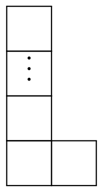
Nichtkommutative Algebra und Symmetrie
SS 2019 — Übungsblatt 5

3. Juni 2019

Informationen zur Vorlesung finden Sie unter:
`http://home.mathematik.uni-freiburg.de/soergel/ss19nkas.html`

Exercise 5.1: Consider the natural representation of S_n on $\mathbb{C}^n$. We have $\mathbb{C}^n = triv \oplus \theta$, where θ is irreducible (cf. Exercise 3.3 and 3.4) Show that θ is the representation corresponding to the following Young diagram



Exercise 5.2: Let Y be an Young diagram and let $L(Y)$ be the corresponding irreducible representation of S_n. Let V_{sgn} be the sign representation. Show that the tableau of $L(Y) \otimes V_{\text{sgn}} \cong L(Y^t)$, where Y^t denotes the transpose Young diagram.

Exercise 5.3: Let Y be a Young diagram with n boxes. The goal of this exercise is to show that the irreducible representations $L(Y)$ has dimension at least the number of standard tableaux of shape Y. (Recall: a standard tableau is a labelling of the boxes in Y with the set $\{1, 2, \ldots, n\}$ such that it is monotone on each column and row.)
Consider the group S_Y of permutations of the boxes of Y (which is clearly isomorphic to S_n). The subgroup S (resp. Z) is the stabilizer of the columns in Y (resp. of the rows in Y). Recall the idempotents

$$E_Y = \frac{1}{|S|} \sum_{g \in S} g \in \mathbb{C}S_Y \qquad A_Y = \frac{1}{|Z|} \sum_{g \in Z} \operatorname{sgn}(g) g \in \mathbb{C}S_Y.$$

We have

$$L(Y) \cong (\mathbb{C}S_Y) E_Y A_Y$$

Consider the set B_Y of tableaux of shape Y. That is, $B_Y \cong Ens^{\times}(Y, \{1, \ldots, n\})$. On $\mathbb{C}B_Y$ we have a right operation of S_Y, given by precomposition.

1. Show that $\mathbb{C}B_Y \cong \mathbb{C}S_Y$ as right $\mathbb{C}S_Y$-modules. In particular we have an isomorphism $L(Y) \cong (\mathbb{C}B_Y) E_Y A_Y$ of $\mathbb{C}$-vector spaces.

Let $D_Y \subset B_Y$ the subset of standard tableaux. Consider the linear map $\operatorname{res} : \mathbb{C}B_Y \to \mathbb{C}D_Y$ defined by

$$\operatorname{res}(\phi) = \begin{cases} \phi & \text{if } \phi \in D_Y \\ 0 & \text{if } \phi \notin D_Y. \end{cases}$$

We want to show that $\mathrm{res} : (\mathbb{C}B_Y)E_Y A_Y \to \mathbb{C}D_Y$ is surjective.

2. Find an example of a standard tableau $\phi \in D_Y$, an element $g \in S$ and an element $h \in S$ such that $\phi \cdot (gh)$ is also a standard tableau, different from ϕ.

Given a tableau ϕ we define $R(\phi)$ to be the series of the sum of the values in its rows, e.g.

$$R\left(\begin{array}{|c|c} \hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\right) = (4, 2).$$

We say that $\phi < \psi$ if $R(\phi) > R(\psi)$ in the lexicographic order.

3. Let $\phi \in D_Y$, $g \in S$ and $h \in Z$, with g and h not both trivial. Assume that $\phi \cdot (gh)$ is a standard tableau. Show that $\phi \cdot (gh) < \phi$.
(Hint: look at the first row which is altered by g).

4. Let $\phi \in D_Y$. Show that

$$\mathrm{res}(\phi E_Y A_Y) = \frac{1}{|S||Z|}\phi + \sum_{\psi < \phi} c_\psi \psi \tag{1}$$

for some $c_\psi \in \mathbb{C}$.

5. Use (1) to deduce that $\mathrm{res} : (\mathbb{C}B_Y)E_Y A_Y \to \mathbb{C}D_Y$ is surjective.

Remark: In fact, $\dim L(Y)$ is exactly the number of standard tableaux of shape Y. To show this, it's enough to show that $\sum_Y |D_Y|^2 = n! = S_n$ (see Section 3.2 of the Skript).