^{-(t-s)|\xi|^2} g(s).$$

Consequently, the solution to (1.2.9) has the form

$$v(t) = \int_0^t e^{-(t-s)|\xi|^2}\, g(s)\, ds.$$

Taking the Fourier transform, the solution u to (1.2.7) has (at least formally) the form

$$u(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, f(y,s)\, dy\, ds \tag{1.2.10}$$

Before we show that (1.2.10) indeed defines a solution for (1.2.7), we need a definition of smoothness.

Definition 1.2.4 (Space-time spaces). A function $f : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is said to belong to $C^k_\ell(\mathbb{R}^{n+1}_+)$ if

$$\underbrace{\partial_t\partial_t\partial_t\partial_t}_{\ell \text{ times}} \underbrace{DDDD}_{k \text{ times}} f$$

exists and is continous.

A function $f \in C^k_\ell(\mathbb{R}^n \times [0,\infty))$ if that derivative can be continuously extended to $t = 0$.

Theorem 1.2.5. *Let $f \in C^2_1(\mathbb{R}^n \times [0,\infty))$, and assume that f has compact support. Let u be defined as in (1.2.10). Then*

(1) $u \in C^2_1(\mathbb{R}^{n+1}_+)$,
(2) $(\partial_t - \Delta)u = f(x,t)$ in $\mathbb{R}^{n+1}_+$
(3) For each $x_0 \in \mathbb{R}^n$,

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = 0$$

Proof. Observe that there is a singularity in the integral when $s = t$. To see that u is C^2_1 we change variables, and have

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(z,r)\, f(x-z,t-r)\ dz\, dr$$

Now we can compute the derivatives,

$$u_t(x,t) = \int_{\mathbb{R}^n} \Phi(z,t)\, f(x-z,0)\ dz + \int_0^t \int_{\mathbb{R}^n} \Phi(z,r)\, f_t(x-z,t-r)\ dz\, dr$$

$$D^2u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(z,r)\, D^2 f(x-z,t-r)\ dz\, dr.$$

Both right-hand sides are bounded if $f \in C^2_1(\mathbb{R}^n)$ and f has compact support.

In order to compute the equation note that for any $t > 0$,

$$u_t(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} \Phi(z,t)\, f(x-z,0)\ dz + \int_0^t \int_{\mathbb{R}^n} \Phi(z,r)\ (\partial_t - \Delta_x)\, f(x-z,t-r)\ dz\, dr.$$

For any small ε we decompose $u_t(x,t) - \Delta u(x,t)$ into three components I_ε, II_ε, III,

$$I_\varepsilon := \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(z,r)\ (\partial_t - \Delta_x)\, f(x-z,t-r)\ dz\, dr$$

$$II_\varepsilon := \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(z,r)\ (\partial_t - \Delta_x)\, f(x-z,t-r)\ dz\, dr$$

$$III := \int_{\mathbb{R}^n} \Phi(z,t)\, f(x-z,0)\ dz$$

For I_ε we compute, in view of Lemma 1.2.2,

$$|I_\varepsilon| \le \varepsilon \left(\|f_t\|_{L^\infty(\mathbb{R}^{n+1}_+)} + \|D^2 f\|_{L^\infty(\mathbb{R}^{n+1}_+)} \right) \xrightarrow{\varepsilon\to 0} 0.$$

For II_ε we do an integration by parts, for this we observe that

$$(\partial_t - \Delta_x)\, f(x-z,t-r) = (-\partial_r - \Delta_z)\, f(x-z,t-r)$$

Integrating by parts, (here we use that $\varepsilon > 0$, so the singularity of Φ is cut away),

$$\begin{aligned} II_\varepsilon = &\int_\varepsilon^t \int_{\mathbb{R}^n} (\partial_r - \Delta_z)\, \Phi(z,r)\, f(x-z,t-r)\; dz\, dr \\ &+ \int_{\mathbb{R}^n} \Phi(z,\varepsilon)\, f(x-z,t-\varepsilon)\; dz - \int_{\mathbb{R}^n} \Phi(z,t)\, f(x-z,0)\; dz \end{aligned}$$

and since Φ solves the heat equation,

$$= 0 + \int_{\mathbb{R}^n} \Phi(z,\varepsilon)\, f(x-z,t-\varepsilon)\; dz - III,$$

We thus have

$$u_t(x,t) - \Delta u(x,t) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(z,\varepsilon)\, f(x-z,t-\varepsilon)\; dz.$$

As in the proof of Theorem 1.2.3, we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(z,\varepsilon)\, f(x-z,t-\varepsilon)\; dz = f(x,t).$$

We thus have shown that $(\partial_t - \Delta)u = f$ in $\mathbb{R}^{n+1}_+$.

For the final claim observe that in view of Lemma 1.2.2

$$\|u\|_{L^\infty} \leq t\|f\|_{L^\infty(\mathbb{R}^n)} \xrightarrow{t\to 0} 0.$$

□

Combining Theorem 1.2.3 and Theorem 1.2.5 we have a full representation formula: let

$$u(x,t) := \int_{\mathbb{R}^n} \Phi(x-y,t)\; g(y)\; dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, f(y,s)\; dy\, ds. \tag{1.2.11}$$

THEOREM 1.2.6. *For f and g as in Theorem 1.2.3 or Theorem 1.2.5, respectively, let u be given by (1.2.11). Then u is a solution of*

$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \mathbb{R}^{n+1}_+ \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

EXERCISE 1. *Für eine Funktion $f : \mathbb{R}^n \to \mathbb{R}$ sei die Fouriertransform $\hat{f} : \mathbb{R}^n \to \mathbb{R}$ definiert als*

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle}\; f(x)\; dx.$$

Zeigen Sie in formalen Rechnungen (also unter Annahme, dass die Integrale alle konvergieren und kommutieren)

(1) dass die Inversionsformel gilt

$$f(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{+i\langle \xi, y\rangle}\ \hat{f}(\xi)\ dx$$

Dabei dürfen Sie benutzen, dass

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle \xi, z\rangle}\ g(z)\ d\xi = g(0).$$

(2) Show that $\hat{f}(0) = \int_{\mathbb{R}^n} f(x)\ dx$.

(3) Sei $f = \partial_{x_i} g$. *Zeigen Sie (formale Rechnung) für alle* $\xi = (\xi_1, \ldots, \xi_n)$ *und alle* $i = 1, \ldots, n$,

$$\hat{f}(\xi) = -i\xi_i\ \hat{g}(\xi).$$

Zeigen Sie auch die Umkehrung, Ist $g(x) := -ix_i f(x)$

$$\partial_{\xi_i} \hat{f}(\xi) = \hat{g}(\xi).$$

(4) Schliessen Sie aus der vorigen Rechnung, dass falls $f = \Delta g$,

$$\hat{f}(\xi) = -|\xi|^2\, \hat{g}(\xi).$$

(5) Sei $f_\lambda(x) := f(\lambda x)$ *für ein* $\lambda \in \mathbb{R}$. *Zeigen Sie*

$$\hat{f}_\lambda(\xi) = \lambda^{-n} \hat{f}(\xi/\lambda).$$

(6) Zeigen sie in einer Dimension, $n = 1$, *dass für* $f(x) := \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{x^2}{2}}$ *gilt*

$$\hat{f}(\xi) = f(\xi).$$

Hinweis: *Zeigen Sie mit obigen Rechnungen, dass gelten muss*

$$\partial_\xi \hat{f}(\xi) = -\xi \hat{f}(\xi) \tag{1.2.12}$$

Verwenden Sie dann

$$\int_{\mathbb{R}} e^{-\xi^2} = \sqrt{\pi}.$$

um zu zeigen, dass $\hat{f}(0) = f(0)$. *Damit ist das Anfangswertproblem (1.2.12) eindeutig lösbar, mit eindeutiger Lösung* $\hat{f} = f$.

Bemerkung: Tatsächlich gilt in allen Dimensionen für $f(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}}$

$$\hat{f}(\xi) = f(\xi).$$

(7) Zeigen Sie nun, dass für festes $t \in (0, \infty)$, *falls* $\hat{f}(\xi) := e^{-t|\xi|^2}$, *so gilt*

$$f(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

(8) Zeigen Sie, dass für $f, g : \mathbb{R}^n \to \mathbb{R}$ *gilt*

$$\widehat{fg}(\xi) = \int_{\mathbb{R}^n} \hat{f}(\xi - \eta)\ \hat{g}(\eta)\ d\eta.$$

EXERCISE 2. *Let Φ be the fundamental solution of the heat equation, that is*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & t < 0 \end{cases}$$

(1) Show that for $t > 0$

$$\partial_t \Phi(x,t) - \Delta\Phi(x,t) = 0.$$

(2) Moreover, show that for $|x| \neq 0$,

$$\lim_{t \to 0_+} \Phi(x,t) = 0.$$

(3) Show that for $|x| = 0$,

$$\lim_{t \to 0_+} \Phi(x,t) = +\infty.$$

1.3. Mean-value formula

(cf. [**Eva98**, Chapter 2.3])

Use the fundamental solution to construct a parabolic ball, or *heat ball*

$$E(x,t;r) \subset \mathbb{R}^{n+1}.$$

DEFINITION 1.3.1 (Heat ball). Let $(x,t) \in \mathbb{R}^{n+1}$. Set

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^{n+1} \colon s \leq t, \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}.$$

THEOREM 1.3.2 (mean value). *Let $X \subset \mathbb{R}^{n+1}$ be open and $u \in C^2_1(X)$ solve $(\partial_t - \Delta)u = 0$ in X. Then there holds*

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dyds$$

for all $E(x,t;r) \subset X$.

PROOF. Without limit of generality u is smooth and $(x,t) = (0,0)$. $E(r) = E(0,0;r)$.

$$\Phi(r) := \frac{1}{r^n} \int_{E(r)} u(y,s) \frac{|y|^2}{s^2} \, dyds.$$

We show $\Phi'(r) = 0$ for $r > 0$.

$$\Phi(r) = \int_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} \, dyds.$$

We calculate

$$\begin{aligned}\Phi'(r) &= \int_{E(1)} \left(u_{y^i}(ry, r^2 s) y^i \frac{|y|^2}{s^2} + 2r u_s(ry, r^2 s) \frac{|y|^2}{s} \right) \, dyds \\ &= r^{-n-1} \int_{E(r)} \left(u_{y^i}(y, s) y^i \frac{|y|^2}{s^2} + 2u_s(y, s) \frac{|y|^2}{s} \right) \, dyds \\ &\equiv A + B\end{aligned}$$

Set

$$\psi_r(y, s) = -\frac{n}{2} \log(-4\pi s) + n \log r + \frac{|y|^2}{4s},$$

then

$$e^{\psi_r(y,s)} = r^n \Phi(y, -s)$$

and

$$\psi_r(y, s) = 0 \quad \text{on } \partial E(r).$$

There holds

$$\psi_{y^i} = \frac{y_i}{2s}$$

and hence

$$\begin{aligned}B &= \frac{1}{r^{n+1}} \int_{E(r)} 4u_s(y, s) y_i \psi_{y^i}(y, s) \, dyds \\ &= -\frac{1}{r^{n+1}} \int_{E(r)} 4\partial_{y^i}(u_s(y, s) y^i) \psi(y, s) \, dsdy \\ &= -\frac{1}{r^{n+1}} \int_{E(r)} 4n u_s(y, s) \psi(y, s) \\ &\quad - \frac{1}{r^{n+1}} \int_{E(r)} 4u_{sy^i}(y, s) y^i \psi(y, s) \, dyds \\ &= -\frac{1}{r^{n+1}} \int_{E(r)} 4n u_s(y, s) \psi(y, s) \\ &\quad + \frac{1}{r^{n+1}} \int_{E(r)} 4u_{y^i}(y, s) y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) \, dsdy \\ &= -\frac{1}{r^{n+1}} \int_{E(r)} 4n u_s(y, s) \psi(y, s) \\ &\quad - \frac{1}{r^{n+1}} \int_{E(r)} \frac{2n}{s} u_{y^i}(y, s) y^i \, dyds - A.\end{aligned}$$

Hence

$$\begin{aligned}\Phi'(r) &= -\frac{1}{r^{n+1}}\int_{E(r)} \Delta u_s(y,s) 4n\psi(y,s)\ dyds \\ &\quad - \frac{1}{r^{n+1}}\int_{E(r)} u_{y^i}(y,s) y^i\ dyds \\ &= \frac{1}{r^{n+1}}\int_{E(r)} u_{y^i}(y,s) 4n\partial_{y^i}\psi(y,s)\ dyds \\ &\quad - \frac{1}{r^{n+1}}\int_{E(r)} \frac{2n}{s} u_{y^i}(y,s) y^i \\ &= 0.\end{aligned}$$

Thus Φ is constant along r and hence

$$\begin{aligned}&\lim_{r\to 0} r^{-n}\int_{E(r)} (u(y,s)-u(0,0))\frac{|y|^2}{s^2}\ dyds + 4u(0,0) \\ \leq &\lim_{r\to 0} Cr(\|\nabla u\|_\infty + \|\partial_t u\|_\infty) = 4u(0,0).\end{aligned}$$

□

1.4. Maximum principle and Uniqueness

Definition 1.4.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and denote with $\Omega_T := \Omega \times (0,T]$ for some time $T > 0$. It is important to note that the top $\Omega \times \{T\}$ belongs to Ω_T. The parabolic boundary Γ_T of Ω_T is the boundary of Ω_T without the top,

$$\Gamma_T = \overline{\Omega_T}\backslash\Omega_T = \partial\Omega \times [0,T) \cup \Omega \times \{0\}.$$

Theorem 1.4.2. *Let U be bounded and $u \in C^2_1(U_T) \cap C^0(\bar{U}_T)$ be a solution of $u_t = \Delta u$ in U_T. Then there holds the weak maximum principle*

(i)

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u$$

and the strong maximum principle:

(ii) If U is connected and if there is $(x_0,t_0) \in U_T$ with

$$u(x_0,t_0) = \max_{\bar{U}_T} u,$$

then

$$u(x,t) = u(x_0,t_0) \quad \forall (x,t) \in U_{t_0}.$$

PROOF. (ii)⇒(i), since if

$$\max_{\bar U_T} u > \max_{\Gamma_T} u \tag{1.4.1}$$

then by (ii) u is constant at all prior times, which contradicts (1.4.1).

Now we prove (ii). Suppose there is $(x_0, t_0) \in U_T$ with

$$u(x_0, t_0) = M = \max_{\bar U_T} u.$$

Since $t_0 > 0$, there exists a small heat ball $E(x_0, t_0, r) \subset U_T$ and we have by 1.3.2

$$M = u(x_0, t_0) = \frac{1}{4r^n} \int_{E(x_0,t_0,r)} u(y, s) \frac{|y - x|^2}{(t - s)^2}\, dsdy \leq M.$$

Hence $u = M$ in E. Now let $(x_1, t_1) \in U_{t_0}$. Then there exists a continuous path $\gamma\colon [0, 1] \to U$ connecting x_0 and x_1. In the spacetime set

$$\Gamma(r) = (\gamma(r), rt_1 + (1 - r)t_0).$$

Let

$$\rho = \max\{r \in [0, 1] \colon u(\Gamma(r)) = M\}.$$

Show that $\rho = 1$. Suppose $\rho < 1$. Then we use the proof above to find a heat ball

$$E = E(\Gamma(\rho), r'),$$

where $u = M$. Since Γ crosses E (time parameter is decreasing along Γ), we obtain a contradiction to the maximality of ρ. □

REMARK 1.4.3. The same holds for $-u$ and hence we have a minimum principle. Hence, if in particular

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } U_T \\ u &= 0 \quad \text{on } \partial U \times [0, T] \\ u &= g \quad \text{in } U \times \{0\} \end{aligned}$$

with $g(x) > 0$ for some $x \in U$ then $u > 0$ in U_T (infinite speed of propagation, non-relativistic).

REMARK 1.4.4. For general $X \subset \mathbb{R}^{n+1}$ open we have a similar result, see exercises.

THEOREM 1.4.5 (Uniqueness on bounded domains). *Let $U \Subset \mathbb{R}^n$ bounded and $g \in C^0(\Gamma_T)$, $f \in C^0(U_T)$. Then there is at most one solution $C^2_1(U_T) \cap C^0(\bar U_T)$ to*

$$\begin{aligned} u_t - \Delta u &= f \quad \textit{in } U_T \\ u &= g \quad \textit{on } \Gamma_T. \end{aligned}$$

PROOF. Apply the maximum (and minimum) principle to show that the difference of two solutions is zero. □

THEOREM 1.4.6. *Let* $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])$ *be a solution of*

$$\begin{aligned}(\partial_t - \Delta)u &= 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\}\end{aligned}$$

with the growth condition

$$u(x,t) \le Ae^{a|x|^2}$$

for some $a, A > 0$. *Then there holds*

$$\sup_{\mathbb{R}^n \times [0,T]} u \le \sup_{\mathbb{R}^n} g.$$

PROOF. Suppose first

$$4aT < 1.$$

Let

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T+\epsilon-t)}}$$

for some $\mu > 0$. Then $v_t - \Delta v = 0$. 1.4.2 implies

$$\forall U \Subset \mathbb{R}^n\colon \max_{\bar{U}_T} v \le \max_{\Gamma_T} v \le \max(\max v(\cdot,0), \max_{\partial U \times [0,T]} v(x,t)).$$

We have

$$v(x,0) = g(x) - \frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T+\epsilon)}} \le \sup_{\mathbb{R}^n} g.$$

Let $U = B_R(0)$, then

$$\max_{\bar{B}_R(0)\times[0,T]} v \le \max\left(\sup_{\mathbb{R}^n} g, \max_{|x|=R, t\in[0,T]} v(x,t)\right).$$

For $|x| = R$ and $t \in (0,T)$

$$\begin{aligned}v(x,t) &= u(x,t) - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\epsilon-t)}} \\ &\le Ae^{a|x|^2} - \frac{\mu}{(T+\epsilon-t)^{\frac{n}{2}}} e^{\frac{R^2}{4(T+\epsilon-t)}}.\end{aligned}$$

Now there exist $\epsilon > 0, \gamma > 0$, such that

$$at\gamma = \frac{1}{4(T+\epsilon)}$$

and hence

$$v(x,t) \le Ae^{aR^2} - \frac{\mu}{(T+\epsilon)^{\frac{n}{2}}} e^{aR^2+\gamma R^2}.$$

If $R >> 0$, then $v(x,t) \le g(0)$. So for large R and $|x| = R$ we have

$$v(x,t) \le \sup_{\mathbb{R}^n} g$$

and so

$$\max_{(x,t)\in\overline{B_R(0)_T}} v(x,t) \le \sup_{\mathbb{R}^n} g \quad \forall R >> 1$$

and with $R \to \infty$

$$\sup_{\mathbb{R}^n\times[0,T]} v(x,t) \le \sup_{\mathbb{R}^n} g$$

for any μ. Letting $\mu \to 0$ for fixed x gives the claim. □

THEOREM 1.4.7. *Let $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n \times [0,T])$. Then there is at most one solution $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])$ of*

$$\begin{aligned}(\partial_t - \Delta)u &= f \quad \text{in } \mathbb{R}^n \times (0,T)\\ u &= g \quad \text{on } \mathbb{R}^n \times \{0\}\end{aligned}$$

with

$$|u(x,t)| \le Ae^{a|x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times (0,T).$$

PROOF. Exercise 4 □

EXERCISE 3. *Wir haben in Theorem 1.4.7 das starke Maximumsprinzip auf parabolischen Zylindern kennengelernt. Benutzen Sie dies um ein starkes Maximumsprinzip auf allgemeinen Mengen X herzuleiten:*

Sei $X \subset \mathbb{R}^{n+1}$ eine beliebige beschränkte, offene Menge. Angenommen es gilt $u \in C^\infty(\overline{X})$ und

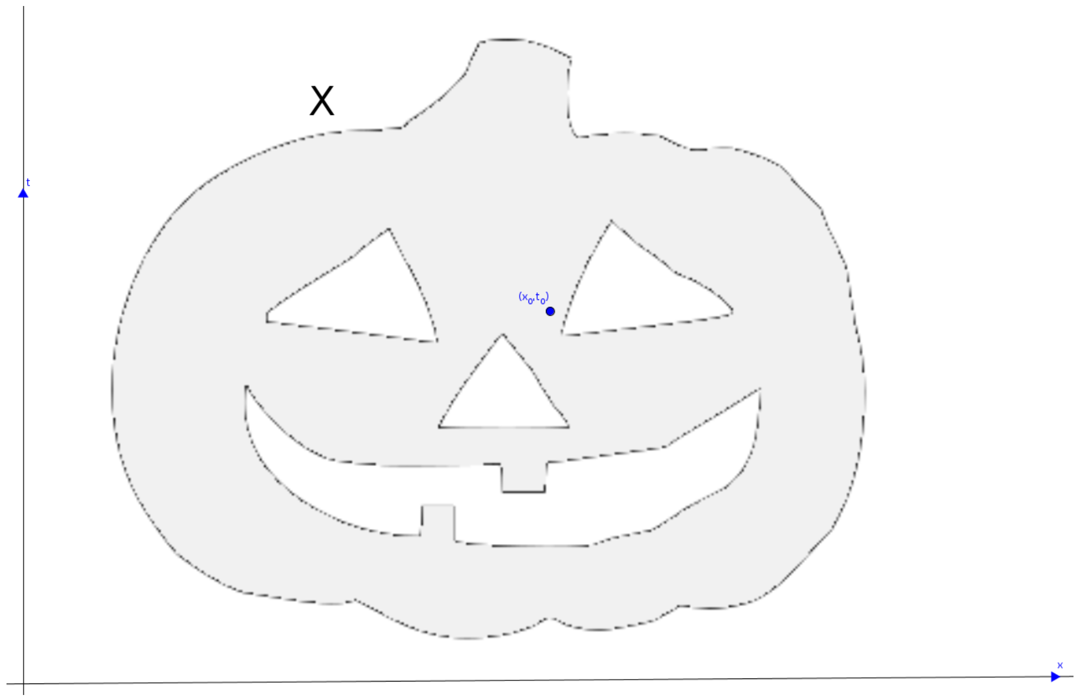
$$\partial_t u - \Delta u \quad \text{in } X.$$

Angenommen es gilt für ein $(x_0, t_0) \in X$, dass

$$M := u(x_0,t_0) = \sup_{(x,t)\in X} u(x,t).$$

(1) *Beschreiben Sie in Worten die Punkte die notwendigerweise zu der Menge C gehören, wobei*

$$C := \{(x,t) \in X : \quad u(x,t) = M\}.$$



(2) *Seien die Menge* X *(grau) und der Punkt* (x_0, t_0) *wie im Bild gegeben. Zeichnen Sie (in orange) die Menge* C *ein.*

EXERCISE 4. *Zeigen Sie Theorem 1.4.7: Seien* $g \in C^0(\mathbb{R}^n)$, $f \in C^0(\mathbb{R}^n \times [0,T])$ *für ein* $T > 0$.

Angenommen es gibt zwei Lösungen u^1 *und* $u^2 \in C_1^2(\mathbb{R}^n \times (0,T)) \cap C^0(\mathbb{R}^n \times [0,T])$ *des Anfangswertproblems*

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x) & \text{für } x \in \mathbb{R}^n. \end{cases}$$

Gibt es weiterhin Konstanten a_1, a_2 *und* $A_1, A_2 > 0$ *so dass*

$$|u^1(x,t)| \leq A_1\, e^{a_1\,|x|^2}, \quad |u^2(x,t)| \leq A_2\, e^{a_2\,|x|^2} \quad \forall (x,t) \in \mathbb{R}^n \times [0,T],$$

so gilt

$$u^1 \equiv u^2 \quad \text{auf } \mathbb{R}^n \times [0,T].$$

Hinweis: *Benutzen Sie Theorem 1.4.6 (Starkes Maximumsprinzip für das Cauchy-Problem) aus der Vorlesung.*

EXERCISE 5. *(cf. [**Joh91**]) Gegeben Sei die folgende* Tychonoff-*Funktion:*

$$u(x,t) := \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!}\; x^{2k},$$

wobei $g^{(k)}$ die k-te Ableitung ist, und

$$g(t) := \begin{cases} e^{(-t^{-\alpha})} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

(1) *Zeigen Sie, $u \in C_1^2(\mathbb{R}_+^2) \cap C^0(\mathbb{R} \times [0, \infty))$.*
(2) *Zeigen Sie nun, dass*

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = 0 & \text{für } x \in \mathbb{R}^n. \end{cases} \tag{1.4.2}$$

(3) *Finden Sie eine andere Lösung $v \not\equiv u$ von (1.4.2).*
(4) *Warum (ohne Beweis) ist dies kein Widerspruch zu Aufgabe 4?*

1.5. Harnack's Principle

In the parabolic setting a Harnack in the whole spacetime is not possible. We have to wait some time. For example for

$$(\partial_t - \Delta)u = 0 \quad \text{in } B_1 \times (0, T))$$

we have a uniformly positive solution at time $t > 0$ if only there is one point at $t = 0$ with $u(x, 0) > 0$.

Theorem 1.5.1 (Parabolic Harnack inequality). *Assume $u \in C_1^2(\mathbb{R}^n \times (0, T]) \cap L^\infty(\mathbb{R}^n \times [0, T])$ and solves*

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0, T)$$

and

$$u \geq 0 \quad \text{in } \mathbb{R}^n \times (0, T)$$

Then for any compactum $K \subset \mathbb{R}^n$ and any $0 < t_1 < t_2 < T$ there exists a constant C, so that

$$\sup_{x \in K} u(x, t_1) \leq C \inf_{y \in K} u(y, t_2)$$

Proof. By the representation formula, Theorem 1.2.3 and uniqueness of the Cauchy problem

$$u(x_2, t_2) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t_2)^{\frac{n}{2}}} e^{-\frac{|x_2 - y|^2}{4t_2}} u_0(y)\, dy.$$

Now, for $t_1 < t_2$ whenever $|x_1|, |x_2| \leq \Lambda < \infty$, there exists a constant $C = C(|t_1 - t_2|, \Lambda)$ so that

$$-\frac{|x_2 - y|^2}{4t_2} \geq -\frac{|x_1 - y|^2}{4t_1} - C.$$

See Exercise 6.

Consequently,

$$u(x_2,t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} e^{-C} \int_{\mathbb{R}^n} \frac{1}{(t_1)^{\frac{n}{2}}} e^{-\frac{|x_1-y|^2}{4t_1}}\ u_0(y)\ dy = \left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} e^{-C} u(x_1,t_1).$$

□

EXERCISE 6. *Zeigen Sie die folgende Abschätzung, die wir für das Harnack-Prinzip, Theorem 1.5.1, verwenden.*

Ist $K \subset \mathbb{R}^n$ kompakt und $0 < t_1 < t_2 < \infty$, dann gibt es eine Konstante $C > 0$ abhängig von K und $(t_2 - t_1)$, so dass

$$\frac{|x_1-y|^2}{t_2} \leq \frac{|x_2-y|^2}{t_1} + C \quad \forall x_1, x_2 \in K,\ y \in \mathbb{R}^n.$$

EXERCISE 7 (Counterexample Harnack). *(1) Sei $u_0 : \mathbb{R}^n \to [0,\infty)$ eine glatte Funktion mit kompaktem support mit $u_0(0) = 1$. Setze*

$$u(x,t) := \int_{\mathbb{R}^n} \Phi(x-y,t)\ u_0(y) \quad t > 0$$

Zeigen Sie,

$$\inf_{x\in\mathbb{R}^n} u(x,t) = 0 \quad \textit{für alle } t > 0.$$

Aber

$$\sup_{x\in\mathbb{R}^n} u(x,t) > 0 \quad \textit{für alle } t > 0.$$

Warum ist dies kein Widerspruch zum Harnack-Prinzip, Theorem 1.5.1?

(2) Zeigen Sie, dass das folgende Sei $\xi \in \mathbb{R}^n$ gegeben, und u definiert als

$$u_\xi(x,t) := (t+1)^{-\frac{1}{2}} e^{-\frac{|x+\xi|^2}{4(t+1)}}.$$

Zeigen Sie dass u eine Lösung von $(\partial_t - \Delta)u = 0$ auf $\mathbb{R}^n \times (0,\infty)$ ist. Zeigen Sie aber auch, dass es jedes feste $t > 0$ keine *Konstante $C = C(t) > 0$ gibt für die gilt*

$$\sup_{x\in[-1,1]} u_\xi(x,t) \leq C \inf_{y\in[-1,1]} u_\xi(y,t) \quad \forall \xi \in \mathbb{R}^n.$$

Warum ist dies kein Widerspruch zum Harnack-Prinzip, Theorem 1.5.1?

Hinweis: *Wählen Sie $x = -\frac{\xi}{|\xi|}$ und $y = 0$. Was passiert, wenn $|\xi| \to \infty$?*

1.6. Regularity and Cauchy-estimates

THEOREM 1.6.1 (Smoothness). *Let $u \in C^2_1(U_T)$ satisfy*

$$u_t = \Delta u \quad \textit{in } U_T.$$

Then $u \in C^\infty(\mathrm{int}(U_T))$.

PROOF. This is a standard technique to transfer local questions to global situations, using a cut-off function. Let

$$C(x,t;r) = \{(y,s)\colon |x-y| \leq r, t-r^2 \leq s \leq t\}$$

and

$$C_1 = C(x_0,t_0;r), \quad C_2 = C\left(x_0,t_0;\frac{3}{4}r\right), \quad C_3 = C\left(x_0,t_0;\frac{r}{2}\right)$$

for some r such that $C_1 \subset U_T$. Choose a cut-off function

$$\eta \in C^\infty(\mathbb{R}^n \times [0,t_0])$$

with $0 \leq \eta \leq 1$, $\eta_{|C_2} \equiv 1$, $\eta \equiv 0$ around $\mathbb{R}^n \times [0,t_0]\backslash C_1$. Suppose first that u is smooth. Set

$$v(x,t) = \eta(x,t)u(x,t) \quad \forall (x,t) \in \mathbb{R}^n \times (0,t_0],$$

extended by 0. Then

$$\begin{aligned}
\partial_t v - \Delta v &= u_t\eta + \eta_t u - \eta\Delta u - u\Delta\eta - 2\langle \nabla u, \nabla\eta\rangle \\
&= \eta_t u - u\Delta\eta - 2\langle \nabla u, \nabla\eta\rangle \\
&=: f(x,t)
\end{aligned}$$

with bounded v and $f \in C_1^2$ by smoothness of u. Let $(x,t) \in C_3$. Then

$$\begin{aligned}
v(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\ dyds \\
&= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\big(u(y,s)\eta_t(y,s) - u(y,s)\Delta\eta(y,s) \\
&\quad - 2\langle \nabla u(y,s), \nabla\eta(y,s)\rangle\big)\ dyds
\end{aligned}$$

We note: The singularity $y = x$ and $s = t$ is cut off due to $(x,t) \in C_3$. Hence

$$\begin{aligned}
v(x,t) = &\int_{C_1} \Phi(x-y,t-s)\big((\partial_t - \Delta)\eta(y,s)u(y,s)\big)\ dyds \\
&+ \int_{C_1} 2D\Phi(x-y,t-s)D\eta(y,s)u(y,s).
\end{aligned}$$

By convolution: If $u \in C_1^2(U_T)$, we have a representation

$$v(x,t) = \int_C K(x,y,s,t)u(y,s)\ dyds$$

with no singularities in the kernel. Thus v is smooth and so is u around (x_0,t_0). □

THEOREM 1.6.2 (Cauchy estimates). *For all $k,l \in \mathbb{N}$ there exists $C > 0$ such that for all $u \in C^{2,1}(U_T)$ ($u \in L^1_{\text{loc}}$ will be sufficient), solving*

$$(\partial_t - \Delta)\, u = 0,$$

there holds

$$\max_{C(x_0,t_0;\frac{r}{2})} |D_x^k \partial_t^l u| \leq \frac{C}{r^{k+2l+n+2}} \|u\|_{L^1(C(x_0,t_0;r))}$$

for all $C(x_0, t_0; r) \subset U_T$.

PROOF. Suppose first $(x_0, t_0) = (0, 0)$ and $r = 1$. Set

$$C(1) = C(0, 0; 1).$$

Then as in the proof of Theorem 1.6.1 we have

$$u(x,t) = \int_{C(1)} K(x,t,y,s)u(y,s)\ dyds \quad \forall (x,t) \in C\left(\frac{1}{2}\right).$$

Then

$$D_x^k \partial_t^l u(x,t) = \int_{C(1)} \left(D_x^k \partial_t^l K(x,t,y,s)\right) u(y,s)\ dyds$$

and hence

$$|D_x^k \partial_t^l u(x,t)| \leq C_{k,l} \|u\|_{L^1(C(1))} \quad \forall (x,t) \in C\left(\frac{1}{2}\right).$$

Thus the claim is proven for $r = 1$. For $r > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$ set

$$v(x,t) = u(x_0 + rx, t_0 + r^2 t).$$

Then

$$\max_{C\left(\frac{1}{2}\right)} |D_x^k \partial_t^l v| \leq C_{k,l} \|v\|_{L^1(C(1))}.$$

Hence

$$\max_{C(x_0,r_0;\frac{r}{2})} |D_x^k \partial_t^l u| r^{k+2l} \leq C_{k,l} r^{-(n+2)} \|u\|_{L^1(C(1))}.$$

□

CHAPTER II

linear parabolic equations

2.1. Definitions

The heat equation is the simplest or most pure *parabolic* equation. In general we want to study equations of the form

$$\partial_t u - Lu,$$

where L is a uniformly elliptic differential operator (for each time t). More precisely, we study L which for given coefficient functions $a_{ij}(x,t)$, $b_i(x,t)$ and $c(x,t)$ has the form

$$Lu(x,t) = a_{ij}(x,t)\,\partial_{ij}u(x,t) + b_i(x,t)\,\partial_i u(x,t) + c(x,t)\,u(x,t).$$

Recall that we use Einstein's summation convention,

$$= \sum_{i,j=1}^{n} a_{ij}(x,t)\,\partial_{ij}u(x,t) + \sum_{i=1}^{n} b_i(x,t)\,\partial_i u(x,t) + c(x,t)\,u(x,t).$$

We want L to be elliptic (and equivalently $\partial_t - L$ to be parabolic), which simply means that the leading order coefficients form a non-degenerate, positive matrix.

DEFINITION 2.1.1 (Parabolic). We say that an operator $\partial_t - L$ is uniformly parabolic, if there exists a constant $\lambda > 0$ so that

$$a_{ij}(x,t)\,\xi_i\,\xi_j \geq \lambda|\xi|^2 \quad \forall (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n.$$

Equivalently, the matrix $A(x,t) = (a_{ij}(x,t))_{1\leq i,j\leq n}$ satisfies

$$\langle A(x,t)\xi, \xi\rangle_{\mathbb{R}^n} \geq \lambda \quad \forall (x,t) \in \Omega_T,\ \xi \in \mathbb{R}^n,\ |\xi| = 1.$$

We also say that L is uniformly elliptic.

The simples example of a parabolic operator is the heat operator. Indeed take

$$a_{ij} := \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and $b \equiv c \equiv 0$. Then $L = +\Delta$. Indeed, parabolic operators have many features similar to $\partial_t - \Delta$.

DEFINITION 2.1.2. Let $X \subset \mathbb{R}^{n+1}$ be an $n+1$-dimensional domain. The *parabolic boundary* $\mathcal{P}X$ of X is defined as follows. For $\rho > 0$, $(x_0, t_0) \in \mathbb{R}^{n+1}$ define the (backwards-in-time) cylinder $Q_\rho(x_0, t_0)$ as

$$Q_\rho(x_0, t_0) = \left\{(x, t) \in \mathbb{R}^{n+1} : \ |x - x_0| < \rho, \ t \in (t_0 - \rho^2, t_0), \right\}.$$

Then the parabolic boundary $\mathcal{P}X$ of X is defined as

$$\mathcal{P}X := \{(x_0, t_0) \in \partial X \quad \text{so that } Q_\rho(x_0, t_0) \cap X^c \neq \emptyset \quad \forall \rho > 0\}$$

EXERCISE 8. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Omega_T = \Omega \times (0, T]$. Show that $\mathcal{P}\Omega_T = \Gamma_T$.*

2.2. Maximum principles

2.2.1. Weak maximum principle. We will always assume that the operators $\partial_t + L$ are uniformly parabolic and the coefficients a_{ij}, b^i, c are continuous. Moreover we assume symmetry,

$$a_{ij} = a_{ji} \quad 1 \leq i, j \leq n.$$

Also $X \subset \mathbb{R}^{n+1}$ bounded.

THEOREM 2.2.1 (Weak maximum principle, $c \equiv 0$). *Let $X \subset \mathbb{R}^{n+1}$ be open and bounded and let L be an elliptic operator with*

$$c = 0. \tag{2.2.1}$$

Let $u \in C^2_1(X) \cap C^0(\bar{X})$.

(1) If u is a subsolution of $\partial_t - L$, i.e.

$$(\partial_t - L)u \leq 0, \tag{2.2.2}$$

then

$$\sup_{\bar{X}} u = \sup_{\partial_P X} u.$$

(2) If u is a supersolution of $\partial_t - L$, i.e.

$$(\partial_t - L)u \geq 0,$$

then

$$\inf_{\bar{X}} u = \inf_{\partial_P X} u.$$

PROOF. We only proof the first claim, the second one follows by replacing u with $-u$. Also we will assume that $X = \Omega_T$

For now assume that we have a *strict subsolution*. That is,

$$(\partial_t - L)u < 0 \quad \text{in } \Omega_T. \tag{2.2.3}$$

Assume that there exists a point $(x_0, t_0) \in \Omega_T$ with $u(x_0, t_0) = \max_{\overline{\Omega_T}} u$. Then $x_0 \in \Omega$ and $t_0 \in (0, T]$, so the maximality condition tells us

$$\partial_t u(x_0, t_0) \geq 0, \quad Du(x_0, t_0) = 0, \quad D^2 u(x_0, t_0) \leq 0.$$

In particular, observing (2.2.1),

$$\partial_t u(x_0, t_0) - Lu(x_0, t_0) \geq a_{ij}(x_0, t_0)\, \partial_{ij} u(x_0, t_0).$$

In view of Exercise 9 this implies

$$\partial_t u(x_0, t_0) - Lu(x_0, t_0) \geq 0,$$

a contradiction to (2.2.3). So what do we do if we had only (2.2.2)? We consider a subsolution slightly below u. Let $u^\varepsilon(x, t) := u(x, t) - \varepsilon t$. Then, again with (2.2.1),

$$\partial_t u^\varepsilon - Lu^\varepsilon = \partial_t u - Lu - \varepsilon < 0 \quad \text{in } \Omega_T.$$

The above argument implies that

$$\max_{\overline{\Omega_T}} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon \quad \forall \varepsilon > 0.$$

In particular we have

$$\max_{\overline{\Omega_T}} u \leq \varepsilon T + \max_{\overline{\Omega_T}} u_\varepsilon \leq \varepsilon T + \max_{\Gamma_T} u_\varepsilon \leq \varepsilon T + \max_{\Gamma_T} u.$$

Letting $\varepsilon \to 0$ we have

$$\max_{\overline{\Omega_T}} u \leq \max_{\Gamma_T} u.$$

The inverse estimate is always true, so the claim is proven. □

EXERCISE 9. *A matrix $A \in \mathbb{R}^{n \times n}$ is nonnegative, $A \geq 0$, if*

$$\langle Av, v \rangle \geq 0 \quad \forall v \in \mathbb{R}^n.$$

A matrix A is symmetric, if $A^T = A$.

Show that

1. *$A \geq 0$ implies $P^T A P \geq 0$ for any matrix $P \in \mathbb{R}^{n \times n}$.*
2. *$A \geq 0$ implies that the diagonal entries $A_{ii} \geq 0$ for any $i \in \{1, \dots, n\}$.*
3. *$A \geq 0$ and $B \geq 0$ and B is symmetric then*

$$A : B := \sum_{i,j=1}^{n} A_{ij} B_{ij} \geq 0.$$

If $c \geq 0$, then we have to adapt the claim. For a function f let $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$.

EXERCISE 10. *Complete the above proof for general domain X.*

THEOREM 2.2.2 (Weak maximum principle, $c \le 0$). *Let u and X as in 2.2.1 and $\partial_t - L$ parabolic with $c \le 0$. Then if $u_t - Lu \le 0$ then*

$$\sup_{\bar{X}} u \le \sup_{\partial_P X} u_+.$$

For $u_t - Lu \ge 0$, then

$$\inf_{\bar{X}} u \ge -\sup_{\partial_P X} u_-,$$

where $u_+ = \max(0, u)$ and $u_- = -\min(u, 0)$. If $u_t = Lu$, then

$$\sup_{\bar{X}} |u| = \sup_{\partial_P X} |u|$$

PROOF. We just prove the first claim, the second and third are simple corollaries.

Again, we assume Ω_T, general X is an exercise. we first simplify the equation, and assume that

$$(\partial_t - L)u < 0 \quad \text{in } \Omega_T.$$

The only situation we have to exclude is that there exists $(x_0, t_0) \in \Omega_T$ at which there is a *positive* maximum value $u(x_0, t_0) > 0$. With the arguments above,

$$u_t(x_0, t_0) + Lu(x_0, t_0) \ge c(x_0, t_0)\, u(x_0, t_0) \ge 0,$$

and we have our contradiction. The full claim is obtained if we consider again $u^\varepsilon(x, t) := u(x, t) - \varepsilon t$. Then

$$\max_{\overline{\Omega_T}} u_\varepsilon \le \max_{\Gamma_T} (u_\varepsilon)_+ \le \max_{\Gamma_T} (u)_+.$$

We let $\varepsilon \to 0$ to conclude. □

A consequence of the weak maximum principle is uniqueness of solutions and the comparison principle.

COROLLARY 2.2.3 (Uniqueness). *Let $X \subset \mathbb{R}^{n+1}$ and L as above with $c \le 0$. Let $u, v \in C^2_1(X) \cap C^0(\bar{X})$ satisfy*

$$u_t - Lu = v_t - Lv.$$

Then if $u = v$ on $\partial_P X$, we have $u = v$ in X.

COROLLARY 2.2.4 (Comparison Principle). *Let X and L as above and $u, v \in C^2_1(X) \cap C^0(\bar{X})$ with*

$$u_t - Lu \le v_t - Lv$$

in X with $u \le v$ on $\partial_P X$, then we have $u \le v$ in X.

We leave the proofs as exercises, Exercise 11.

EXERCISE 11. *Prove Corollaries 2.2.3 and 2.2.4. Hint: What equation does $u - v$ satisfy?*

2.2.2. Strong Maximum principle. Let

$$u_t - Lu = 0 \quad \text{in } \Omega_T$$

We want to understand better the relation between u at different times. We have the following very important "propagation of positivity" property. See [**Lie96**, II, Lemma 2.6]

LEMMA 2.2.5. [PROPAGATION OF POSITIVITY] *For $R > 0$ and $\alpha > 0$ let $B_R(0) \subset \mathbb{R}^n$. Let $Q(R) = B_R \times (0, \alpha R^2)$. Let $0 \le u \in C_1^2(Q(R))$ satisfy*

$$u_t - Lu \geq 0,$$

where L is elliptic with $b = c = 0$. If

$$u(x,0) \geq h \quad \forall |x| < \epsilon R \tag{2.2.4}$$

for some $h > 0$ and $0 < \epsilon < 1$, then

$$u(x, \alpha R^2) \geq c(\epsilon, \lambda, R, \|a_{ij}\|_\infty) h \quad \forall |x| \leq \frac{R}{2}$$

for some positive c.

PROOF. Let $\tilde{Q} \subset \mathbb{R}^{n+1}$ be a cone so that at time $t = 0$, $\tilde{Q} \cap (\mathbb{R}^n \times \{t = 0\})$ is the ball $\{|x| < \varepsilon R\}$ and at time $t = \alpha R^2$, $\tilde{Q} \cap (\mathbb{R}^n \times \{t = \alpha R^2\})$ is the ball $\{|x| < R\}$. See Figure 1. In formulas, $\tilde{Q}$ can be written

$$\tilde{Q} = \left\{(x,t) \in \mathbb{R}^{n+1} : \quad |x|^2 < \psi(t), 0 < t < \alpha R^2\right\}$$

for

$$\psi(t) := \frac{(1-\varepsilon^2)}{\alpha} t + \varepsilon^2 R^2.$$

On $\tilde{Q}$ we will construct a comparison ("barrier") function v with the following properties:

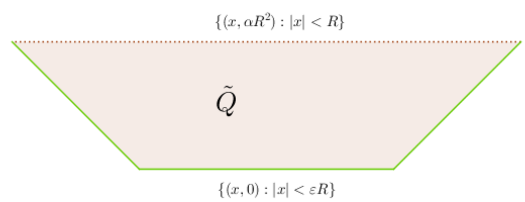


FIGURE 1. $\tilde{Q}$ and its parabolic boundary $\mathcal{P}\tilde{Q}$ (green)

$$\begin{cases} v_t - Lv \leq 0 & \text{in } \tilde{Q} \\ v \leq u & \text{on } \mathcal{P}\tilde{Q} \end{cases} \tag{2.2.5}$$

and moreover

$$v(x, \alpha R^2) \geq c\, h \quad \text{whenever } |x| \leq \tfrac{R}{2} \tag{2.2.6}$$

If we have such a v, then by Corollary 2.2.4 (the general domain version)

$$u(x, \alpha R^2) \geq v(x, \alpha R^2) \geq ch \quad \text{whenever } |x| \leq \tfrac{R}{2}$$

So how do we construct such a v? We essentially rescale (in time) the map $(1-|x|^2)^2$. Choose the Ansatz

$$v(x,t) := \mu(t)\,(\nu(t)-|x|^2)^2.$$

For μ, ν nonnegative functions. In general, away from $t=0$, we only know that $u \geq 0$, so to make v as large as possible, it seems reasonable to set $v(x,t) \equiv 0$ on the positive part of the parabolic boundary $\mathcal{P}\tilde{Q} \cap \{t>0\}$. That is,

$$\nu(t) := \psi(t).$$

Now we compute the equation. Firstly

$$\partial_{x^i x^j} v(x,t) = 8\mu(t)\, x^j\, x^i - 4\mu(t)\,(\psi(t)-|x|^2)\delta_{ij}$$

Consequently, by ellipticity

$$-a_{ij}(x,t)\,\partial_{x^i x^j} v(x,t) \leq \mu(t)\left(-8\,\psi(t)\,\lambda + 8\,(\psi(t)-|x|^2)\,\lambda + 4(\psi(t)-|x|^2)\,\mathrm{tr}(A)\right).$$

Also,

$$v_t(x,t) = \mu'(t)\,(\psi(t)-|x|^2)^2 + 2\mu(t)\,(\psi(t)-|x|^2)\psi'(t).$$

This v_t has to be the positive guy, so we would like to be able to compare $\mu'(t)$ and $\nu'(t)$. We thus choose (note that $\psi(t)>0$) for some constant $\eta>0$,

$$\mu(t) := \eta\psi(t)^{-q}.$$

Then

$$-a_{ij}(x,t)\,\partial_{x^i x^j} v(x,t) \leq \eta\psi^{1-q}(t)\left(-8\,\lambda + 8\,\left(\frac{(\psi(t)-|x|^2)}{\psi(t)}\right)\lambda + 4\left(\frac{(\psi(t)-|x|^2)}{\psi(t)}\right)\mathrm{tr}(A)\right).$$

and (observe that $\psi'(t) = \frac{1-\varepsilon^2}{\alpha}R$,

$$\begin{aligned} v_t(x,t) =& \eta\left(-q\psi^{-q-1}(t)\,(\psi(t)-|x|^2)^2 + 2\psi(t)^{-q}\,(\psi(t)-|x|^2)\right)\ \frac{1-\varepsilon^2}{\alpha}R \\ =& \eta\psi(t)^{1-q}\left(-q\,\left(\frac{(\psi(t)-|x|^2)}{\psi(t)}\right)^2 + 2\psi(t)\,\left(\frac{(\psi(t)-|x|^2)}{\psi(t)}\right)\right)\ \frac{1-\varepsilon^2}{\alpha}R. \end{aligned}$$

We see a quadratic structure in

$$\xi(t) := \left(\frac{(\psi(t)-|x|^2)}{\psi(t)}\right),$$

namely

$$\begin{aligned} & v_t(x,t) - a_{ij}(x,t)\partial_{x^i x^j} v(x,t) \\ \leq& \eta\psi^{1-q}(t)\left(-\left(q\,\frac{1-\varepsilon^2}{\alpha}R\right)\ \xi(t)^2 + \left(2\frac{1-\varepsilon^2}{\alpha}R\ \psi(t)^2\ + 8\lambda + 4\,\mathrm{tr}(A)\right)\xi(t) - 8\,\lambda\right). \end{aligned}$$

Observe that the leading order term and the zero-order term are negative, hence (see Exercise 12) there exists a large $q>0$ so that

$$v_t(x,t) - a_{ij}(x,t)\,\partial_{x^i x^j} v(x,t) \leq 0 \quad \text{in } \tilde{Q}.$$

On the other hand, for $t = 0$, in view of (2.2.4),

$$v(x,0) = \eta \varepsilon^{-2q} R^{-2q} \left(\varepsilon^2 R^2 - |x|^2\right)^2 \le \eta\, (\varepsilon R)^{4-2q} \le \frac{1}{h} \eta\, (\varepsilon R)^{4-2q}\, u(x,0).$$

So we choose

$$\eta := h\, (\varepsilon R)^{2q-4}.$$

Then v satisfies (2.2.5). It remains to check (2.2.6). For $|x| \le \frac{R}{2}$,

$$v(x, \alpha R) = h\, (\varepsilon R)^{2q-4}\, R^{-2q} \left(R^2 - |x|^2\right)^2 \ge h \varepsilon^{2q-4}\, \frac{9}{16}.$$

This finishes the proof of Lemma 2.2.5. It is worth noting that we actually get an estimate of the form ε^κ, where κ is a uniform constant depending on R, λ, etc. For this assume w.l.o.g. that $\varepsilon < \frac{1}{2}$, for any $\varepsilon > \frac{1}{2}$ the claim follows from the $\varepsilon < \frac{1}{2}$ case since the positivity set is larger than required. ☐

Exercise 12. *Assume that $a, b, c \in \mathbb{R}$ be fixed. To any $\lambda \in \mathbb{R}$ we associate the polynomial*

$$p_\lambda(x) := \lambda a x^2 + bx + c \quad x \in \mathbb{R}.$$

Show that if $a < 0$ and $c < 0$ then there exists a $\lambda > 0$ so that

$$\sup_{x \in \mathbb{R}} p_\lambda(x) < 0.$$

Hint: *p-q formula*

Theorem 2.2.6 (Strong Maximum Principle). *Let $b, c = 0$, L elliptic, $X \subset \mathbb{R}^{n+1}$ open and bounded, $u \in C^2_1(X) \cap C^0(\bar{X})$ and assume in X:*

$$(\partial_t - L)u \le 0.$$

Assume there is $(x_0, t_0) \in X$, such that

$$u(x_0, t_0) = \sup_X u,$$

then

$$u(x,t) = u(x_0, t_0) \quad \forall (x,t) \in S(x_0, t_0),$$

where

$$S(x_0, t_0) = \{(x,t) \colon \exists\ g \in C^0\left([0,1], X \backslash \partial_p X\right), g(0) = (x_0, t_0),\ g(1) = (x,t), g \text{ decreasing in } t\}.$$

Proof. Set

$$M := \max_{\bar{X}} u.$$

Claim: Assume a maximal point $(y_0, t_0) \in X$, $r > 0$, such that

$$Q(y_0, t_0, 3r) \subset X$$

and such that there is $(y_1, t_1) \in Q(y_0, t_0, r)$ with

$$u(y_1, t_1) < M.$$

Then $u(y_0, t_0) < M$. Set $v = M - u$ and

$$R = 2|y_1 - y_0| < 2r, \quad \alpha := \frac{t_0 - t_1}{R^2}.$$

By continuity there exists $\epsilon > 0$ and $h > 0$ such that

$$v(x, t_1) > h, \quad |y| < \epsilon R.$$

By 2.2.5 there exists $c > 0$, such that $v(y, t_0) > ch > 0$ for all $|y - y_1| < R/2$, a contradiction. Hence if $u(x_0, t_0) = M$, then $u(y, t) = M$ for all $(y, t) \in Q(x_0, t_0; r)$, whenenver $Q(x_0, t_0; 3r) \subset X$. Hence $\{u = M\} \cap S(x_0, t_0)$ is (parabolically) open and closed and hence all of $S(x_0, t_0)$. □

2.3. Hopf Lemma

This section follows the presentation in [**And11**].

Definition 2.3.1. [Spherical cap condition] Let $X \subset \mathbb{R}^{n+1}$. We say $(x_0, t_0) \in \partial_P X$ satisfies the *spherical cap condition*, if there exist $r > 0$ and $(x_1, t_1) \in \mathbb{R}^{n+1}$ with $x_1 \neq x_0$, such that

$$(x_0, t_0) \in \partial B_r^{n+1}(x_1, t_1)$$

and

$$\emptyset \neq B_r^{n+1}(x_1, t_1) \cap \{t < t_0\} \subset X.$$

Theorem 2.3.2 (Hopf Lemma). *Let $X \subset \mathbb{R}^{n+1}$ open and bounded, L elliptic, $b, c = 0$ and $u \in C_1^2(X) \cap C^0(\bar{X})$ with*

$$(\partial_t - L)u \leq 0$$

in X. Assume $(x_0, t_0) \in \partial_P(X)$ satisfying the spherical cap condition with cap A and

$$u(x, t) < u(x_0, t_0) \quad \forall (x, t) \in A.$$

Then

$$\limsup_{h \to 0} \frac{u((x_0, t_0) + he) - u(x_0, t)}{h} < 0 \quad \forall e \; \forall h \ll 1 \colon (x_0, t) + he \in A. \tag{2.3.1}$$

Observe that the inequality (2.3.1) with "$\leq$" is trivial. The strict inequality "$<$" is the main result.

PROOF. Set

$$M = u(x_0, t_0).$$

We also know that from the strong maximum principle

$$u(x, t_0) < M \quad \forall (x, t_0) \in \partial A.$$

Obviously (2.3.1) holds with with the weak inequality. Wlog

$$u(x, t) < M \quad \forall (x, t) \in \partial A \backslash \{(x_0, t_0)\}.$$

Set

$$w(x, t) = e^{-\alpha\left(|x-x_1|^2+|t-t_1|^2\right)} - e^{-\alpha r^2}, \quad \alpha > 0.$$

then

$$w(x, t) \in [0, 1] \quad \forall (x, t) \in B_r^{n+1}(x_1, t_1),$$

$$w(x, t) = 0 \quad \forall (x, t) \in \partial B_r^{n+1}(x_1, t_1).$$

Then

$$\dot{w} = -2\alpha(t - t_1)e^{-\alpha\left(|x-x_1|^2+|t-t_1|^2\right)},$$

$$\partial_i w = -2\alpha(x^i - x_1^i)e^{-\alpha\left(|x-x_1|^2+|t-t_1|^2\right)},$$

$$\partial_j\partial_i w = -2\alpha e^{-\alpha\left(|x-x_1|^2+|t-t_1|^2\right)} \left(\delta_{ij} - 2\alpha(x^i - x_1^i)(x^j - x_1^j)\right).$$

Hence

$$\begin{aligned}\dot{w} - Lw &= 2\alpha e^{-\alpha\left(|x-x_1|^2+|t-t_1|^2\right)} \left(-(t - t_1) + a^{ij}\delta_{ij} - 2\alpha a^{ij}(x^i - x_1^i)(x^j - x_1^j)\right) \\ &\leq 2\alpha e^{-\alpha\left(|x-x_1|^2+|t-t_1|^2\right)} \left(-(t - t_1) + \|\mathrm{tr}(A)\|_\infty - 2\alpha\lambda|x - x_1|^2\right).\end{aligned}$$

Set

$$\Omega_\epsilon = A \cap \{|x - x_0| < \epsilon\}.$$

Hence for all $(x, t) \in \Omega_\epsilon$ we have $|x - x_1| \geq \frac{1}{2}|x_1 - x_0| > 0$. Thus choose α large such that

$$\dot{w} - Lw \leq 0 \quad \forall (x, t) \in \Omega_\epsilon.$$

Put

$$v = u + \mu w, \quad \mu > 0.$$

Then $\dot{v} - Lv \leq 0$ in Ω_ϵ. We have

$$\partial_P\Omega_\epsilon = S_1 \cup S_2,$$

with

$$S_1 = \partial_P A \cap \partial B_r(x_1, t_1), \quad S_2 = \bar{A} \cap \{|x - x_0| = \epsilon\}.$$

On S_1 we have $v \leq M$. On S_2 there exists $\sigma > 0$, such that $u(x,t) < M - \sigma$. Hence $v = u + \mu w \leq M - \sigma + \mu < M$ for small μ. Thus

$$v(x,t) \leq M \quad \forall (x,t) \in \partial_P \Omega_\epsilon.$$

Also

$$\dot{v} - Lv \leq 0 = (\dot{u} - Lu)(x_0, t_0)$$

and hence

$$v(x,t) \leq M = v(x_0,t_0) \quad \forall (x,t) \in \Omega_\epsilon.$$

We deduce for all e with $(x_0,t_0) + he \in A$ for small h, that

$$\limsup_{h \to 0} \frac{v((x_0,t_0)+he) - v((x_0,t_0))}{h} \leq 0.$$

But

$$\partial_e w = 2\alpha e^{-\alpha|x_0 - x_1|^2 + |t_0 - t_1|^2} \left\langle e, (x_1 - x_0, t_1 - t_0) \right\rangle > 0,$$

and hence (2.3.1) follows. □

2.4. Harnack's inequality

Later we prove some weak Harnack estimates. Without proof, now we state:

THEOREM 2.4.1 (Parabolic Harnack inequality). *Assume $u \in C^2_1(U_T)$ and solves*

$$(\partial_t - L)u = 0 \quad \textit{in } U_T$$

and

$$u \geq 0 \quad \textit{in } U_T$$

Assume moreover that $b \equiv 0$ and $c \equiv 0$ and a is smooth.

If $V \Supset U$ is connected, then for each time $0 < t_1 < t_2 \leq T$ there is a constant C such that

$$\sup_{x \in V} u(x,t_1) \leq C \inf_{x \in V} u(x,t_2).$$

PROOF. See [**Eva98**, Theorem 10, p.391]. □

CHAPTER III

A short look at Semi-group theory

As references we refer to [**Eva98**, §7.4] and [**CH98**].

In Section 1.2 we looked at $(\partial_t - \Delta)\, u = 0$ and naively we should have

$$u = e^{t\Delta} u(0).$$

We made this precise with the help of the Fourier Transform.

Is there a similar relation if we look at L instead of Δ?

Generally: Let X be a real Banach space and a linear map A,

$$A\colon D(A) \subset X \to X,$$

where $D(A)$ is the domain of A, a linear (usually dense) subset of X. We are looking for solutions $u \in C^1((0,T),X)$ of

$$\begin{aligned} \dot{u} &= Au, \quad t \in (0,T), \\ u(0) &= \varphi. \end{aligned} \tag{3.0.1}$$

A is in general not bounded, but closed. Assume there exists a solution to (3.0.1), then

$$T(t)\varphi := u(t)$$

defines an operator. Properties of T:

- $T(t)\colon X \to X$ is linear,
- $T\colon [0,\infty) \to L(X)$.
- $T(0) = \mathrm{id}$,
- $T(t+s) = T(t) \circ T(s)$,
- $t \mapsto T(t)\varphi$ is continuous.

The latter three properties are characteristic for a semigroup.

Assume now that we have a semigroup

$$T\colon [0,\infty) \times X \to X.$$

Then we find some A such that T is the semigroup of A. A will then be called the generator of T.

$$\begin{aligned}\dot{u}(t) &= \lim_{s\to 0}\frac{u(t+s)-u(t)}{s} = \lim_{s\to 0}\frac{T(t+s)\varphi - T(t)\varphi}{s}\\ &= \lim_{s\to 0}\frac{T(s)-T(0)}{s}u(t)\\ &\equiv Au(t).\end{aligned}$$

Hence let

$$Au = \lim_{s\to 0}\frac{T(s)-T(0)}{s}u,$$

whenever the limit exists. Call $D(A)$ the set of $u \in X$ where this limit exists.

One might conjecture there is some sort of equivalence between generators A and semigroups T.

Questions: Which generators A allow semigroups? Which generators are implies by semigroups?

The main theorem which gives us an answer to this question is the Hille-Yoshida Theorem at the end of this Section.

3.1. m-dissipative operators

We want to solve

$$\begin{aligned} u'(t) &= Au, \quad t > 0\\ u(0) &= \varphi \end{aligned} \tag{3.1.1}$$

with some operator

$$D(A) \subset X \to X,$$

where X is a Banach space and $D(A)$ a linear subspace, e.g. $X = L^2$ and $D(A) = H^2$. In general A will not be bounded.

3.1.1. linear bounded operators. (i) Let $X = \mathbb{R}^n$ or $\mathbb{C}^n$, $A\colon X \to X$ linear (and thus bounded), then

$$u(t) = e^{tA}\varphi$$

is the unique solution to (3.1.1), where

$$e^{tA} = \sum_{k=0}^{\infty}\frac{1}{k!}t^k A^k.$$

(ii) Let X be a general Banach space and $A \in L(X)$, where $L(X)$ is the space of bounded linear operators. Here e^{tA} also makes sense.

LEMMA 3.1.1. *Let $A, B \in L(X)$. Then*

(i) e^A *converges absolutely,*

(ii) $e^0 = \mathrm{id}$,

(iii) $AB = BA \quad \Rightarrow \quad e^{A+B} = e^A e^B$,

(iv) $e^{-A} = \left(e^A\right)^{-1}$.

THEOREM 3.1.2. *Let $A \in L(X)$, $\varphi \in X$, $T > 0$. Then there exists a unique solution $u \in C^1((0,T), X)$ of*

$$\begin{aligned} u'(t) &= Au(t) \\ u(0) &= \varphi. \end{aligned}$$

PROOF. Put

$$u(t) = e^{tA}\varphi.$$

Then

$$u'(t) = e^{tA}A\varphi = Au(t).$$

For a second solution v set

$$w(t) = e^{-tA}v(t),$$

then $w'(t) = 0$ and hence $w(t) = w(0) = \varphi$. □

3.1.2. unbounded operators. Let X be a real or complex Banach space. An operator

$$A\colon D(A) \subset X \to X$$

is called linear, if and only if $D(A)$ is a linear subspace and A ist linear on $D(A)$. We say A is densely defined, if

$$\overline{D(A)} = X.$$

A is bounded, if and only if

$$\|A\| := \sup_{\|x\| \leq 1} \|Ax\| < \infty.$$

Otherwise it is called unbounded.

examples

(1) $X = L^2(\mathbb{R}^n)$, $A = \Delta$, $D(A) = H^2(\mathbb{R}^n)$ or $D(A) = C^\infty$.

(2) $X = C^0([0,1])$, $D(A) = X$, $K \in C^0([0,1] \times [0,1])$

$$Au(x) = \int_0^1 K(x,y)u(y)\, dy$$

is bounded.

We use the following notation.

$$G(A) = \{(u, Au) \subset X \times X : u \in D(A)\}$$

is the graph of A,

$$R(A) = \{Au : u \in D(A)\}$$

the range of A. An extension of A is

$$\tilde{A} \colon D(\tilde{A}) \subset X \to X,$$

such that

$$D(A) \subset D(\tilde{A}) \quad \text{and} \quad Au = \tilde{A}u \quad \forall u \in D(A).$$

A is called closed, if $G(A)$ is closed in $X \times X$. A is called closable, if there exists a closed extension $\tilde{A}$.

Theorem 3.1.3 (Closed Graph Theorem). *Let $A \colon X \to X$ be linear. Then A is continuous (i.e. bounded) if and only if A is closed.*

3.1.3. Notion of m-dissipative operators.

X Banach space, $A \colon D(A) \to X$ linear.

Definition 3.1.4. A is *dissipative*, if

$$\|u - \lambda Au\| \geq \|u\| \quad \forall u \in D(A), \lambda > 0.$$

A is called *accretive*, if $-A$ is dissipative.

Lemma 3.1.5. *Let X be a Hilbert space,*

$$A \colon D(A) \subset X \to X$$

linear, then A is dissipative if and only if

$$\operatorname{Re} \langle u, Au \rangle \leq 0 \quad \forall u \in D(A).$$

If for example $A = \Delta$, $X = L^2(\mathbb{R}^n)$, $D(A) = H^2(\mathbb{R}^n)$, then

$$\langle u, \Delta u \rangle = -\int_{\mathbb{R}^n} |\nabla u|^2 \leq 0.$$

For Schroedinger equation:

$$\langle u, \pm i\Delta u \rangle = \mp i \int_{\mathbb{R}^n} |\nabla u|^2$$

and hence the real part is 0 and both $i\Delta$ and $-i\Delta$ are dissipative.

PROOF OF LEMMA 3.1.5. Let A dissipative, then:

$$\|u\|^2 + \lambda^2 \|Au\|^2 - 2\lambda \text{Re}\,\langle u, Au\rangle - \|u\|^2 = \|u - \lambda Au\|^2 - \|u\|^2 \geq 0.$$

Dividing by λ and letting $\lambda \to 0$ gives

$$\text{Re}\,\langle u, Au\rangle \leq 0.$$

Let

$$\text{Re}\,\langle Au, u\rangle \leq 0,$$

then

$$\|u - \lambda Au\|^2 = \|u\|^2 + \lambda^2 \|Au\|^2 - 2\lambda \text{Re}\,\langle u, Au\rangle \geq \|u\|^2.$$

□

DEFINITION 3.1.6 (m-dissipative). A linear operator $A\colon D(A) \subset X \to X$ is called *m-dissipative*, if A is dissipative and $I - \lambda A$ is surjective for all $\lambda > 0$. (hence $I - \lambda A$ is continuously invertible.)

Our aim is to show that for any m-dissipative A we can define (some sort of) e^A. We also call A m-accretive, if $-A$ is m-dissipative. Set

$$J_\lambda = (I - \lambda A)^{-1}\colon X \to D(A).$$

Then

$$\|J_\lambda v\| \leq \|v\| \quad \forall v \in X.$$

LEMMA 3.1.7. *Let A be dissipative, then A is m-dissipative if and only if there exists $\lambda_0 > 0$ such that $I - \lambda_0 A$ is surjective.*

PROOF. Let $\lambda \in (0, \infty)$ and $v \in X$. Find $u \in D(A)$ such that $u - \lambda Au = v$.

$$u - \lambda_0 Au = \frac{\lambda_0}{\lambda} v + \left(1 - \frac{\lambda_0}{\lambda}\right) u$$

is equivalent to

$$u = J_{\lambda_0}\left(\frac{\lambda_0}{\lambda} v + \left(1 - \frac{\lambda_0}{\lambda}\right) u\right) \equiv F(u).$$

We show the right hand side is a contraction in u. Then

$$\|F(u) - F(w)\| = \left\| J_{\lambda_0}\left(\left(1 - \frac{\lambda_0}{\lambda}\right)(u - w)\right)\right\| \leq \left|1 - \frac{\lambda_0}{\lambda}\right| \|u - w\|.$$

Hence F is a contraction, if $\lambda < \lambda_0/2$. Then there is a unique $u \in D(A)$ with $F(u) = u$. Iteration give the result. □

PROPOSITION 3.1.8. *All m-dissipative operators are closed.*

PROOF. J_1 exists and is continuous, hence $I - A$ is closed and hence A is closed. □

example:
$X = L^2$, $A = \Delta$, $D(A) = H^2$. Then A is m-dissipative. We only have to show that

$$\forall v \in L^2 \; \exists u \in H^2 \colon u - \Delta u = v.$$

Here we see that the choice of $D(A)$ is important (the above will not work for $D(A) = C^\infty$.) We solve this by Fourier-transform.

$$\hat{u}(\xi) + |\xi|^2 \hat{u}(\xi) = \hat{v}(\xi)$$

and hence we conjecture

$$\hat{u}(\xi) := \frac{1}{1 + |\xi|^2} \hat{v}(\xi).$$

Hence $\hat{u} \in L^2$ and

$$\frac{\xi^1 \xi^2}{1 + |\xi|^2} \hat{v}(\xi) \in L^2$$

implies that $u, \nabla^2 u \in L^2$.

PROPOSITION 3.1.9. *Let A be m-dissipative, then*

$$\forall u \in \overline{D(A)} \colon \quad \|J_\lambda u - u\| \xrightarrow{\lambda \to 0} 0.$$

PROOF. There holds

$$\|J_\lambda - I\| \leq \|J_\lambda\| + \|I\| \leq 2.$$

Hence it suffices to prove the result for $u \in D(A)$.

$$\|J_\lambda u - u\| = \|J_\lambda \left(u - (I - \lambda A) u \right) \| \leq \lambda \|Au\| \to 0, \quad \lambda \to 0.$$

□

Set

$$A_\lambda := A J_\lambda = \frac{1}{\lambda}(J_\lambda - I).$$

This $A_\lambda \in L(X)$ will serve as an "approximation" for A, so that we can make (certain) sense of an operator e^{tA} in terms of $\lim_{\lambda \to 0} e^{tA_\lambda}$. This is justified by the following

PROPOSITION 3.1.10. *Let A be m-dissipative and $\overline{D(A)} = X$. Then*

$$A_\lambda u \to Au, \quad \forall u \in D(A).$$

PROOF.

$$J_\lambda A u \to Au,$$

since $D(A)$ is dense. Furthermore, we have

$$(I - \lambda A) A = A (I - \lambda A).$$

Thus, multiplying both sides with J_λ from the left and also from the right, we have $A_\lambda = AJ_\lambda = J_\lambda A$. $\square$

3.2. Semigroup Theory

Let X be a Banach space. A semigroup is an operator

$$T\colon [0,\infty) \to L(X),$$

such that

(i) $T(0) = I$,
(ii) $T(t+s) = T(t)T(s)$.

T is called C^0-semigroup (strongly continuous semigroup), if

(iii) $\lim_{t\to 0} \|T(t)u - u\| = 0 \quad \forall u \in X$.

Note, that $T(s)T(t) = T(t)T(s)$.

Examples

(1) $A \in L(X)$, $T(t) = e^{tA}$.
(2) $X = L^p(\mathbb{R})$, $p \in [1,\infty]$.

$$T(t)u(x) = u(t+x).$$

If $p < \infty$, then T is a continuous semigroup, since C_c^∞ is dense and hence for $u \in L^p$ and $\epsilon > 0$ there exists $f \in C_c^\infty$ with

$$\|f - u\|_p < \epsilon/3.$$

We have for small t,

$$\sup_x |f(x-t) - f(x)| < t\|\nabla f\|_\infty < \epsilon/3$$

Then

$$\begin{aligned}\|T(t)u - u\|_p &\leq \|T(t)f - f\|_p + \|T(t)(u-f)\|_p + \|u - f\|_p \\ &\leq \frac{2\epsilon}{3}\end{aligned}$$

and

$$\left(\int_{\mathbb{R}} |T(t)f - f|^p\right)^{\frac{1}{p}} < \frac{\epsilon}{3}\left(\operatorname{diam}(\operatorname{supp} f) + 1\right).$$

For $p = \infty$ let $u = \chi_{[0,1]}$, then

$$\|u - T(t)u\|_\infty = \sup_x |u(x) - u(x+t)| \geq 1 \quad \forall t > 0.$$

Thus T is no C^0-semigroup for $p = \infty$.

PROPOSITION 3.2.1. *Let $T(t)$ be a C^0-semigroup. Then $\exists M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t}.$$

PROOF. Show that there exists $\delta > 0$ such that

$$\sup_{0<t<\delta} \|T(t)\| < \infty. \tag{3.2.1}$$

If this was not the case, then there exists a sequence $t_n \to 0$ with $\|T(t_n)\| \to \infty$. Recall Banach-Steinhaus: If for a sequence $A_n \in L(X)$ we have

$$\forall u \in X\colon \sup_n \|A_n u\| < \infty,$$

then $\sup_n \|A_n\| < \infty$.

Hence in our case we find $u \in X$ such that $\|T(t_n)u\| \to \infty$, in contradiction to the C^0-property. Hence (3.2.1) must be true. Now let $t > 0$, then there exists $n \in \mathbb{N}$ and $s \in (0, \delta)$, such that

$$t = n\delta + s.$$

Then

$$T(t) = T(\delta) \circ \dots \circ T(\delta) \circ T(s).$$

Then

$$\|T(t)\| \leq \|T(\delta)\|^n \|T(s)\| \leq M^{n+1} \leq MM^{\frac{t}{\delta}} = Me^{t \log \frac{M}{\delta}}.$$

□

PROPOSITION 3.2.2. *Let $T(t)$ be a C^0-semigroup. Then the map*

$$(t, u) \mapsto T(t)u$$

is continuous.

PROOF. Exercise. □

DEFINITION 3.2.3. Let $T(t)$ be a C^0-semigroup. Then

$$\omega_0 = \inf\{w \in \mathbb{R}\colon \exists M \geq 1, \|T(t)\| \leq Me^{\omega t}\}$$

ist called the *growth bound* of the semigroup.

DEFINITION 3.2.4. A C^0-semigroup is called *contraction semigroup*, if

$$\forall t > 0\colon \|T(t)\| \leq 1.$$

Recall that

$$\|J_\lambda\| \leq 1, \quad \|A_\lambda\| \leq \frac{2}{\lambda}.$$

We define

$$T_\lambda(t) = e^{tA_\lambda},$$

which is a C^0-semigroup and we have

$$\|T_\lambda(t)\| \leq \|e^{tJ_\lambda \frac{1}{\lambda}} e^{-\frac{t}{\lambda} I} - e^{-\frac{t}{\lambda}} \| e^{\frac{t}{\lambda} J_\lambda}\| \leq e^{-\frac{t}{\lambda}} e^{\frac{t}{\lambda}} = 1.$$

THEOREM 3.2.5 (Hille Yoshida (Part I)). *Let $A\colon D(A) \subset X \to X$ m-dissipative and densely defined. Then for all $u \in X$ the limit*

$$T(t)u = \lim_{\lambda \to 0} T_\lambda(t)u$$

exists and the convergence is uniform on intervals of the form $[0,T]$. Furthermore $(T(t))_{t\geq 0}$ is a contraction semigroup and for all $u \in D(A)$,

$$u(t) := T(t)u$$

is the unique solution $u \in C^0([0,\infty), D(A)) \cap C^1((0,\infty), X)$ to

$$\begin{cases} \dot{u}(t) & = Au(t) \quad t > 0 \\ u(0) & = u \end{cases} \tag{3.2.2}$$

PROOF. **Step (1): On the contraction semigroup property**

There holds $J_\lambda J_\mu = J_\mu J_\lambda$ and the same for A_λ. Let $\lambda, \mu > 0$, then

$$\begin{aligned} T_\lambda(t)u - T_\mu(t)u &= \left(e^{tA_\lambda} - e^{tA_\mu}\right) u \\ &= e^{tA_\lambda}(I - e^{t(A_\mu - A_\lambda)})u \end{aligned}$$

and hence

$$\begin{aligned} \|T_\lambda(t)u - T_\mu(t)u\| &\leq \|I - e^{t(A_\mu - A_\lambda)}u\| \\ &\leq |t| \left(\|e^{tA_\mu}\| + \|e^{tA_\lambda}\|\right) \|(A_\mu - A_\lambda)u\| \\ &\leq 2|t| \|\left(A_\mu - A_\lambda\right) u\| \to 0, \quad |\mu - \lambda| \to 0 \end{aligned}$$

uniformly on bounded intervals. Hence the proposed limit exists, if $u \in D(A)$. Since $T(t)$ is a uniformly bounded linear operator and hence extends to all of X, since $D(A)$ is dense.

Now let $u \in X$ with approximating sequence $u_n \in D(A)$.

$$\begin{aligned} \|T_\lambda(t)u - T(t)u\| &\leq \|T_\lambda(t)u - T_\lambda(t)u_n\| + \|T_\lambda(t)u_n - T(t)u_n\| \\ &\quad + \|T(t)(u_n - u)\| \\ &\leq 2\|u_n - u\| + \|T_\lambda(t)u_n - T(t)u_n\|. \end{aligned}$$

Hence $T_\lambda(t)u \to T(t)u$. Furthermore

$$\begin{aligned} \|T(t)T(s)u - T(t+s)u\| &\leq \|T(t)T(s)u - T(t)T_\lambda(s)u\| \\ &\quad + \|T(t)T_\lambda(s)u - T_\lambda(t)T_\lambda(s)u\| \\ &\quad + \|T_\lambda(t+s)u - T(t+s)u\| \\ &\to 0. \end{aligned}$$

Step (2): On the equation (3.2.2)

Let $u \in D(A)$ and set

$$u_\lambda(t) = e^{tA_\lambda} u.$$

Then

$$\frac{d}{dt} = e^{tA_\lambda} A_\lambda u = T_\lambda(t) A_\lambda u.$$

Equivalently, also using $A_\lambda u \to Au$ and $T_\lambda \to T$,

$$u(t) \leftarrow u_\lambda(t) = u + \int_0^t T_\lambda(s) A_\lambda u \; ds \to u + \int_0^1 T(s) \; Au \; ds.$$

Thus $u \in C^1$ and

$$\dot{u}(t) = T(t)Au = Au(t).$$

Uniqueness proceeds as in Theorem 3.1.2. □

3.2.1. Generators of semigroups. Let $T(t)$ be a contraction semigroup. Define

$$D(L) := \left\{ u \in X \colon \lim_{h \to 0} \frac{T(h)u - u}{h} \text{ exists} \right\}.$$

For $u \in D(L)$ set

$$Lu = \lim_{h \to 0} \frac{T(h)u - u}{h}.$$

Example: $X = C_{ub}(\mathbb{R})$ be the set of uniformly continuous, bounded functions with the L^∞-norm.

$$T(t)u(x) := u(x + t).$$

Then $T(t)$ is a contraction semigroup. Then

$$Lu = u', \quad D(L) = \{u, u' \in C_{ub}(\mathbb{R})\}.$$

PROOF. It is clear that $u, u' \in C_{ub}(\mathbb{R})$ implies

$$\left\| \frac{u(x+h) - u(x)}{h} - u'(x) \right\|_\infty \to 0.$$

Now let $u \in D(L)$, then $u'_+ \in C_{ub}(\mathbb{R})$ and hence $u'_+ = u' \in C_{ub}(\mathbb{R})$. □

THEOREM 3.2.6 (Hille Yoshida Part II). *Let $T(t)$ be a contraction semigroup with generator L. Then L is m-dissipative and densely defined.*

PROOF. (i) L is dissipative, i.e. for all $\lambda > 0$, $\|u - \lambda Lu\| \geq 0$.

$$\begin{aligned}\left\|u - \lambda\frac{T(h)u - u}{h}\right\| &= \left\|\left(1 + \frac{\lambda}{h}\right)u\right\| - \left\|\frac{\lambda}{h}T(h)u\right\| \\ &= \left(1 + \frac{\lambda}{h}\right)\|u\| - \frac{\lambda}{h}\|T(h)u\| \\ &\geq \left(1 + \frac{\lambda}{h}\|u\| - \frac{\lambda}{h}\|u\|\right) = \|u\|.\end{aligned}$$

$h \to 0$ on the left hand side shows L is dissipative.

(ii) L is m-dissipative. It suffices to show that $(I - L)$ is surjective. Thus we want to find Ju, such that

$$(I - L)Ju = u.$$

Ansatz:

$$Ju = \int_0^\infty e^{-t}T(t)\ dt.$$

Then

$$\|Ju\| \leq \int_0^\infty e^{-t}\|T(t)u\|\ dt \leq \|u\|$$

and hence $\|J\| = 1$. We claim that

$$(I - L)Ju = u$$

and therefore calculate

$$\begin{aligned}(T(h) - I)\,Ju &= \int_0^\infty e^{-t}T(t+h)u\ dt - \int_0^\infty e^{-t}T(t)u\ dt \\ &= \int_h^\infty e^{-t+h}T(t)u\ dt - \int_0^8 e^{-t}T(t)u\ dt \\ &= \int_0^\infty \left(e^{-t+h} - e^{-t}\right)T(t)u - \int_0^h e^{-t+h}T(t)u\ dt \\ &= (e^h - 1)\int_0^\infty e^{-t}T(t)u\ dt - e^h\int_0^h e^{-t}T(t)u\ dt \\ &= (e^h - 1)Ju - e^h\int_0^h e^{-t}T(t)u\ dt.\end{aligned}$$

Hence

$$\frac{T(h) - I}{h}Ju = \frac{e^h - 1}{h}Ju - \frac{e^h}{h}\int_0^h e^{-t}T(t)u\ dt.$$

Thus $Ju \in D(L)$ and

$$LJu = Ju - u,$$

which is the claim.

(iii) $D(L)$ is dense. Set

$$u_h = \frac{1}{h}\int_0^h T(s)u\ ds.$$

There holds

$$\begin{aligned}\|u_h - u\| &= \left\|\frac{1}{h}\int_0^h (T(s)-I)\,u\ ds\right\| \\ &\leq \frac{1}{h}\int_0^h \|(T(s)-I)\,u\| \to 0.\end{aligned}$$

Thus we show $u_h \in D(L)$ for all $h > 0$ and $u \in X$. Now let $t << h$, we calculate

$$\begin{aligned}\frac{T(t)-I}{t}u_h &= \frac{1}{ht}\int_t^{t+h} T(s)u\ ds - \frac{1}{ht}\int_0^h T(s)u\ ds \\ &= \frac{1}{ht}\int_h^{t+h} T(s)u\ ds + \frac{1}{ht}\int_t^h T(s)u\ ds \\ &\quad - \frac{1}{ht}\int_0^t T(s)u\ ds - \frac{1}{ht}\int_t^h T(s)u\ ds \\ &\to \frac{1}{h}T(h)u - \frac{1}{h}T(0)u \in X\end{aligned}$$

and hence the left hand side converges in X. □

CHAPTER IV

Schauder estimates

References: [**IS13**] and [**Kry96**]

Our aim is that for some solution of

$$(\partial_t - \Delta)u = f$$

we want to obtain $C^{2+\alpha}$ estimates in dependence of $f \in C^\alpha$.

4.1. Parabolic Hölder spaces

$X \subset \mathbb{R}^{n+1}$, Also here, the philosophy is that functions have half smoothness in time compared to space.

For $(x_i, t_i) \in \mathbb{R}^{n+1}$ put

$$\rho\left((x_1,t_1),(x_2,t_2)\right) = \sqrt{|t_1 - t_2|} + |x_1 - x_2|.$$

Definition 4.1.1. Let $X \subset \mathbb{R}^{n+1}$, $\alpha \in (0,1)$. Set

$$[u]_{\alpha,X} := \sup_{(x_1,t_1)\neq(x_2,t_2)\in X} \frac{|u(t_1,x_1) - u(t_2,x_2)|}{\rho\left((x_1,t_1),(x_2,t_2)\right)^\alpha}$$

and

$$\|u\|_{\alpha,X} = [u]_{\alpha,X} + \|u\|_\infty.$$

Also let

$$[u]_{2+\alpha,X} := [\dot{u}]_{\alpha,X} + [D^2u]_{\alpha,X}$$

and

$$\|u\|_{2+\alpha,X} = \|u\|_\infty + [u]_{2+\alpha,X}.$$

The spaces $(C^{2+\alpha}(X), \|\cdot\|_{2+\alpha})$, $(C^\alpha(X), \|\cdot\|_\alpha)$ are Banach spaces.

Lemma 4.1.2 (Computations). *For all $\alpha \in (0,1)$ there hold:*

(1)

$$[uv]_{\alpha,X} \le \|u\|_\infty[v]_{\alpha,X} + \|v\|_\infty[u]_{\alpha,X},$$

(2) $k \in \{0, 2\}$,

$$[u+v]_{k+\alpha} \leq [u]_{k+\alpha,X} + [v]_{k+\alpha,X}.$$

There is an alternative description for the Hölder norms. We define

$$\mathcal{P}_2 = \{\text{polynomials in } t, x \text{ of the form } p(t,x) = \lambda_1 t + \lambda_2^i x_i + \lambda_3^{ij} x_i x_j + \lambda_4\}$$

and

$$[u]'_{2+\alpha,\mathbb{R}^{n+1}} = \sup_{(t_1,x_1)\in\mathbb{R}^{n+1}} \sup_{\rho>0} \frac{1}{\rho^{2+\alpha}} \inf_{p\in\mathcal{P}_2} \|u-p\|_{\infty,Q_\rho((x_1,t_1))},$$

where Q is the parabolic cylinder of radius ρ.

THEOREM 4.1.3 (Equivalence of Hölder norms). *There exists $C > 0$, such that for all $u \in C^{2+\alpha}(\mathbb{R}^{n+1})$*

$$[u]'_{2+\alpha,\mathbb{R}^{n+1}} \leq C[u]_{2+\alpha,\mathbb{R}^{n+1}} \tag{4.1.1}$$

and

$$[u]_{2+\alpha,\mathbb{R}^{n+1}} \leq C[u]'_{2+\alpha,\mathbb{R}^{n+1}}. \tag{4.1.2}$$

PROOF. (4.1.1) is an exercise (take p a Taylor polynomial).

As for (4.1.2), let $h > 0$ and set

$$\sigma_h(\partial_t)u(t,x) = \frac{u(t,x) - u(t-h^2,x)}{h^2}$$

$$\sigma_h(\partial_{ij})u(t,x) = \frac{1}{h^2}\left(u(t,x+he_i+he_j) - u(t,x+he_i) - u(t,x+he_j) + u(t,x)\right)$$

Observe that

$$\sigma_h(\partial_t)(p) = c, \quad \sigma_h(\partial_{ij})p = c$$

and, due to Taylor,

$$|\sigma_h(\partial_t)u(t,x) - \partial_t u(t,x)| \leq Ch^\alpha [u]_{2+\alpha,\mathbb{R}^{n+1}}$$

and similarly in ∂_{ij}. Now let $(x_i,t_i) \in \mathbb{R}^{n+1}$ and

$$\rho = \rho((x_1,t_1),(x_2,t_2)), \quad h := \epsilon\rho,$$

where ϵ will be chosen.

Then

$$\begin{aligned}
|\partial_t u(x_1,t_1) - \partial_t u(t_2,x_2)| &\leq |\sigma_h(\partial_t)u(t_1,x_1) - \sigma_h(\partial_t)u(t_2,x_2)| \\
&\quad + |\sigma_h(\partial_t)u(t_1,x_1) - \partial_t u(x_1,t_1)| \\
&\quad + |\sigma_h(\partial_t)u(t_2,x_2) - \partial_t u(x_2,t_2)| \\
&\leq 2Ch^\alpha [u]_{2+\alpha,\mathbb{R}^{n+1}} \\
&\quad + |\sigma_h(\partial_t)(u-p)(t_1,x_1) - \sigma_h(\partial_t)(u-p)(t_2,x_2)|.
\end{aligned}$$

Suppose $t_1 \le t_2$. Then (t_1, x_1), $(t_1 - h^2, x_1)$, (t_2, x_2), $(t_2 - h^2, x_2) \in Q_{3\rho}(t_2, x_2)$ and hence

$$|\sigma_h(\partial_t)(u-p)(t_1,x_1)| + |\sigma_h(\partial_t)(u-p)(x_2,t_2)| \le \frac{4}{h^2}\|u-p\|_{\infty,Q_{3\rho}}$$

for all $p \in \mathcal{P}_2$. Taking the infimum gives

$$\begin{aligned}\frac{1}{\rho^\alpha}|\partial_t u(t_1,x_1) - \partial_t u(t_2,x_2)| &\le 2C\frac{h^\alpha}{\rho^\alpha}[u]_{2+\alpha,\mathbb{R}^{n+1}} + \frac{4}{\rho^\alpha h^2}\inf_{p\in\mathcal{P}_2}\|u-p\|_{\infty,Q_{3\rho}} \\ &\le 2C\epsilon^\alpha[u]_{2+\alpha,\mathbb{R}^{n+1}} + \frac{4}{\epsilon^2}[u]'_{2+\alpha,\mathbb{R}^{n+1}}.\end{aligned}$$

An analogueous estimate holds for spatial derivatives. Absorbing the $[u]$-part into the right hand side gives the result. □

PROPOSITION 4.1.4. *(Interpolation)*

$$\forall \alpha \in (0,1), \gamma > 0\colon \|\partial_t u\|_{\infty,X} \le C(\gamma)\|u\|_\infty + \gamma[u]_{2+\alpha,X}.$$

The same holds for Du and $[u]_{\alpha,X}$.

PROPOSITION 4.1.5 (Arzela-Ascoli). *Let $X \subset \mathbb{R}^{n+1}$ be bounded and $u_k \in C^{2,\alpha}(X)$ uniformly bounded. Then there exists a subsequence converging in $C^{2,\beta}$ for all $\beta < \alpha$.*

4.2. Schauder estimates with constant coefficients

References: [**IS13**, Chapter 2.4], [**Kry96**, Chapter 8.6]

First, we prove the (interior) Schauder estimate for the heat equation. The general case is a consequence of this theorem.

THEOREM 4.2.1. *(Schauder) Let $\alpha \in (0,1)$, $T \in \mathbb{R} \cup \{\infty\}$, $u \in C^\infty(\mathbb{R}^n \times (-\infty, T])$. Set*

$$f := (\partial_t - \Delta)u.$$

Then there exists $C = C(n,\alpha) > 0$ such that

$$[u]_{2+\alpha,\mathbb{R}^n\times(-\infty,T)} \le C[f]_{\alpha,\mathbb{R}^n\times(-\infty,T)}.$$

There are several proofs of this theorem. A popular one is due to Safanov and can be found in [**Kry96**]. We use here the blow-up approach due to Simon [**Sim97**].

PROOF. We prove the case $T = \infty$, the case $T < \infty$ is an exercise the reader is urged to do, Exercise 13.

Assume the claim is false, that is for any $k \in \mathbb{N}$ there exists a smooth $u_k \in C^\infty(\mathbb{R}^{n+1})$ so that

$$[u_k]_{C^{2+\alpha}(\mathbb{R}^{n+1})} \ge k\,[(\partial_t - \Delta)u_k]_{C^\alpha(\mathbb{R}^{n+1})}.$$

Our goal is to produce a contradiction from this assumption. For this we first modify the sequence $(u_k)_{k\in\mathbb{N}}$ appropriately, then we pass to the limit as $k \to \infty$.

- Firstly, without loss of generality, we can assume

$$[u_k]_{C^{2+\alpha}(\mathbb{R}^{n+1})} = 1, \tag{4.2.1}$$

$$[(\partial_t - \Delta)u_k]_{C^{\alpha}(\mathbb{R}^{n+1})} < \frac{1}{k}, \tag{4.2.2}$$

  otherwise we rescale $\tilde{u}_k := u_k/[u_k]_{C^{2+\alpha}(\mathbb{R}^{n+1})}$ and work with $\tilde{u}_k$ instead of u_k.
- The condition (4.2.1) implies for some $(x_k, t_k) \in \mathbb{R}^{n+1}$ and some $\vec{v}_k \in \mathbb{R}^{n+1}\backslash\{0\}$

$$\frac{1}{2} \leq \frac{|D^2u_k((t_k,x_k)+\vec{v}_k) - D^2u_k(t_k,x_k)|}{\rho(\vec{v}_k,0)^\alpha} + \frac{|\partial_t u_k((t_k,x_k)+\vec{v}_k) - \partial_t u_k(t_k,x_k)|}{\rho(\vec{v}_k,0)^\alpha}.$$

  Let $e_i = (0,\dots,0,1,0,\dots,0)^T$ the i-th unit vector in $\mathbb{R}^{n+1}$. By decomposing $\vec{v}_k$ into its components we may simplify and for $c_0 := \frac{1}{2(n+1)}$ we necessarily find some $i_k \in \{1,\dots,n+1\}$ and some $h_k > 0$ so that

$$\begin{aligned} c_0 \leq & \frac{|D^2u_k((t_k,x_k)+h_k e_{i_k}) - D^2u_k((t_k,x_k))|}{\rho(h_k e_{i_k},0)^\alpha} \\ & + \frac{|\partial_t u_k((t_k,x_k)+h_k e_{i_k}) - \partial_t u_k((t_k,x_k))|}{\rho(h_k e_{i_k},0)^\alpha}. \end{aligned}$$

- Up to taking a subsequence $k \to \infty$ (again denoted by k), we may assume that $e_{i_k} = e_{i_0}$ for some fixed $i_0 \in \{1,\dots,n+1\}$: there must be a constant subsequence of $i_k \in \{1,\dots,n+1\}$.
- W.l.o.g. $(t_k, x_k) = 0$, otherwise replace u_k by $\tilde{u_k}(t,x) := u_k(t+t_k, x+x_k)$.
- W.l.o.g.

$$u_k(0) = \partial_t u_k(0) = \partial_{x^i} u_k(0) = \partial_{x^i x^j} u_k(0) = (\partial_t - \Delta)u_k(0) = 0,$$

  otherwise we add a polynomial $p \in \mathcal{P}_2$, i.e. of the form

$$p(t,x) = c_1 + tc_2 + xc_3 + x^T c_4 x,$$

  so that $\tilde{u_k} := u_k - p$ satisfies these conditions.
- Furthermore we may assume $h_k = 1$. Otherwise we scale

$$\tilde{u}_k(t,x) = \begin{cases} h^{-2-\alpha} u_k(h^2 t, hx), & \text{if } e_{i_0} \in \{0\} \times \mathbb{R}^n \\ \sqrt{h}^{-2-\alpha} u_k(ht, \sqrt{h}x), & \text{if } e_{i_0} \in \mathbb{R} \times \{0\}. \end{cases}$$

All these assumptions yield that without loss of generality, $u_k \in C^\infty(\mathbb{R}^{n+1})$ satifies (4.2.1) and (4.2.2) and moreover

$$|D^2u_k(e_{i_0})| + |\partial_t u_k(e_{i_0})| \geq c_0 \quad \forall k \in \mathbb{N}. \tag{4.2.3}$$

Observe that the latter condition is stable under *local* $C^{2,\beta}$-convergence ($\beta < \alpha$), while (4.2.1) is not, which is the main reason we did these simplifciations. Now we can pass to the limit:

For large $R > 1$ to be chosen later, we set

$$\Gamma(R) = \{(t,x) \in \mathbb{R}^{n+1} \colon |x| \leq R, |t| \leq R^2\}.$$

For any $(t,x) \in \Gamma(R)$ there holds

$$\begin{aligned}
|u_k(t,x)| &= |u_k(t,x) - u_k(0,0)| \\
&\leq |u_k(t,x) - u_k(0,x)| + |u_k(0,x) - u_k(0,0)| \\
&\leq R^2 \|\partial_t u_k\|_{\infty,\Gamma(R)} + C\ R \|Du_k\|_{\infty,\Gamma(R)\cap\{t=0\}} \\
&\leq R^2 \|\partial_t u_k\|_{\infty,\Gamma(R)} + C\ R \|Du_k - Du_k(0)\|_{\infty,\Gamma(R)\cap\{t=0\}} \\
&\leq R^2 \|\partial_t u_k\|_{\infty,\Gamma(R)} + C\ R^2 \|D^2 u_k\|_{\infty,\Gamma(R)} \\
&\leq C\ R^{2+\alpha} [u_k]_{2+\alpha},
\end{aligned}$$

For some dimensional constant $C > 0$.

In particular, in view of (4.2.1),

$$\sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(\Gamma(R))} \leq C\ R^{2+\alpha}. \tag{4.2.4}$$

In particular

$$\sup_{k\in\mathbb{N}} \|u_k\|_{2+\alpha,\Gamma(R)} \leq C(1 + R^{2+\alpha}).$$

With Arzela-Ascoli, Proposition 4.1.5 we find some $u \in C^{2,\alpha}$ and have w.l.o.g. (otherwise we take a subsequence),

$$u_k \to u, \quad \text{in } C^{2,\beta}$$

for any $\beta < \alpha$.

In particular, we have pointwise convergence of first and second derivatives and thus by (4.2.3),

$$|D^2 u(e_{i_0})| + |\partial_t u(e_{i_0})| \geq c_0. \tag{4.2.5}$$

Moreover, by locally uniform convergence, (4.2.4) takes over and we have

$$\|u\|_{L^\infty(\Gamma(R))} \leq C\ R^{2+\alpha}.$$

In particular, we have an L^1-estimate we can later use for the Cauchy estimates (observe that the size of $\Gamma(R)$ is $|\Gamma(R)| = C\ R^{n+2}$)

$$\|u\|_{L^1(\Gamma(R))} \leq C\ R^{n+4+\alpha}.$$

Furthermore by (4.2.2), $(\partial_t - \Delta)u$ is constant in $\Gamma(R)$, and since $(\partial_t - \Delta)u(0) = 0$, we have

$$(\partial_t - \Delta)u = 0 \quad \text{in } \Gamma(R).$$

We thus may apply the Cauchy-estimates, Theorem 1.6.2, (they are written for C^2_1 but they can easily be extended to $C^{2+\beta}$). Assume that $R > 1$ is so large that $B_1(0)^{n+1} \subset \Gamma(R/4)$. For this we estimate

$$
\begin{aligned}
&|D^2u(e_{i_0})| + |\partial_t u(e_{i_0})| \\
\leq &\|D^2u\|_{\infty,B_1^{n+1}(0)} + \|\partial_t u\|_{\infty,B_1^{n+1}(0)} \\
&\leq \|D^2u - D^2u(0)\|_{\infty,B_1^{n+1}(0)} + \|\partial_t u - \partial_t u(0)\|_{\infty,B_1^{n+1}(0)} \\
&\leq C\left(\|D^3u\|_{\infty,B_1^{n+1}(0)} + \|\partial_t D^2u\|_{\infty,B_1^{n+1}(0)} + \|\partial_t Du\|_{\infty,B_1^{n+1}(0)} + \|\partial_t\partial_t u\|_{\infty,B_1^{n+1}(0)}\right),
\end{aligned}
$$

and with the Cauchy-estimates, Theorem 1.6.2, we then have

$$
|D^2u(e_{i_0})| + |\partial_t u(e_{i_0})| \leq C\left(R^{-n-5} + R^{-n-6}\right)\|u\|_{L^1(\Gamma(R))}.
$$

In view of (4.2) we then finally obtain

$$
|D^2u(e_{i_0})| + |\partial_t u(e_{i_0})| \leq C\left(R^{-n-5} + R^{-n-6}\right)R^{n+4+\alpha} \leq 2C\ R^{\alpha-1},
$$

which (since $\alpha < 1$) for large enough $R > 1$ contradicts (4.2.5). □

EXERCISE 13. *Zeigen Sie Theorem IV.3.2 (Schauder für konstante Koeffizienten) aus der Vorlesung für $T < \infty$:*

Sei $\alpha \in (0,1)$, $T < \infty$, $u \in C^\infty(\mathbb{R}^n \times (-\infty, T])$ und

$$
f := (\partial_t - \Delta)u.
$$

Dann gilt für eine Konstante $C = C(\alpha, n)$,

$$
[u]_{2+\alpha,\mathbb{R}^n\times(\infty,T)} \leq C\ [f]_{\alpha,\mathbb{R}^n\times(\infty,T)}.
$$

Hinweise:

- *Zeigen Sie, dass Sie Ohne Einschränkung annehmen können: $T = 0$*
- *Die Cauchy-Abschätzungen, Theorem I.6.2, gelten rückwärts in der Zeit!*

COROLLARY 4.2.2 (Schauder with constant coefficient)). *Let $\alpha \in (0,1)$, $L = a^{ij}\partial_{ij}$ elliptic and a^{ij} symmetric and constant. Then there exists $C = C(\alpha, n, |a^{ij}|, \lambda) > 0$ such that for all $u \in C^\infty(\mathbb{R}^n \times (-\infty, T))$ we have*

$$
[u]_{2+\alpha,(-\infty,T)\times\mathbb{R}^n} \leq C[\dot{u} - Lu]_{\alpha,(-\infty,T)\times\mathbb{R}^n}.
$$

PROOF. There exists $P \in \mathrm{SO}(n)$ and a diagonal matrix D with

$$
A = P^TDP = P^T\sqrt{D}PP^T\sqrt{D}P \equiv B^2.
$$

Put

$$
v(t,x) = u(t, Bx).
$$

Then

$$\begin{aligned}\Delta v(t,x) &= \partial_i^2(u(t,Bx))\\ &= \partial_i\left(B^{ij}\partial_j u(t,Bx)\right)\\ &= (B^2)^{ij}\partial_{ij}u(t,Bx)\\ &= a^{ij}\partial_{ij}u(t,Bx).\end{aligned}$$

Hence

$$\partial_t v - \Delta v = \partial_t u - a^{ij}\partial_{ij}u$$

and Theorem 4.2.1 gives the result. □

4.3. Schauder Estimate for variable coefficient

PROPOSITION 4.3.1. *Let* $X = \Omega \times (0,T) \subset \mathbb{R}^{n+1}$, $u \in C^2(\bar{X})$, $u \in C^0(X \cup \partial_P X)$. *For* $g = u_{|\partial_P X}$ *and*

$$f = \partial_t u - Lu,$$

where a^{ij} *is continuous,* $b = c = 0$. *Then*

$$\|u\|_\infty \le T\|f\|_\infty + \|g\|_\infty.$$

PROOF. Set

$$v^\pm(t,x) = u \pm \left(\|g\|_\infty + t\|f\|_\infty\right).$$

Then

$$\left(\partial_t - L\right)v^+ = f + \|f\|_\infty \ge 0$$

and reversed for v^-. Furthermore

$$v^+ \ge 0, \quad v^- \le 0$$

on $\partial_P X$. By the maximum principle

$$v^+ \ge 0, v^- \le 0$$

throughout X, which implies the claim. □

THEOREM 4.3.2 (Schauder (interior)). *Let* $u \in C^{2,\alpha}(\overline{(0,T)\times\mathbb{R}^n})$, $a \in (0,1)$, $h = u_{|\{0\}\times\mathbb{R}^n}$, $\partial_t u - Lu = f$ *for*

$$L = a^{ij}\partial_{ij} + b^i\partial_i + c,$$

with coefficients in C^α. *Then there exists* $C = C(\alpha, n, \lambda, \|a\|_\infty, [a^{ij}]_\alpha, [b]_\alpha, [c]_\alpha)$ *such that*

$$\|u\|_{2+\alpha,(0,T)\times\mathbb{R}^n} \le C\left([f]_{\alpha,(0,T)\times\mathbb{R}^n} + [h]_{2+\alpha,\mathbb{R}^n} + \|u\|_{\infty,\mathbb{R}^n\times(0,T)}\right).$$

Proof. First suppose $b = c = 0$ and $h \in C^{2,\alpha}(\mathbb{R}^{n+1})$ and $u = h$ on $(\mathbb{R}^n \times \{0\})$. We freeze the a^{ij}. Let $0 < \gamma < 1$ be chosen later. Let $(x_1, t_1), (x_2, t_2) \in (0, T) \times \mathbb{R}^n$ such that

$$\|\partial_t u\|_{\alpha,(0,T)\times\mathbb{R}^n} \leq 2\frac{|\partial_t u(x_1, t_1) - \partial_t u(x_2, t_2)|}{\rho((x_1, t_1), (x_2, t_2))^\alpha}.$$

Case 1: $\rho \geq \gamma$. Then

$$\begin{aligned}[\partial_t u]_{\alpha,(0,T)\times\mathbb{R}^n} &\leq 4\gamma^{-\alpha}\|\partial_t u\|_{\infty,(0,T)\times\mathbb{R}^n} \\ &\leq \frac{1}{4}[u]_{2+\alpha,(0,T)\times\mathbb{R}^n} + C(\gamma)\|u\|_{\infty,(0,T)\times\mathbb{R}^n}.\end{aligned}$$

Case 2: $\rho < \gamma$. Let $\xi \in C_c^\infty(\mathbb{R}^{n+1})$ with

$$\xi((y, t)) = 1, \quad \rho((y, t), 0) < 1$$

and

$$\xi((y, t)) = 0, \quad \rho((y, t), 0) \geq 2.$$

Set

$$\eta(t, x) = \xi\left(\frac{t - t_1}{\gamma^2}, \frac{x - x_1}{\gamma}\right).$$

Then by 4.2.2

$$\begin{aligned}[\partial_t u]_{\alpha,(0,T)\times\mathbb{R}^n} &\leq 2\rho((x_1, t_1), (x_2, t_2))^{-\alpha}|\partial_t(u\eta)(x_1, t_1) - \partial_t(u\eta)(x_2, t_2)| \\ &\leq 2[u\eta]_{2+\alpha,(0,T)\times\mathbb{R}^n} \\ &\leq C[(\partial_t - L)(x_1, t_1)(u\eta)]_{\alpha,\mathbb{R}^n\times(-\infty,T)} \\ &\leq C[(\partial_t - L)(x_1, t_1)(u\eta)]_{\alpha,(0,T)\times\mathbb{R}^n} + \|h\|_{2+\alpha,\mathbb{R}^n} \\ &\leq C[(\partial_t - L)(u\eta)]\alpha, (0, T) \times \mathbb{R}^n \\ &\quad + [((\partial_t - L)(x_1, t_1) - (\partial_t - L))(u\eta)]_{\alpha,(0,T)\times\mathbb{R}^n} \\ &\quad + \|u\|_\infty + [h]_{2+\alpha,\mathbb{R}^n} \\ &\equiv I + II + \|u\|_\infty + [h]_{2+\alpha,\mathbb{R}^n}.\end{aligned}$$

$$(\partial_t - L)(u\eta) = \eta f + u(\partial_t - L)\eta - 2a^{ij}\partial_i u\partial_j u$$

and hence

$$\begin{aligned}I &\leq C(\gamma, a^{ij})\left([f]_\alpha + [u]_2 + [Du]_\alpha\right) \\ &\leq \gamma^\alpha[u]_{2+\alpha} + C(\gamma)[f]_\alpha + \|u\|_{\infty,(0,T)\times\mathbb{R}^n}.\end{aligned}$$

Also with Proposition 4.1.4,

$$[\left(a^{ij}(x_1, t_1) - a_{ij}\right)\partial_{ij}(u\eta)]_{\alpha,(0,T)\times\mathbb{R}^n} \leq C\gamma^\alpha[u]_{2+\alpha} + C(\gamma)\|u\|_\infty,$$

since

$$\|a^{ij}(x_1, t_1) - a_{ij}\|_{\infty,\text{supp } \eta} \leq C\gamma^\alpha[a]_\alpha$$

and hence

$$II \le C\gamma^\alpha [u]_{2+\alpha} + C(\gamma)\|u\|_\infty.$$

The same argument holds for D^2u and thus

$$\begin{aligned}[u]_{2+\alpha,(0,T)\times\mathbb{R}^n} \le& \left(C\gamma^\alpha + \frac{1}{2}\right)[u]_{2+\alpha,(0,T)\times\mathbb{R}^n} \\ &+ C(\gamma)\left([f]_\alpha + \|u\|_\infty + [h]_{2+\alpha}\right).\end{aligned}$$

Choose γ such that the first term of the right hand side is absorbed in the left hand side, which gives the result in case $b = c = 0$. In general:

$$\partial_t u - a^{ij}\partial_{ij}u = f + b^i\partial_i u + cu$$

and thus

$$\begin{aligned}[u]_{2+\alpha} &\le C\left(\|u\|_\infty + [h]_{2+\alpha} + [f + b^i\partial_i u + cu]_{\alpha,(0,T)\times\mathbb{R}^n}\right) \\ &\le \|u\|_\infty + [h]_{2+\alpha} + [f]_\alpha \\ &\quad + [b]_\alpha\|\partial_i u\|_\infty + [c]\|u\|_\infty + \|b\|_\infty[\partial_i u]_\alpha + \|c\|_\infty[u]_\alpha \\ &\le \|u\|_\infty + [h]_{2+\alpha} + [f]_\alpha + C(b,c,\epsilon)\|u\|_\infty + \epsilon[u]_{2+\alpha}.\end{aligned}$$

□

CHAPTER V

Viscosity Solutions

Viscosity solutions were introduced by Crandall and Lions. A standard reference is [**CIL92**]. See also [**Koi12**] and [**IS13**, Chapter 3].

Consider the equation

$$\partial_t u + F(t,x,Du,D^2u) = 0. \tag{5.0.1}$$

Observe that there is no u-term here, and thus corresponds to the linear equation $(\partial_t + L)u$ with $c \equiv 0$.

F is called *degenerately elliptic*, if

$$F(t,x,p,A) \geq F(t,x,p,B) \quad \forall (t,x) \in \mathbb{R}^{n+1}, p \in \mathbb{R}^n, A \leq B, \tag{5.0.2}$$

with symmetric matrices A, B.

It is a simple observation, see also Exercise 9, that for parabolic linear operators $L = a_{ij}\partial_{ij} + b_j\partial_j$ with $c \equiv 0$, the operator F given as

$$F(t,x,p,A) := -a_{ij}A_{ij} + b_j p_j$$

is degenerate elliptic in the above sense.

Also, we observe that if a smooth u is a solution to

$$\partial_t u + F(t,x,Du,D^2u) = 0 \quad \text{in a point } (t_0,x_0) \in \mathbb{R}^{n+1}$$

then for any test-function φ "touching u from above", i.e. so that $\varphi \geq u$ and $\varphi(x_0,t_0) = u(x_0,t_0)$ then $\partial_t\varphi(x_0,t_0) = \partial_t u(x_0,t_0)$, $D\varphi(x_0,t_0) = D\varphi u(x_0,t_0)$ and $D^2\varphi(x_0,t_0) \geq D^2u(x_0,t_0)$ and consequently

$$\partial_t\varphi(t_0,x_0)+F(t_0,x_0,D\varphi(x_0,t_0),D^2\varphi(x_0,t_0) \leq \partial_t u(t_0,x_0)+F(t_0,x_0,Du(t_0,x_0),D^2u(t_0,x_0)) = 0$$

In words, if u is a smooth solution of (5.0.1) in (t_0,x_0), then any φ touching u from above in (t_0,x_0) is a subsolution of (5.0.1) in (t_0,x_0).

The same way, if u is a smooth solution of (5.0.1) in (t_0,x_0) then any φ touching u from below in (t_0,x_0) is a supersolution of (5.0.1) in (t_0,x_0).

The converse trivially holds true: If any φ touching u from above in (t_0,x_0) is a subsolution of (5.0.1) in (t_0,x_0), then taking $\varphi := u$ so is u. The same holds of course for supersolutions.

Also for merely continuous functions u we can define what it means to be touched above or below from some test-function φ, thus for thus functions u will can define the following weak notion of subsolution (in the Viscosity sense). If any testfunction φ touching from u above in a point (t_0, x_0) is a subsolution, then we say that u is a (Viscosity-)subsolution. Similar definitions hold for supersolution. A Viscosity solution is then simply a function which is sub- and supersolution.

5.1. Definitions and first properties

A function u is lower semicontinuous (lsc) , if

$$u(x) \le \liminf_{y \to x} u(y)$$

and upper semicontinuous (usc) if

$$u(x) \ge \limsup_{y \to x} u(y).$$

For a function u the *upper semicontinuous envelope* is

$$u^* = \lim_{r \to 0} \sup\{u(y)\colon |y - x| \le r\}.$$

u^* is the smallest upper semicontinuous function with $u \le u^*$. The *isc envelope* is

$$u_* = \lim_{r \to 0} \inf\{u(y)\colon |y - x| \le r\},$$

which is the largest isc function with $u_* \le u$. Cf. Exercise 15.

Definition 5.1.1 (Test-function). A test function on an open $Q \subset \mathbb{R}^{n+1}$ is a function $\varphi\colon Q \to \mathbb{R}$ which is C^1 in time and C^2 in space.

A test function φ *touches* a function $u\colon Q \to \mathbb{R}$ from above (below) in (t_0, x_0), if

$$\varphi \ge u, \quad (\varphi \le u)$$

and

$$\varphi(x_0, t_0) = u(x_0, t_0).$$

Definition 5.1.2 (Viscosity solution). Let $Q \subset \mathbb{R}^{n+1}$ open and $u\colon Q \to \mathbb{R}$ a function. We define (super-, sub-)solutions of the equation

$$\partial_t v + F(t, x, Dv, D^2 v) = 0. \tag{5.1.1}$$

(1) u is a subsolution of (5.1.1), if u is upper semicontinuous and for all $(x, t) \in Q$ and for all test functions φ touching u from above in (x, t) we have

$$\partial_t \varphi + F(t, x, D\varphi, D^2\varphi) \le 0.$$

(2) u is a supersolution of (5.1.1), if u is lsc and for all $(x, t) \in Q$ and for all test functions φ touching u from below in (x, t) we have

$$\partial_t \varphi + F(t, x, D\varphi, D^2\varphi) \ge 0.$$

(3) u is a vixcosity solution of (5.1.1), if u is a sub- and supersolution. Observe, that in particular u is supposed to be continuous.

DEFINITION 5.1.3 (2^{nd} order sub/super differentials).

$$\begin{aligned}\mathcal{P}^{\pm}(u)(t,x) = \{(\alpha,p,X) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_{\text{sym}} :\\ (\alpha,p,X) = (\partial_t\varphi(x,t), D\varphi(x,t), D^2\varphi(x,t))\\ \text{for some test function from above (below) } \varphi\}.\end{aligned}$$

Observe that if $(\alpha,p,X)\in\mathcal{P}^+(u)(t,x)$ and φ is the associated test-function then we have by $u(y,s)\le\varphi(y,s)$ and by Taylor

$$u(y,s)\le u(x,t)+\alpha(s-t)+p\cdot(y-x)+\frac{1}{2}(y-x)^TX(y-x)+o(|y-x|^2+|s-t|)$$

In particular u being viscosity subsolution is equivalent to saying u is usc and for all $(\alpha,p,X)\in\mathcal{P}^+(u)$ we have

$$\alpha+F(x,t,p,X)\le 0.$$

A similar characterization holds for supersolutions.

DEFINITION 5.1.4 (Limit of (sub-) superdifferentials).

$$\begin{aligned}\bar{\mathcal{P}}^{\pm}(u)(t,x) = \{(\alpha,p,X) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_{\text{sym}} : \exists(t_n,x_n\to(t,x))\\ \exists(\alpha_n,p_n,X_n)\in\mathcal{P}^{\pm}(u)(t_n,x_n),\\ (\alpha_n,p_n,X_n)\to(\alpha,p,X)\\ u(t_n,x_n)\to u(t,x)\}.\end{aligned}$$

We suppose from now on that F is continuous and degenerately elliptic.

PROPOSITION 5.1.5. *(1) Let $Q\subset\mathbb{R}^{n+1}$ open and assume that $(u_\alpha)_{\alpha\in\mathcal{A}}$ be a family of subsolutions for*

$$\partial_t u+F(t,x,Du,D^2u)=0 \quad \textit{in } Q$$

Let u be the upper semicontinuous envelope of $\sup_\alpha u$ (which itself needs not to be upper semicontinuous), that is

$$u=\left(\sup_\alpha u_\alpha\right)^*$$

and suppose u is pointwise finite, then u is a subsolution.

(2) Let $(u_n)_{n\in\mathbb{N}}$ a sequence of subsolutions. The upper relaxed limit *$\bar{u}$ is defined by*

$$\bar{u}(t,x)=\limsup_{(s,y)\to(t,x),n\to\infty}u_n(s,y).$$

If $\bar{u}$ is pointwise finite, then $\bar{u}$ is a subsolution in Q.

PROOF. We only show (1), the argument for (2) is ananlogous.

Fix $(t_0, x_0) \in Q$ and $(\alpha_0, p_0, X_0) \in \mathcal{P}^+(u)(t_0, x_0)$ throughout this proof.

We want to show that

$$\alpha_0 + F(t_0, x_0, p_0, X_0) \leq 0.$$

By the definition of u we find a sequence in $(u_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and points $(x_n, t_n) \in Q$ so that

$$(x_n, t_n, u_n(x_n, t_n)) \to (x_0, t_0, u(x_0, t_0)).$$

For small $r \in (0, 1)$ let $(\hat{x}_n, \hat{t}_n)$ be a maximizer of $B(r) := \overline{B_r^{n+1}(x_0, t_0)}$ of the function

$$(s, y) \mapsto u_n(s, y) - p \cdot (y - x_0) - \alpha(s - t_0) - \frac{1}{2}(y - x_0)^T X(y - x_0)$$

The maximum is attained because of upper semicontinuity of u_n.

Then we have

$$\begin{aligned} u_n(s, y) &\leq u_n(\hat{x}_n, \hat{t}_n) + p \cdot (y - \hat{x}_n) + \alpha(s - \hat{t}_n) + \frac{1}{2}(y - x_0)^T X(y - x_0) \\ &\quad - \frac{1}{2}(\hat{x}_n - x_0)^t X(\hat{x}_n - x_0) \\ &=: \varphi_n(s, y), \end{aligned}$$

and we also have

$$u_n(\hat{x}_n, \hat{t}_n) = \varphi_n(\hat{x}_n, \hat{t}_n).$$

That is, φ_n is a (smooth) test function from above for u_n in $(\hat{x}_n, \hat{t}_n)$. In particular,

$$\partial_s \varphi_n(\hat{x}_n, \hat{t}_n) + F(\hat{x}_n, \hat{t}_n, D\varphi_n(\hat{x}_n, \hat{t}_n), D^2\varphi(\hat{x}_n, \hat{t}_n)) \leq 0.$$

Computing the derivatives of φ_n, this becomes

$$\alpha + F(x_0 + (\hat{x}_n - x_0), t_0 + (\hat{t}_n - t_0), p_0 + X_0(\hat{x}_n - x_0), X_0) \leq 0.$$

Up to a subsequence we may assume that $\hat{x}_n \to \bar{x} \in B(r)$ and $\hat{t}_n \to \bar{t} \in B(r)$. With the continuity of F, we then have

$$\alpha + F(x_0 + (\bar{x} - x_0), t_0 + (\bar{t} - t_0), p_0 + X_0(\bar{x} - x_0), X_0) \leq 0.$$

This holds for any small $r > 0$, and $(\bar{x}, \bar{t}), (x_0, t_0) \in B(r)$. Letting $r \to 0$, and again with the continuity of F, we conclude

$$\alpha + F(x_0, t_0, p_0, X_0) \leq 0.$$

□

EXERCISE 14. *Zeigen Sie:*

- $$u_*(x) := \sup\{\tilde{u}(x) : \quad \tilde{u} \le u, \quad \tilde{u} \textit{ unterhalbstetig}\}$$
  ist unterhalbstetig.
- *Ist* $(u_\alpha)_\alpha$ *eine Familie von oberhalb stetigen Funktionen, so ist* $u := \inf_\alpha u_\alpha$ *oberhalb stetig*
- *Ist* $(u_\alpha)_\alpha$ *eine Familie von unterhalb stetigen Funktionen, so ist* $u := \sup_\alpha u_\alpha$ *unterhalb stetig*
- *überlegen Sie sich ein Beispiel einer Familie von oberhalb stetigen Funktionen, so dass* $u := \sup_\alpha u_\alpha$ *beschränkt ist, aber* nicht *oberhalb stetig ist.*

EXERCISE 15. *Zeigen Sie, dass der upper semicontinuous envelope* $u^*(x)$ *für eine Funktion* $u : \mathbb{R}^n \to \mathbb{R}$*,definiert als*

$$u^*(x) := \lim_{r \to 0_+} \sup_{|y-x|<r} u(y),$$

tatsächlich die kleinste oberhalbstetige Funktion oberhalb u *ist. Dazu zeigen Sie:*

- *Für jedes feste* $x \in \mathbb{R}^n$ *und jede Funktion* $u : \mathbb{R}^n \to \mathbb{R}$ *gilt*
  $$\limsup_{y \to x} u(y) = \lim_{r \to 0_+} \sup_{|y-x|<r} u(y)$$
- $u^*(x) \ge u(x)$
- $u^*(x)$ *ist oberhalb stetig*
- *Für jedes oberhalbstetige* v *mit* $v \ge u$ *gilt* $v \ge u^*$*.*

CHAPTER VI

Harnack inequality for fully nonlinear parabolic equations

Reference: [**IS13**, Chapter 4].

6.1. Setup

We look at

$$\partial_t u + F(D^2u, (x,t)) = f$$

and assume F to be uniformly elliptic, see Definition 6.1.2 below. We aim to prove an equality of the form

$$\sup_K u(\cdot, t_1) \leq C \inf_K u(\cdot, t_2) + C\|f\|,$$

for $t_2 > t_1$.

Definition 6.1.1 (Pucci-operator). Let $M \in \mathbb{R}^{n\times n}$ be symmetric, $0 < \lambda \leq \Lambda$. Then

$$P^+(M) = \sup_{\lambda I \leq A \leq \Lambda I} (-\operatorname{tr}(AM))$$

and

$$P^-(M) = \inf_{\lambda I \leq A \leq \Lambda I} (-\operatorname{tr}(AM))$$

Observe, if u satisfies

$$\partial_t u - A^{ij}\partial_{ij} u = f$$

with

$$\lambda |\xi|^2 \leq A^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2,$$

then

$$\partial_t u(x,t) + P^+(D^2u(x,t)) \geq f(x,t) \geq \partial_t + P^-(D^2u(x,t)).$$

Compare the following with degenerate ellipticity (5.0.2).

Definition 6.1.2. (Uniformly elliptic) Let

$$F\colon \mathbb{R}^{n\times n}_{\text{sym}} \times X \to \mathbb{R}$$

is uniformly elliptic with (λ, Λ), if

$$P^-(X-Y) \leq F(X,(x,t)) - F(Y,(x,t)) \leq P^+(X-Y).$$

Observe that then

$$P^-(X) \leq F(X,(x,t)) - F(0,(x,t)) \leq P^+(X)$$

and hence if

$$\partial_t u + F(D^2u(x,t),(x,t)) = f,$$

then

$$\partial_t u - P^+(D^2u) \geq f(x,t) + F(0,(x,t))$$

and similarly for P^-.

6.2. Alexandrov-Bakelman-Pucci maximum principle

Recall the elliptic case. For u we define the *contact set* $\{u = \Gamma(u)\}$, where $\Gamma(u)$ is the convex envelope of u, i.e. the largest convex function below u. Then there holds: **Elliptic ABP maximum principle:** Let $Lu \leq f$ in Ω. Then

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + C_\Omega \left(\int_{\{u=\Gamma(u)\}} |f|^n \right)^{\frac{1}{n}}.$$

We state (without proof) the parabolic version.

Definition 6.2.1. (Monotone envelope) Let $\Omega \subset \mathbb{R}^n$ be convex, (a,b) an open interval and assume

$$u\colon (a,b) \times \Omega \to \mathbb{R}$$

to be l.s.c. Then $\Gamma(u)$ is the monotone envelope, defined as the largest function

$$v\colon (a,b) \times \Omega \to \mathbb{R},$$

such that

- $v \leq u$
- $v(t,\cdot)$ is convex for all $t \in (a,b)$
- v is nonincreasing in time.

One can show

$$\Gamma(u)(t,x) = \sup\{\xi \cdot x + h\colon \xi \in \mathbb{R}^n, h \in \mathbb{R}, \\ \xi \cdot y + h \leq u(s,y)\ \forall y \in \Omega\ \forall s \in (a,t)\}.$$

Theorem 6.2.2. *(Parabolic ABP) Let u be a supersolution of*

$$\partial_t u + P^+(D^2u) = f$$

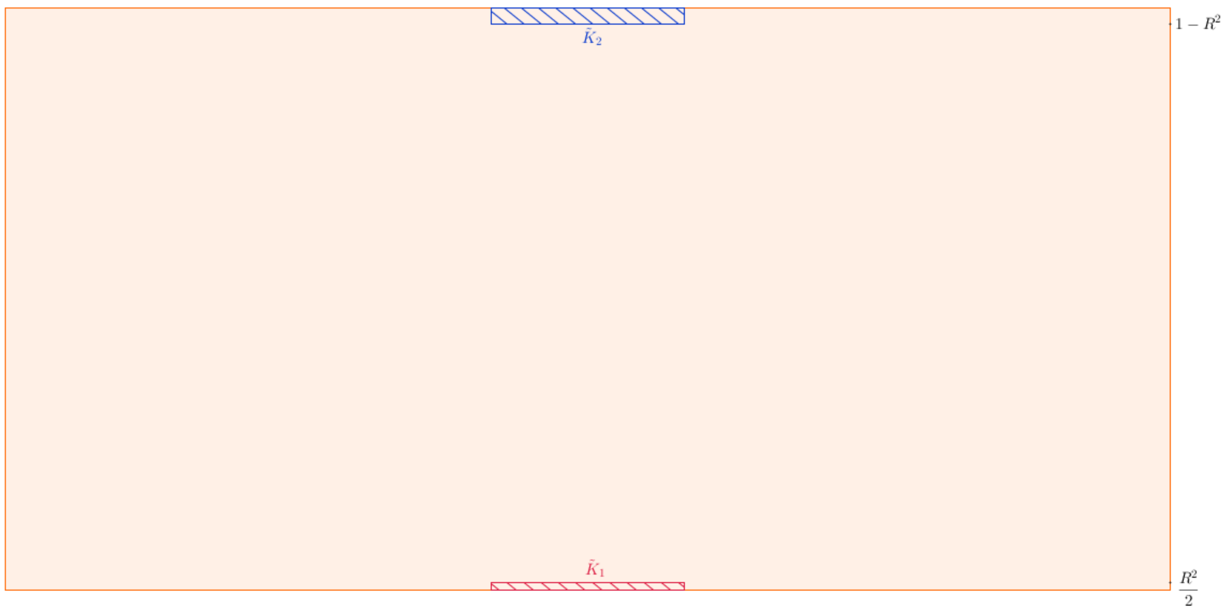


FIGURE 1. The sets $\tilde{K}_1$, $\tilde{K}_2$

in $Q_\rho = (-\rho^2, 0) \times B^n_\rho(0)$. *If* $u \geq 0$ *on* $\partial_P Q_\rho$, *then*

$$\sup_{Q_\rho} u^- \leq C\rho^{\frac{n}{n+1}} \left(\int_{u=\Gamma(u)} |f^+|^{n+1} \right)^{\frac{1}{n+1}},$$

where $\Gamma(u)$ *is the monotone envelope in* $Q_{2\rho}$ *of*

$$\begin{cases} \min(0,u), & Q_\rho \\ 0, & Q_{2\rho} \backslash Q_\rho. \end{cases}$$

6.3. The L^ε-estimate

We want to prove:

THEOREM 6.3.1 (L^ϵ-estimate). *There exists* $\epsilon > 0$, $R \in (0,1)$, $C > 0$, *depending on* λ, Λ *and* n *such that for all nonnegative supersolutions* u *of*

$$\partial_t u + P^+(D^2 u) = f \quad \text{in } (0,1) \times B^n_{\frac{1}{R}}(0),$$

then

$$\left(\int_{\tilde{K}_1} u^\epsilon \right)^{\frac{1}{\epsilon}} \leq C \left(\inf_{\tilde{K}_2} u + \|f\|_{L^{n+1}((0,1)\times B^n_{\frac{1}{R}}(0))} \right),$$

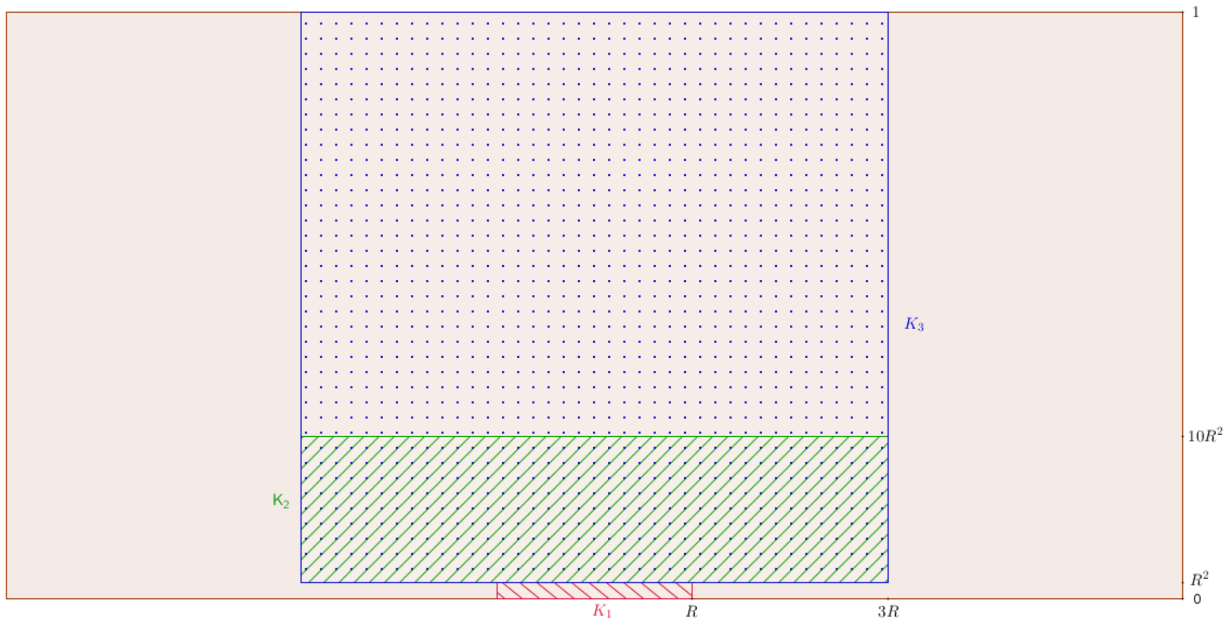


FIGURE 2. The sets K_1, K_2, K_3

where (see Figure 1)

$$\tilde{K}_1 = \left(0, \frac{R^2}{2}\right) \times (-R, R)^n,$$

$$\tilde{K}_2 = (1 - R^2, 1) \times (-R, R)^n.$$

Further sets, see Figure 2

$$K_1 = K_1(R) = (0, R^2) \times (-R, R)^n,$$

$$K_2 = (R^2, 10R^2) \times (-3R, 3R)^n,$$

$$K_3 = (R^2, 1) \times (-3R, 3R)^n.$$

LEMMA 6.3.2. *(Barrier for L^ϵ) For all $R \in \left(0, \min\left(\frac{1}{3\sqrt{n}}, \frac{1}{\sqrt{10}}\right)\right)$ there exists a Lipschitz function*

$$0 \le \Phi \colon Q_1(0,1) \to \mathbb{R}$$

such that Φ is C^2 in x where $\Phi > 0$ and

$$\partial_t \Phi + P^+(D^2\Phi) \le g$$

for $g \colon Q_1 \to \mathbb{R}$ continuous and bounded with

$$\text{supp } g \subset K_1,$$

$\Phi \geq 2$ in K_3 and $\Phi = 0$ on $\partial_p Q$.

Proof. It suffices to construct φ, such that

$$\partial_t\varphi + P^+(D^2\varphi) \leq 0,$$

$$\varphi = 0, \quad \partial_p Q_1 \backslash \{(0,0)\},$$

$$\varphi > 0 \quad \text{in } \overline{K_3}$$

and

$$\varphi \to \infty \quad \text{in } (0,0).$$

Then we set

$$\Phi(x,t) = \begin{cases} 2\frac{\varphi(t,x)}{\min_{K_3}\varphi}, & (t,x) \notin K_1 \\ \text{Lipschitz ext. with zero on } \partial_p Q_1 \text{ in } K_1. \end{cases}$$

For some $T \in (0,1)$ we first construct φ on $(0,T)$. Take in $(0,T) \times B_1$:

$$\varphi(t,x) = t^{-p}\psi\left(\frac{x}{\sqrt{t}}\right).$$

$$\begin{aligned} &\partial_t\varphi + P^+(D^2\varphi) \\ &= t^{-p-1}\left(-p\psi\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2}D\psi\left(\frac{x}{\sqrt{t}}\right)\frac{x}{\sqrt{t}} + P^+(D^2\psi)\left(\frac{x}{\sqrt{t}}\right)\right) \end{aligned} \tag{6.3.1}$$

We want the bracket to be nonpositive. Substitute $z = x/\sqrt{t}$. If $(x,t) \in K_2$, then

$$|z| = \frac{|x|}{\sqrt{t}} \leq \frac{3R\sqrt{n}}{R} = 3\sqrt{n}.$$

Choose ψ such that $\psi(z) = 1$ for $|z| = 3\sqrt{n}$ and $\psi(z) = 0$ for $|z| > 6\sqrt{n}$. For $q > 0$ let:

$$\psi(z) = \begin{cases} (6\sqrt{n})^q(2^q-1)\left(|z|^{-q} - (6\sqrt{n})^{-q}\right), & 3\sqrt{n} \leq |z| \leq 6\sqrt{n} \\ \text{smooth} \in [1,2], & |z| \leq 3\sqrt{n} \\ 0, & |z| > 6\sqrt{n}. \end{cases}$$

For $|z| \in (3\sqrt{n}, 6\sqrt{n})$ compute:

$$-\frac{1}{2}zD\psi(z) = (6\sqrt{n})^q(2^q-1)\frac{q}{2}|z|^{-q},$$

$$P^+(D^2\psi)(z) = (6\sqrt{n})^q(2^q-1)^{-1}q\frac{\left(\Lambda(n-1) - \lambda(q+1)\right)|z|^{-q}}{|z|^2}.$$

For large q we have

$$-\frac{1}{2}zD\psi(z) + P^+(D^2\psi) \leq 0$$

in the set $(3\sqrt{n}, 6\sqrt{n})$. For $|z| < 3\sqrt{n}$ note that $\psi(z) \in [1,2]$ and hence

$$-p\psi(z) - \frac{1}{2}D\psi(z)z + P^+(D^2\psi)(z) < 0.$$

Hence, in view of (6.3.1),

$$\partial_t\varphi(x,t) + P^+(D^2\varphi)(x,t) \leq 0 \quad \text{for } t \in (0,T].$$

Recall $\psi = 0$ for $|z| > 6\sqrt{n}$ and hence if $x \in \partial B_1$ and $t \in (0,T)$ for $T = \frac{1}{36n}$, then

$$\frac{x}{\sqrt{t}} \geq \frac{1}{6\sqrt{n}}$$

and hence

$$\varphi(x,t) = 0 \quad \forall x \in \partial B_1^n, t \in (0,T).$$

Also, we have

$$\lim_{t\to 0}\varphi(t,x) = 0$$

uniformly in $B_1(0)\backslash B_\epsilon(0)$ for any $\epsilon > 0$, since then $\frac{x}{\sqrt{t}} \to \infty$.

Then $\varphi(t,x)$ is properly defined for $t \in (0,T]$,

Now we need to give a definition for $\varphi(t,x)$ for $t \geq T$, which we do by a continuation argument. Note that by construction of ψ,

$$\varphi(T,x) \geq T^{-p} > 0 \quad \text{whenever } |x| \leq \frac{1}{2} \tag{6.3.2}$$

Moreover

$$\varphi(T,x) \geq 0, \quad \mathcal{P}^+(D^2\varphi) \leq 0 \quad \text{for } |x| \in (\tfrac{1}{2}, 1). \tag{6.3.3}$$

Set

$$C = \max\left\{0, \sup_{x\in B_{\frac{1}{2}}(0)} \frac{P^+(D^2\varphi(T,x))}{\varphi(T,x)} < \infty\right\}$$

For $t > T$ we simply define

$$\varphi(t,x) := e^{-C(t-T)}\varphi(T,x).$$

Then

$$\begin{aligned}\partial_t\varphi(t,x) + P^+(D^2\varphi) &= -Ce^{-C(t-T)}\varphi(T,x) + P^+(D^2\varphi(T,x))e^{-C(t-T)}\\ &= e^{-C(t-T)}\left(-C\varphi(T,x) + P^+(D^2\varphi(T,x))\right)\\ &\leq 0\end{aligned}$$

for $|x| \in (1/2, 1)$ by (6.3.3) and for $|x| < 1/2$ by (6.3.2). Thus φ is a subsolution and since $\varphi > 0$ on $K_3 \cap \{t = T\}$, we have still that $\inf_{K_3}\varphi > 0$.

□

PROPOSITION 6.3.3 (Basic measure estimate). *There exists* $\epsilon_0 \in (0,1)$, $M > 1$, $\mu = \mu(R, \lambda\Lambda, n) \in (0,1)$, *so that for all supersolutions* $u \geq 0$ *of*

$$\partial_t u + P^+(D^2 u) = f \quad \text{in } Q_1(0,1),$$

then, if $\inf_{K_3} u \leq 1$ *and* $\|f\|_{L^{n+1}(Q_1(0,1))} \leq \epsilon_0$, *then*

$$|\{u \leq M\} \cap K_1| \geq \mu |K_1|.$$

PROOF. Let ϕ be from Lemma 6.3.2 and set

$$w = u - \phi.$$

Then

$$\begin{aligned}\partial_t w + P^+(D^2 w) &\geq \partial_t u + P^+(D^2 u) - \partial_t \phi - P^+(D^2 \phi)\\ &\geq f - g,\end{aligned}$$

where g is also from Lemma 6.3.2. Also $w = u \geq 0$ on $\partial_p Q_1(1,0)$ and

$$\inf_{K_3} w \leq \inf_{K_3} u - 2 \leq -1.$$

Hence

$$\sup_{K_3} w^- \geq 1.$$

Let $\Gamma(w)$ be the monotone envelope in Q_1 of

$$\begin{cases}\min(w,0), & Q_1\\ 0, & Q_2 \backslash Q_1.\end{cases}$$

Then $\Gamma(w) = w$, if $w \leq 0$ and hence

$$\{\Gamma(w) = w\} \cap K_1 \subset \{u \leq \phi\} \cap K_1.$$

With the ABP principle, Theorem 6.2.2,

$$\begin{aligned}1 \leq \sup_{K_3} w^- \leq \sup_{Q_1} w^- &\leq C_{ABP} \|f\|_{L^{n+1}(Q_1(1,0))}\\ &\quad + C\left(\int_{\{\Gamma(w)=w\}\cap K_1} |g|^{n+1}\right)^{\frac{1}{n+1}}.\end{aligned}$$

Put

$$M = \max\{\max_{K_1} \phi, 1\}.$$

Then

$$1 \leq C\epsilon_0 + C\|g\|_{L^\infty(Q_1)} |\{u \leq M\} \cap K_1|^{\frac{1}{n+1}}$$

and thus, if $\varepsilon_0 > 0$ is chosen small enough,

$$|\{u \leq M\} \cap K_1| \geq \frac{c}{|K_1|} |K_1| \equiv \mu |K_1|.$$

□

REMARK 6.3.4.
- An equivalent formulation of Lemma 6.3.3 is:
If $\|f\|_{L^{n+1}(Q_1(0,1))} \leq \epsilon_0$, then for nonnegative supersolutions the following holds:
$$|\{u > M\} \cap K_1| \geq (1-\mu)|K_1| \Rightarrow u \geq 1 \text{ on } K_3.$$
One should compare this to the propagation of positivity from Lemma 2.2.5. There we had that $u > M$ for some time t_1 implies $u > cM$ for some time t_2. In Lemma 6.3.3 we obtained a finer assumption: $u > M$ just has to hold on a substantial part of K_1 and then $u > 1$ on all of K_3.
- This estimate also holds on $B^n(0,1)\times(0,T)$ instead of $B^n(0,1)\times(0,1)$. Let $u \geq 0$, $\partial_t u + P^+(D^2u) \geq f$ in $(0,T)\times B_1$. If
$$\inf_{(R^2,T)\times(-3R,3R)^n}$$
and then
$$|\{u \leq M\} \cap K_1| \geq \mu |K_1|.$$

COROLLARY 6.3.5. *(Scaled basic measure estimate) Same* ϵ, M, μ *as in* (6.3.3)*,* $x_0 \in \mathbb{R}^n$*,* $t_0 \in \mathbb{R}$*,* $h > 0$*. If* $u \geq 0$ *and* $\partial_t u + P^+(D^2u) \geq f$ *in* $(t_0,x_0) + \rho Q_1(1,0)$ *and*
$$\|f\|_{L^{n+1}((t_0,x_0)+\rho Q_1(1,0))} \leq \epsilon_0 \frac{h}{M\rho^{\frac{n}{n+1}}},$$
then, if
$$|\{\{u > h\} \cap \{(t_0,x_0) + \rho K_1\}\}| < (1-\mu)|(t_0,x_0) + \rho K_1|,$$
then
$$u > \frac{h}{M} \quad \text{in } (t_0,x_0) + \rho K_3.$$

PROOF.
$$v(t,x) = Mh^{-1}u(t_0 + \rho^2 t, x_0 + \rho x),$$
then
$$\partial_t v + P^+(D^2 v) \geq f \quad \text{in } Q_1(1,0).$$
$$\tilde{f} = \frac{M}{h}\rho^2 f(t_0 + \rho^2 t, x_0 + \rho x).$$
Apply 6.3.3. □

Now we stack those cubes K_2, see Figure 3: Define
$$K_2^{(k)} = (\alpha_k R^2, a_{k+1}R^2) \times (-3^k R, 3^k R)^n,$$
where
$$\alpha_k = \sum_{i=0}^{k-1} g^i = \frac{g^k - 1}{8}.$$

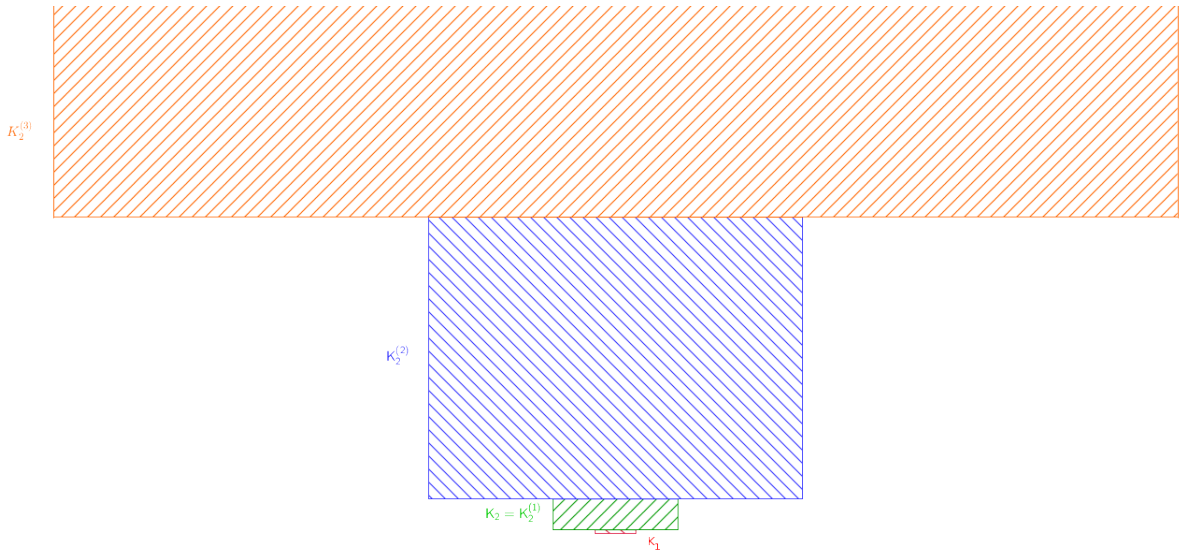


FIGURE 3. Stacked K_2

Now scale K_1 and $K_2^{(k)}$.

$$\rho K_1 = (0, \rho^2 R^2) \times (-\rho R, \rho R)^n,$$

$$\rho K_2 = (\rho^2 R^2, 10\rho^2 R^2) \times (-3\rho R, 3\rho R)^n,$$

$$\rho K_2^{(k)} = (\alpha_k \rho^2 R^2, \alpha_{k+1}\rho^2 R^2) \times (-3^k \rho R, 3^k, \rho R).$$

For $\rho > 0$, $(t_0, x_0) \in \mathbb{R}^{n+1}$ let

$$L_1 = (t_0, x_0 + \rho K_1)$$

and

$$L_2^{(k)} = (t_0, x_0) + \rho K_2^{(k)}.$$

As one can see already from Figure 3, the stacked cubes grow very quickly. It will be important to understand how the stacked cubes $L_2^{(k)}$ eventually leave the set $(0,1) \times (-3,3)^n$. The following Lemma essentially states: If the initial scaled cube L_1 belongs to K_1 then the stacked cubes $\bigcup_{k\geq 1} L_2^{(k)}$ do not leave the the cube $(0,1)\times(-3,3)^n$ sideways, but only through the top $1 \times (-3,3)^n$, see Figure 4. Moreover, any such stacked cube $\bigcup_{k\geq 1} L_2^{(k)}$ will eventually completely cover $\tilde{K}_2$ from Figure 1.

LEMMA 6.3.6 (Stack of cubes). *(1) Let* $R \leq \min(3 - 2\sqrt{2}, \sqrt{2/5}) = 3 - 2\sqrt{2}$, *then for all* $(x_0, t_0), \rho > 0$ *such that* $L_1 \subset K_1$,

$$\bigcup_{k\geq 1} L_2^{(k)} \cap ((0,1) \times (-3,3)^n) = \bigcup_{k \geq 1} L_2^{(k)} \cap \{0 < t < 1\}.$$

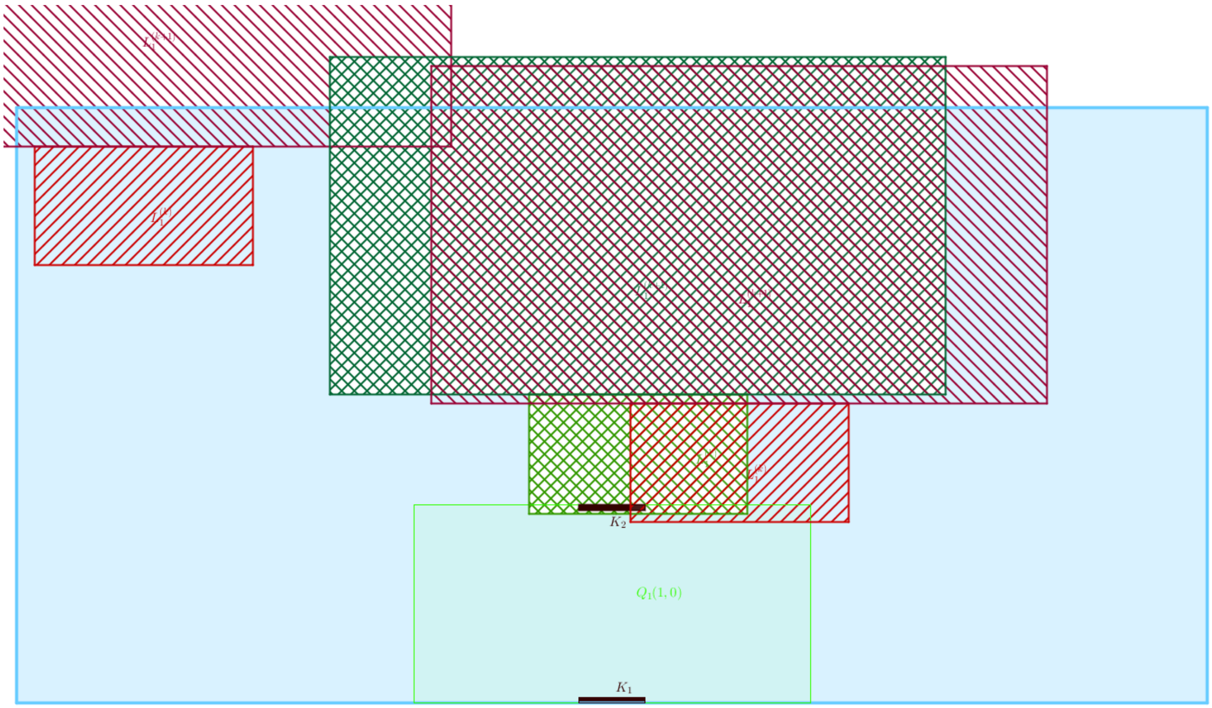


FIGURE 4. How the stacks $\bigcup_k L_2^{(k)}$ leaves the big box $(0,1)\times(-3,3)^n$: What cannot happen (red): leave the big box sideways or not cover $\tilde{K}_2$. What has to happen (green), the stack leaves through the top and covers $\tilde{K}_2$

(2) In particular if $R < \frac{1}{3\sqrt{n}}$, *then*

$$\{t \in (0,1)\} \cap \bigcup_{k\geq 1} L_2^k \subset (0,1) \times B^n_{\frac{1}{R}}(0).$$

(3)

$$\tilde{K}_2 \subset \bigcup_{k\geq 1} L_2^{(k)}.$$

(4) Moreover if k^* *is minimal so that*

$$L_2^{(k^*+1)} \cap \{t = 1\} \neq \emptyset,$$

then

$$\rho^2 R^2 \leq \frac{1}{\alpha_{k^*}}.$$

PROOF. We define paraboloids inside and outside of the stacked cubes $\bigcup_{k\geq 1} L_2^k$. More precisely we find S_+ and S_- so that

$$(t_0, x_0) + S_- \subset \bigcup_{k\geq 1} L_2^{(k)} \subset S_+ + (t_0, x_0).$$

Indeed, define for some s_+, s_- in $\mathbb{R}$,

$$S_\pm = \bigcup_{s>s_\pm} p_\pm(s) \times (-s, s)^n,$$

where

$$p_\pm(z) = a_\pm z^2 + b_\pm \rho^2 R^2,$$

so that

$$p_+(3^k \rho R) = \alpha_k \rho^2 R^2,$$

$$p_-(3^k \rho R) = \alpha_{k+1} \rho^2 R^2$$

and

$$p_\pm(s_\pm) = \rho^2 R^2.$$

Hence

$$a_+ = \frac{1}{8}, \quad b_\pm = -\frac{1}{8}, s_+ = s_- = \sqrt{\frac{9}{8}}\rho R, \quad a_- = \frac{9}{8}.$$

These paraboloids are useful, since we can use the following characterization:

$$(x, s) \in (x_0, t_0) + S_\pm \Leftrightarrow p_\pm(r_x) \leq s - t_0.$$

where $r_x > 0$ is the minimal positive number so that $x - x_0 \in (-r, r)^n$.

ad (i) We need to show

$$x \in \mathbb{R}^n \backslash (-3, 3)^n \ \wedge \ (x, s) \in S_+ + (t_0, x_0) \Rightarrow s \geq 1. \tag{6.3.4}$$

which should hold for any $(t_0, r_0), \rho$ such that $L_1 \subset K_1$. Now $L_1 \subset K_1$ simply means that $\rho \in (0, 1)$ arbitrary, $0 \leq t_0 \leq (1 - \rho^2)R^2$, and $x_0 + (-\rho R, \rho R)^n \subset (-R, R)^n$. Moreover $x = (x^1, \ldots, x^n) \in \mathbb{R}^n \backslash (-3, 3)^n$ implies that there exists at least one $i \in \{1, \ldots, n\}$ so that

$$|(x - x_0)^i| \geq 3 - (1 - \rho)R$$

Thus we need to show that for any $\rho \in (0, 1)$, $t_0 \in (0, (1 - \rho^2)R^2)$ and for any $r > 3 - (1 - \rho)R)$ it holds that

$$p_+(r) + t_0 \geq 1$$

Clearly, $t_0 = 0$, $r = 3-(1-\rho)R$ is the worst case, so we need to show that for any $\rho \in (0,1)$,

$$\frac{1}{8}(3-(1-\rho)R)^2 - \frac{1}{8}\rho^2R^2 \geq 1$$
$$\Leftrightarrow \frac{1}{8}(3-R)^2 + \frac{3}{4}\rho R(3-R) \geq 1$$

Now we see that the worst case is $\rho = 0$, and (6.3.4) holds if and only if

$$\frac{1}{8}(3-R)^2 \geq 1,$$

which is equivalent to $R \leq 3-2\sqrt{2}$. This proves (i)

ad (ii) easy consequence of (i)

ad (iii) Show: starting with $L_1 = (t_0,x_0)+\rho K_1 \subset K_1$, then $(s,x) \in \bigcup_{k=1}^{\infty} L_2^{(k)}$, for every $(s,x) \in \tilde{K}_2$. The worst case is

$$x = -R, \quad s = 1-R^2, \quad x_0 = R(1-\rho), \quad t_0 = (1-\rho^2)R^2.$$

So we have to show that for all $0<\rho<1$:

$$p_-((2-\rho)R) \leq 1-R^2-(1-\rho^2)R^2.$$

Compute the derivative w.r.t ρ to deduce that $\rho = 0$ is the worst case. Hence provide

$$p_-(2R) \leq 1-2R^2 \Leftrightarrow R \leq 3-\sqrt{8}.$$

ad (iv) If $L_2^{(k^*+1)} \cap \{t=1\} \neq \emptyset$, then

$$t_0 + \alpha_{k^*}R^2s^2 \leq 1 \leq t_0 + \alpha_{k^*+1}R^2\rho^2$$

and thus

$$R^2\rho^2 \leq \frac{1-t_0}{\alpha_{k^*}} \leq \frac{1}{\alpha_{k^*}}.$$

□

Now we want to iterate the basic measure estimate.

PROPOSITION 6.3.7. *(Stacked measure estimate) Let ϵ_0, M, μ as in 6.3.3. Assume $u \geq 0$ and*

$$\partial_t u + P^+(D^2u) \geq f \quad \text{in } (0,1)\times B_{\frac{1}{R}}(0).$$

Assume that $(t_0,x_0) \in \mathbb{R}^{n+1}$ and $\rho \in (0,1)$ satisfy

$$(t_0,x_0)+\rho K_1 \subset K_1.$$

Assume that for some $k \in \mathbb{N}$ and $h > 0$ we have

$$\|f\|_{L^{n+1}((0,1)\times B_{\frac{1}{R}}(0))} \leq \epsilon_0 \frac{h}{M^k \rho^{\frac{n}{n+1}}}.$$

Then, if $|\{u > h\} \cap L_1| > (1-\mu)|L_1|$, then

$$\inf_{L_2^{(k)} \cap \{0<t<1\}} u > \frac{h}{M^k}.$$

Proof. Induction on k. $k = 1$ is the rescaled basic measure estimate, because

$$(t_0, x_0) + \rho Q_1(1,0)) \subset (0,1) \times B_{\frac{1}{R}}(0).$$

Assume we know

$$\inf_{L_2^{(k-1)} \cap \{0<t<1\}} u > \frac{h}{M^{k-1}}.$$

If $L_2^{(k-1)}$ is not contained in $(0,1) \times B^n_{\frac{1}{R}}(0)$, then

$$L_2^{(k)} \cap \{0 < t < 1\} = \emptyset.$$

Otherwise by induction hypothesis

$$|\{u > \frac{h}{M^{k-1}}\} \cap L_2^{(k-1)}| = |L_2^{(k-1)}| \geq (1-\mu)|L_2^{(k-1)}|.$$

We have $L_2^{(k-1)} = (t_0, x_0) + \rho K_2(k-1) = (t_1, x_0) + \rho_1 K_1$, where $t_1 = t_0 + \alpha_{k-1} R^2 \rho^2$ and $\rho_1 = 3^{k-1}\rho$. Furthermore

$$L_2^{(k)} = (t_1, x_0) + \rho_1 K_2.$$

Then by hypothesis

$$|\{u > \frac{h}{M^{k-1}}\} \cap (t_1, x_0) + \rho_1 K_1| > (1-\mu)|(t_1, x_0) + \rho_1 K_1|$$

and

$$\inf_{L_2^{(k)} \cap \{0<t<1\}} > \frac{h}{M^k}.$$

□

Corollary 6.3.8. *(Straightly stacked estimate) Under the assumption of 6.3.7 let $k \in \mathbb{N}$ and*

$$R \leq \frac{1}{\sqrt{10(k+1)}}.$$

Assume $L_1 \subset K_1$ and $\bar{L}_1(m)$ be a straight stack. Then, if $|\{u > k\} \cap L_1| > (1-\mu)|L-1|$, then

$$u > \frac{h}{M^k}$$

in $\bigcup_{l=2}^{k} \bar{L}_1^{(l)}$.

PROOF. $\bar{L}_1^{(k)} \subset L_2^{(k)}$. □

Coverings. A cube is always a set

$$Q = (t_0, x_0) + (0, s^2) \times (-s, s)^n.$$

Every cube Q can be decomposed in 2^{n+2} subcubes K of sidelength $s^2/4$ in time and $s/2$ in space and so that the interiors are disjoint, see Figure 5. We say Q is precedessor/father of K and K is the successor/child of Q.K is a dyadic cube of Q, if it can be constructed in finitely many steps from Q.

Let K be a dyadic cube of Q. Then call $\bar{K}$ its precedessor and $\bar{K}^m$ the stack of m copies over $\bar{K}$, see Figure 6.

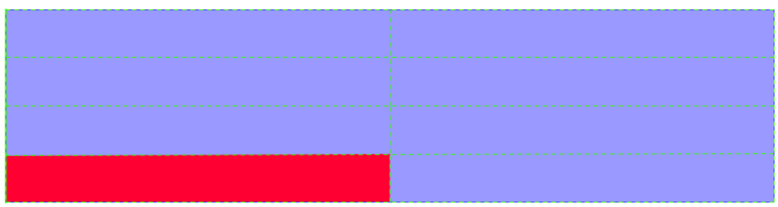


FIGURE 5. Dyadic decomposition of a (parabolic) cube $Q = (0, s^2) \times (-s, s)^2$

LEMMA 6.3.9. *(Stacked covering lemma) Let $m \in \mathbb{N}$, $A, B \subset Q$ be measurable. Assume that $|A| \leq \delta |Q|$ for some $\delta \in (0,1)$, that for all dyadic $K \subset Q$*

$$|K \cap A| > \delta |A| \Rightarrow \bar{K}^m \subset B.$$

Then

$$|A| \leq \delta \frac{m+1}{m} |B|.$$

PROOF. Pick a family of dyadic cubes $(K_i)_{i=1}^{\infty}$, possibly finite. Pick them with the algorithm: Subdivide Q in 2^{n+2} successors $\tilde{K}$. Add a cube to the family if

$$|\tilde{K}_i \cap A| \geq \delta |\tilde{K}_i|,$$

otherwise subdivide $\tilde{K}_i$ and repeat. Then, since $|A| \leq \delta |Q|$, for all $i \in \mathbb{N}$

$$|K_i \cap A| \geq \delta |K_i|, \quad |\bar{K}_i \cap A| < \delta |\bar{K}_i|.$$

We claim, for some subset N with $|N| = 0$.

$$A \subset \bigcup_{i=1}^{\infty} K_i \cup N, \tag{6.3.5}$$

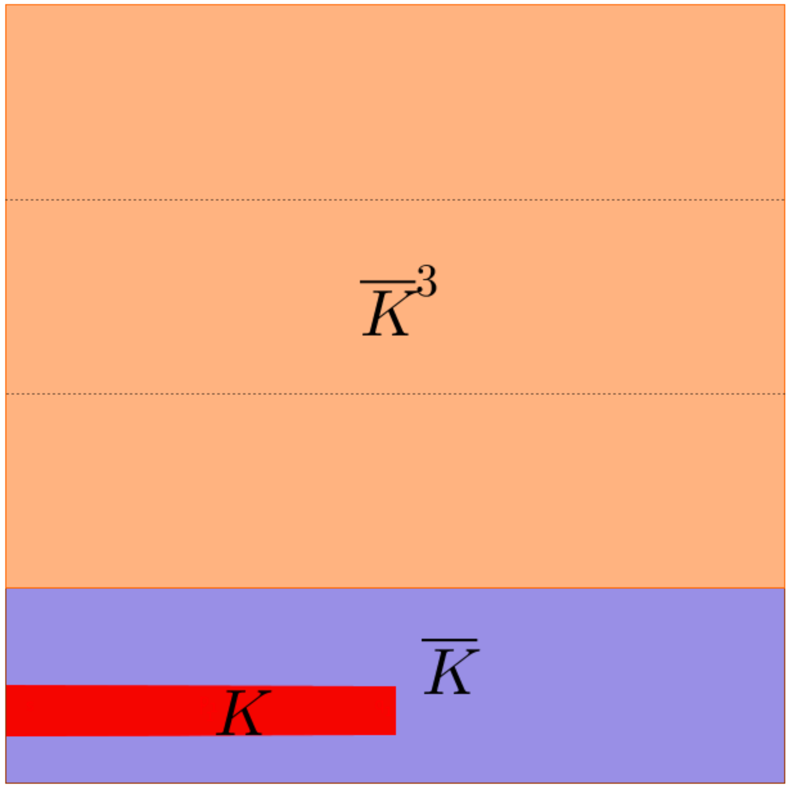


FIGURE 6. Stack of dyadic cubes

If this was false, there existed $N \subset A \setminus \bigcup_{i=1}^\infty K_i$ with positive measure. We observe: For a.e. $(t,x) \in \mathbb{R}^{n+1}$ we have

$$\fint_{(t,x)+(-r^2,r^2)\times(-2r,2r)^n} (1-\chi_A) \to 1 - \chi_A(t,x).$$

Hence, since $|N| > 0$, there is $(t,x) \in N$ with

$$\fint_{(t,x)+(-r^2,r^2)\times(-2r,2r)^n} (1-\chi_A) \to 0.$$

On the other hand $(t,x) \notin \bigcup_{i=1}^\infty K_i$ and hence there exists a sequence of dyadic bad cubes

$$L_i = (t_i, x_i) \times (-r_i^2, r_i^2) \times (-r_i, r_i)^n$$

with $r_0 \to 0$,

$$(t,x) \in \bigcap_{i=1}^\infty L_i$$

and

$$|L_i \cap A| \leq \delta |L_i|.$$

Hence

$$\fint_{L_i} (1-\chi_A) \geq 1-\delta. \tag{6.3.6}$$

Observe $(t,x) \in L_i$ and hence

$$L_i \subset (t,x) + (-r_i^2, r_i^2) \times (-2r_i, 2r_i)^n =: \tilde{L}_i$$

and we have $|\tilde{L}_i| \sim |L_i|$. Hence

$$\fint_{L_i} |1-\chi_A| \leq \frac{|\tilde{L}_i|}{|L_i|} \fint_{\tilde{L}_i} (1-\chi_A) \to 0. \tag{6.3.7}$$

(6.3.6) and (6.3.7) are a contradiction, and the claim (6.3.5) is established.

Now let $\bigcup_{j=1}^{\infty} \bar{K}_j$ be the collection of father cubes of K_i (doubly appearing cubes removed). Then the claim implies

$$|A| \leq \sum_{j=1}^{\infty} |A \cap \bar{K}_j| \leq \delta \sum_{j=1}^{\infty} |\bar{K}_j|.$$

To show

$$\left|\bigcup_{j=1}^{\infty} \bar{K}_j\right| \leq \frac{m+1}{m} \left|\bigcup_{j=1}^{\infty} K_j^m\right|.$$

We write

$$\bigcup_{j=1}^{\infty} \bar{K}_j = \bigcup_{l=1}^{\infty} C_l \times \bigcup_{k=1}^{\infty} (a_k^l, a_k^l + h_k^l),$$

where $C_l \subset \mathbb{R}^n$ are p.d. cubes, then

$$\bigcup_{j=1}^{\infty} \bar{K}_j^m = \bigcup_{l=1}^{\infty} C_l \times \bigcup_{k=1}^{\infty} (a_k^l + h_k^l, a_k^l + (m+1)h_k^l).$$

Thus

$$\begin{aligned}
\left|\bigcup_{j=1}^{\infty} \bar{K}_j^m\right| &= \sum_{l=1}^{\infty} |C_l| \cdot \left|\bigcup_{k=1}^{\infty} (a_k^l, a_k^l + h_k^l)\right| \\
&\leq \sum_{l=1}^{\infty} |C_l| \left|\bigcup_{k=1}^{\infty} (a_k^l + h_k^l, a_k^l + (m+1)h_k^l)\right|,
\end{aligned}$$

where the latter estimate is shown in the next lemma.

□

LEMMA 6.3.10. *Let* $(a_k)_{k=1}^\infty$, $(h_k)_{k=1}^\infty$, $m \in \mathbb{N}$. *Then*

$$\left|\bigcup_{k=1}^\infty (a_k, a_k + h_k)\right| \le \frac{m+1}{m} \left|\bigcup_{k=1}^\infty (a_k + h_k, a_k + (m+1)h_k)\right|.$$

PROOF. We write

$$\bigcup_{k=1}^\infty (a_k + h_k, a_k + (m+1)h_k) = \bigcup_{l=1}^\infty I_l,$$

where I_l are disjoint intervals. I_l has the form

$$\begin{aligned} I_l &= \bigcup_{i=1}^{N_l} (b_i + \mu_i, b_i + (m+1)\mu_i) \\ &= \left(\inf_{i=1,\dots,N_l} (b_i + \mu_i), \sup_{i=1,\dots,N_l} (b_i + (m+1)\mu_i)\right) \\ &=: (b_{\inf} + \mu_{\inf}, b_{\sup} + (m+1)\mu_{\sup}), \end{aligned}$$

where we assumed wlog that $N_l < \infty$. Assume there is $(a, a+h)$ and l so that

$$(a+h, a+(m+1)h) \subset I_l.$$

Hence

$$a + (m+1)h \le b_{\sup} + (m+1)\mu_{\sup}, \quad -a - h \le -b_{\inf} - \mu_{\inf}$$

and by summing we get

$$h \le \frac{1}{m}|I_l|.$$

$$b_{\inf} + \mu_{\inf} \le a + h \le a + \frac{1}{m}|I_l|$$

and hence

$$a \ge b_{\inf} + \mu_{\inf} - \frac{1}{m}|I_l|.$$

Thus

$$(a, a+h) \subset (b_{\inf} + \mu_{\inf} - \frac{1}{m}|I_l|, b_{\sup} + (m+1)\mu_{\sup}).$$

We obtain

$$\bigcup_{a,h\colon (a+h, a+(m+1)h) \subset I_l} (a, a+h) \subset \left(b_{\inf} + \mu_{\inf} - \frac{1}{m}|I_l|, b_{\sup} + (m+1)\mu_{\sup}\right)$$

and

$$\left|\bigcup_{a,h\colon (a+h, a+(m+1)h) \subset I_l} (a, a+h)\right| \le \left(1 + \frac{1}{m}\right)|I_l|.$$

Since the I_l are disjoint we obtain the estimate. □

Proof of Theorem 6.3.1. The idea is to use the stacked covering lemma and the stacked measure estimate for $\{u > M^k\} \cap \tilde{K}_1$.

First observation: It suffices to show, that if

$$\inf_{\tilde{K}_2} u \leq 1, \quad \|f\|_{L^{n+1}((0,1)\times B_{\frac{1}{R}}(0))} \leq \epsilon_0, \tag{6.3.8}$$

then

$$\left(\int_{\tilde{K}_1} u^\epsilon\right)^{\frac{1}{\epsilon}} \leq C. \tag{6.3.9}$$

Proof that (6.3.9) implies Theorem 6.3.1. Take

$$v_\delta = \frac{u}{\inf_{\tilde{K}_2} u + \epsilon_0^{-1}\|f\|_{L^{n+1}((0,1)\times B_{\frac{1}{R}}(0))} + \delta}.$$

which satisfies (6.3.8). (6.3.9) then gives the claim, letting $\delta \to 0$. □

From now on, assume (6.3.8) to hold. (6.3.9) follows once we show

$$\begin{aligned}&\exists k_0 \in \mathbb{N}, m \in \mathbb{N}, B > 0, C_1 > 0\ \forall k \geq k_0:\\ &|A_k| := \left|\left\{u > M^{km}\right\} \cap \left(\left(0, \frac{R^2}{2} + C_1 B^{-k}\right) \times (-R,R)^n\right)\right| \leq C\left(1 - \frac{\mu}{2}\right)^k,\end{aligned} \tag{6.3.10}$$

where M and μ are from 6.3.7.

Proof that (6.3.9) follows from (6.3.10). From (6.3.8) the claim follows via: For $\tau > M^{k_0 m}$ let $k \geq k_0$ such that $\tau \in (M^{km}, M^{(k+1)m})$, hence

$$|\{u > \tau\} \cap \tilde{K}_1| \leq |A_k| \leq C\left(1 - \frac{\mu}{2}\right)^k \leq C\tau^{-2\epsilon},$$

for

$$\epsilon = -\frac{1}{2}\frac{\log\left(1 - \frac{\mu}{2}\right)}{m \log M}.$$

Since $|\tilde{K}_1| < \infty$ we have

$$|\{u < \tau\} \cap \tilde{K}_1| \leq C\tau^{-2\epsilon} \quad \forall \tau > 0.$$

Then

$$\begin{aligned}\int_{\tilde{K}_1} (u(t,x))^\epsilon &= \epsilon \int_0^\infty \tau^{\epsilon-1} |\{u > \tau\} \cap \tilde{K}_1| \, d\tau \\ &\le \epsilon \int_0^1 |\tilde{K}_1| \, d\tau + \epsilon \int_1^\infty \tau^{-2\epsilon} \tau^{\epsilon-1} \, d\tau \\ &\le C.\end{aligned}$$

□

So we need to show (6.3.10), which we do by induction.

For $k = k_0$, simply take

$$C \ge \left(1 - \frac{\mu}{2}\right)^{-k_0} |\tilde{K}_1|.$$

Now we proceed with the induction step:

Suppose there holds

$$|A_k| \le C \left(1 - \frac{\mu}{2}\right)^k$$

then we need to show that

$$|A_{k+1}| \le C \left(1 - \frac{\mu}{2}\right)^{k+1}$$

.

Firstly, take $k_0 >> 1$ such that

$$2C_1 B^{-k} \le \frac{R^2}{2} \quad \forall k \ge k_0,$$

thus $A_k, A_{k+1} \subset K_1$.

We want to apply Lemma 6.3.9. The first assumption we need to satisfy is the following:

Lemma 6.3.11.

$$|A_{k+1}| \le (1 - \mu)|K_1|.$$

Proof.

$$\inf_{\tilde{K}_2} u \le 1$$

and hence

$$\inf_{K_3} u \le 1.$$

Proposition 6.3.3 implies

$$|\{u \le M\} \cap K_1| \ge \mu |K_1|.$$

Thus

$$|A_{k+1}| \leq |\{u > M\} \cap K_1| = |K_1| - |\{u \leq M\} \cap K_1| \leq (1-\mu)|K_1|.$$

□

The second assumption for Lemma 6.3.9 is the following:

LEMMA 6.3.12. *Let K be a dyadic cube of K_1. If $|K \cap A_{k+1}| > (1-\mu)|K|$, then $\bar{K}^m \subset A_k$.*

PROOF. From 6.3.8 we have

$$\bar{K}^m \subset \{u > M^{km}\}.$$

Show

$$\bar{K}^m \subset \left(0, \frac{R^2}{2} + C_1 B^{-k}\right) \times (-R, R)^n.$$

There holds

$$\left(K \cap \left(0, \frac{R^2}{2} + C_1 B^{-k-1}\right) \times (-R, R)^n\right) \neq \emptyset$$

and hence

$$\bar{K}^m \subset \left(0, \frac{R^2}{2} + C_1 B^{-k-1} + \text{height}(\bar{K}) + \text{height}(\bar{K}^m)\right) \times (-R, R)^n.$$

Thus the desired estimate holds iff

$$R^2 \rho^2 \leq \frac{C_1(B-1)}{4(m+1)} B^{-k-1}.$$

Let $L_1 = K$. By the stacking of cubes we have

$$\tilde{K}_2 \subset \bigcup_{i=1}^{\infty} L_2^{(l)}.$$

But we know

$$\inf_{\tilde{K}_2} u \leq 1.$$

Letting k^* be the first index with $L_2^{k^*} \cap \{t > 1\} \neq \emptyset$, we get

$$\inf_{\bigcup_{l=1}^{k^*} L_2^{(l)}} u \leq 1. \tag{6.3.11}$$

On the other hand for all $l \leq (k+1)m$ the assumptions of 6.3.7 are fulfilled ($h = M^l$). We obtain

$$\inf_{\bigcup_{l=1}^{(k+1)m} L_2^{(l)}} u > 1.$$

Thus, in view of (6.3.11)

$$(k+1)m \leq k^* + 1$$

and there holds

$$R^2\rho^2 \leq \frac{1-t_0}{\alpha_{k^*}} \leq \frac{9}{4^{(k+1)m}}.$$

Setting $B = 9^m$ and

$$C_1 = \frac{36(m+1)}{9^m - 1},$$

the desired estimate holds. □

Having Lemma 6.3.11 and Lemma 6.3.12 we can now apply Lemma 6.3.9, and find

$$|A_{k+1}| \leq (1-\mu)\frac{m}{m+1}|A_k|$$

For large m we have

$$\leq \left(1 - \frac{\mu}{2}\right)|A_k|$$

and with the induction hypotesis on A_k

$$\leq C_1\left(1 - \frac{\mu}{2}\right)^{k+1}.$$

This concludes the induction, and thus the proof of Theorem 6.3.1.

6.4. Harnack inequality

PROPOSITION 6.4.1 (Local maximum principle). *Let u be a subsolution of*

$$\partial_t u + F(D^2u, t, x) = 0 \quad \text{in } Q_1(0,0).$$

Then

$$\sup_{Q_{\frac{1}{2}}(0,0)} u \leq C\left(\left(\int_{Q_1} |u|^\epsilon\right)^{\frac{1}{\epsilon}} + \|f\|_{L^{n+1}(Q_1)}\right),$$

where $f = F(0,t,x)$ and ϵ is coming from the L^ϵ-estimate.

PROOF. We may assume $u \geq 0$, since u^+ is a subsolution. For $\gamma > 0$ put

$$\psi(t,x) = h\max\left((1-|x|)^{-2\gamma}, (1+t)^{-\gamma}\right)$$

for $h > 0$ which is minimal such that $u \leq \psi$ in Q_1. There holds

$$h = \min_{(t,x)\in Q_1} \frac{u(t,x)}{\max\left((1-|x|)^{-2\gamma}, (1+t)^{-\gamma}\right)}$$

and

$$\sup_{Q_{\frac{1}{2}}(0)} u \leq Ch.$$

Thus we have to calculate h. Let $(t_0, x_0) \in Q_1$ such that

$$h = \frac{u(t_0, x_0)}{\max\left((1-|x_0|)^{-2\gamma}, (1+t_0)^{-\gamma}\right)}.$$

Set

$$\delta = \min\left((1-|x_0|)^{-2}, (1+t_0)\right),$$

i.e.

$$h = \delta^\gamma u(t_0, x_0).$$

$$Q_\delta(t_0, x_0) = (t_0 - \delta^2, \delta^2) \times B^n_\delta(x_0) \subset Q_1.$$

Set

$$v(t, x) = C - u(t, x),$$

where

$$C = \sup_{Q_{\beta\delta}(t_0,x_0)} \psi \in (h\delta^{-2\gamma}, h\left((1-\beta)\delta\right)^{-2\gamma}),$$

β to be chosen. Then $v \geq 0$ in $Q_{\beta\delta}(t_0, x_0)$ and

$$\partial_t v + P^+(D^2 v) + |f| \geq 0.$$

The L^ϵ-estimate gives

$$\int_{(t_0-\beta\delta,t_0)+\beta\delta\tilde{K}_1} v^\epsilon \leq C(\beta\delta)^{n+2}\left(\inf_{(t_0-\beta\delta,x_0)+\beta\delta\tilde{K}_2} v + (\beta\delta)^{\frac{n}{n+1}}\|f\|_{L^{n+1}}\right).$$

We know

$$v(t_0, x_0) \leq h\left((1-\beta)\delta\right)^{-2\gamma} - h\delta^{-2\gamma}.$$

So

$$\int_{(t_0-\beta\delta,t_0)+\beta\delta\tilde{K}_1} v^\epsilon \leq C(\beta\delta)^{n+2}\left(h\left((1-\beta)^{-2\gamma} - 1\right)\delta^{-2\gamma} + (\beta\delta)^{\frac{n}{n+1}}\|f\|_{L^{n+1}}\right).$$

Let

$$L = (t_0 - \beta\delta, t_0) + \beta\delta\tilde{K}_1$$

and

$$A = \left\{(t, x) \in L \colon u(t, x) \leq \frac{1}{2}u(t_0, x_0) = \frac{1}{2}h\delta^{-2\gamma}\right\}.$$

Then

$$\int_A v^\epsilon \geq |A|\left(h\delta^{-2\gamma} - \frac{1}{2}h\delta^{-2g}\right)^\epsilon = |A|\left(\frac{h\delta}{2}\right)^{-2\gamma\epsilon}$$

and thus

$$|A| \leq C|L|\left(\left((1-\beta)^{-2g} - 1\right)^\epsilon + \left(\frac{\delta^{2\gamma}}{h}\right)^\epsilon (\beta\delta)^{\frac{\epsilon}{n+1}}\|f\|_{L^{n+1}}\right).$$

Furthermore

$$\int_{Q_1} u^\epsilon \geq \int_{L\setminus A} u^\epsilon \geq (|L| - |A|)\, 2^{-\epsilon}(h\delta^{-2\gamma})^\epsilon,$$

so

$$\begin{aligned}\beta^{2+n} C_1 h^\epsilon &= |L| 2^\epsilon \left(h\delta^{-2\gamma}\right)^\epsilon \\ &\leq \int_Q u^\epsilon + C\beta^{n+2+\frac{n\epsilon}{n+1}} \|f\|_{L^{n+1}} + C\beta^{n+2} h^\epsilon \left((1-\beta)^{-2\gamma} - 1\right)^\epsilon,\end{aligned}$$

hence for small β

$$h^\epsilon \leq C_\beta \left(\int_Q u^\epsilon + \|f\|_{L^{n+1}}\right).$$

□

THEOREM 6.4.2 (Harnack inequality). *Let $u \geq 0$ be solution of*

$$\partial_t u + F(x,t,D^2 u) = 0 \quad \textit{in } (-1,0) \times B^n_{\frac{1}{R}}(0),$$

then

$$\sup_{\tilde{K}_3} u \leq C \inf_{Q_R} u + C\|f\|_{L^{n+1}((-1,0)\times B^n_{\frac{1}{R}}(0))},$$

where

$$\tilde{K}_3 = \left(-1 + \frac{3}{8}R^2, -1 + \frac{R^2}{2}\right) \times B_{\frac{R}{2\sqrt{2}}}(0).$$

PROOF. By the L^ϵ-estimate:

$$\int_{\left(-1,-1+\frac{R^2}{2}\right)\times B_{\frac{R}{\sqrt{2}}}} u^\epsilon \leq C\left(\inf_{Q_R} u^\epsilon\right) + \|f\|_{L^{n+1}}.$$

Rescale:

$$v(t,x) = t\left(\frac{t+1-\frac{R^2}{2}}{\frac{R^2}{2}}, \frac{\sqrt{2}}{R}x\right).$$

Then

$$\sup_{Q_{\frac{1}{2}}} \leq C\left(\left(\int_{Q_1} v^\epsilon\right)^{\frac{1}{\epsilon}} + \|f\|_{L^{n+1}(Q_1)}\right).$$

□

Bibliography

[And11] D. Andreucci, *Lecture notes on free boundary problems for parabolic equations*, Doctoral School of Cisterna, 2011.

[CH98] Thierry Cazenave and Alain Haraux, *An introduction to semilinear evolution equations*, Oxford Lecture Series in Mathematics and its Applications, vol. 13, The Clarendon Press, Oxford University Press, New York, 1998, Translated from the 1990 French original by Yvan Martel and revised by the authors. MR 1691574

[CIL92] M. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 1–67. MR 1118699

[Eva98] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR 1625845

[IS13] C. Imbert and L. Silvestre, *An introduction to fully nonlinear parabolic equations*, An introduction to the Kähler-Ricci flow, Lecture Notes in Math., vol. 2086, Springer, Cham, 2013, pp. 7–88. MR 3185332

[Joh91] F. John, *Partial differential equations*, fourth ed., Applied Mathematical Sciences, vol. 1, Springer-Verlag, New York, 1991. MR 1185075

[Koi12] Shigeaki Koike, *A beginner's guide to the theory of viscosity solutions*, www.math.tohoku.ac.jp/∼koike/evis2012version.pdf, 2012.

[Kry96] N. V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, Graduate Studies in Mathematics, vol. 12, American Mathematical Society, Providence, RI, 1996. MR 1406091

[Lie96] G M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. MR 1465184

[Sim97] L. Simon, *Schauder estimates by scaling*, Calc. Var. Partial Differential Equations **5** (1997), no. 5, 391–407. MR 1459795

Index