Nichtkommutative Algebra und Symmetrie
SS 2019 — Ubungsblatt 11

16. Juli 2019

Informationen zur Vorlesung finden Sie unter:
`http://home.mathematik.uni-freiburg.de/soergel/ss19nkas.html`

Exercise 11.1: Let $\mathfrak{g}$ be a reductive Lie algebra. Show that $[\mathfrak{g},\mathfrak{g}]$ is semisimple and that $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus Z(\mathfrak{g})$, where

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0\ \forall y \in \mathfrak{g}\}$$

Exercise 11.2: Show that a Lie algebra $\mathfrak{g}$ is reductive if and only if the representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is completely reducible (i.e. it is a direct sum of irreducible representations).

Exercise 11.3:

- Let $\mathfrak{g}$ be a simple Lie algebra and let $\beta(-,-)$ be a non-trivial bilinear form such that
$$\beta([x, y], z]) = \beta(x, [y, z])$$
for all $x, y, z \in \mathfrak{g}$. Show that β is non-degenerate.
- Assume that $\mathfrak{g}$ is a complex simple Lie algebra and β as before. Show that β is a scalar multiple of the Killing form $\kappa(-,-)$.
- Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$. Show that $\kappa(x, y) = 2n\mathrm{Tr}(xy)$.

Exercise 11.4: Let $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ (cf. Exercise 8.2).

- Show that the set diagonal matrices $\mathfrak{h}$ in $\mathfrak{g}$ form a Cartan subalgebra of $\mathfrak{g}$.
- Compute the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Bonus Exercise 11.5: Let β be the bilinear form on $\mathbb{C}^n$ defined by the matrix

$$B = \begin{pmatrix} & & & 1 \\ & 0 & 1 & \\ & \iddots & & \\ & 1 & 0 & \\ 1 & & & \end{pmatrix}$$

(Recall that all non-degenerate bilinear forms on $\mathbb{C}^n$ are equivalent, we choose B because it is more convenient). Let

$$\mathfrak{o}(n,\mathbb{C}) = \{A \in \mathfrak{gl}(n,\mathbb{C}) \mid \beta(Ax, y) + \beta(x, Ay) = 0\ \forall x, y \in \mathbb{C}^n\} =$$
$$= \{A \in \mathfrak{gl}(n,\mathbb{C}) \mid A^{\perp}B + BA = 0\}$$

- Show that the set diagonal matrices $\mathfrak{h}$ in $\mathfrak{o}(n, \mathbb{C})$ form a Cartan subalgebra of $\mathfrak{o}(n, \mathbb{C})$.
- Compute the root system of $\mathfrak{o}(n, \mathbb{C})$ with respect to $\mathfrak{h}$.

Achtung: the answer varies considerably depending on the parity of n!