

Kazhdan-Lusztig-polynomials and indecomposable bimodules over polynomial rings

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Abstract

We develop a strategy to prove the positivity of coefficient of Kazhdan-Lusztig polynomials for arbitrary Coxeter groups. Thanks go to Martin Härterich and Catharina Stroppel for corrections to previous versions. In particular I thank Patrick Polo, whose Remarks led to a sustancial improvement of this article.

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1 Realization of Hecke algebras via bimodules

Notation 1.1. Given a svelte additive category \mathcal{A} denote its split Grothendieck-group, thus the free abelian group over the objects with relations $M = M' + M''$ whenever we have $M \cong M' \oplus M''$. Given a object M let $\langle M \rangle$ denote its class in $\langle \mathcal{A} \rangle$.

Notation 1.2. Given a \mathbb{Z} -graded ring A let $A\text{-mod}_{\mathbb{Z}}^f$ denote the category of all finitely generated \mathbb{Z} -graded A -modules. We write $M[n]$ for the object M with its \mathbb{Z} -grading shifted by n , in formulas $(M[n])_i = M_{i+n}$.

Remark 1.3. Given a field k and a finitely generated commutative nonnegatively \mathbb{Z} -graded k -algebra A with $A_0 = k$ the endomorphism ring of a finitely \mathbb{Z} -graded A -module is always of finite dimension. The endomorphism ring of an indecomposable such module has no idempotents except zero and one and thus is indecomposable as a right module over itself. But it is also the endomorphism ring of this indecomposable right module of finite length and is thus local. Hence we have in $A\text{-mod}_{\mathbb{Z}}^f$ the Krull-Schmid theorem with the same proof as in [Pie82] 5.4 and the isomorphism classes of indecomposable objects form a basis of $\langle A\text{-mod}_{\mathbb{Z}}^f \rangle$, so in particular we have $\langle N \rangle = \langle M \rangle \Leftrightarrow N \cong M$.

Notation 1.4. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system with a finite number of generators, $|\mathcal{S}| < \infty$. Let $l : \mathcal{W} \rightarrow \mathbb{N}$ be its length function and \leq the Bruhat order on \mathcal{W} . In particular $x < y$ means $x \leq y$, $x \neq y$. On the free $\mathbb{Z}[v, v^{-1}]$ -module

$$\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathcal{S}) = \bigoplus_{x \in \mathcal{W}} \mathbb{Z}[v, v^{-1}]T_x$$

over \mathcal{W} there is exactly one structure of associative $\mathbb{Z}[v, v^{-1}]$ -algebra with $T_x T_y = T_{xy}$ if $l(x) + l(y) = l(xy)$ and $T_s^2 = v^{-2}T_e + (v^{-2} - 1)T_s$ for all $s \in \mathcal{S}$, see [Bou81], IV, §2, Exercice 23. This associative algebra \mathcal{H} is called the Hecke algebra of $(\mathcal{W}, \mathcal{S})$. It is unitary with unit T_e , we abbreviate often $T_e = 1$. In fact it might be more natural to use $q = v^{-2}$, and the first sections of this article would gain clarity thereby. In the later sections however it is essential for a transparent exposition to have a root of q at our disposal. wichtig, eine Wurzel von q zur Verfügung zu haben. The Hecke algebra can also be described as the associative unitary $\mathbb{Z}[v, v^{-1}]$ algebra with generators $\{T_s\}_{s \in \mathcal{S}}$, the quadratic relations $T_s^2 = v^{-2}T_e + (v^{-2} - 1)T_s$ and the so-called braid relations $T_s T_t \dots T_s = T_t T_s \dots T_t$ resp. $T_s T_t T_s \dots T_t = T_t T_s T_t \dots T_s$ in case $st \dots s = ts \dots t$ resp. $sts \dots t = tst \dots s$ for $s, t \in \mathcal{S}$.

Definition 1.5. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system. Let $\mathcal{T} \subset \mathcal{W}$ denote the set of all “reflections”, i.e. all elements of \mathcal{W} , which are conjugate to elements of \mathcal{S} . By a reflection faithful representation of our Coxeter system we mean a representation $\mathcal{W} \hookrightarrow \mathrm{GL}(V)$ in a finite dimensional vector space over a field k of characteristic char $k \neq 2$ with the following properties:

1. Our representation is faithful.
2. For $x \in \mathcal{W}$ we have $\dim(V/V^x) = 1 \Leftrightarrow x \in \mathcal{T}$. Thus exactly the reflections from \mathcal{W} have a fixed point set of codimension one in V .

Remark 1.6. Given a reflection faithful representation the elements of \mathcal{T} are precisely those elements of \mathcal{W} , which act as reflections on V , thus which decompose in V in a onedimensional eigenspace of eigenvalue -1 and a hyperplane of invariand vectors. Indeed $x \in \mathcal{T}$ cannot act unipotently, since an automorphism of order two of a vector space over a field of characteristic $\neq 2$ is diagonalizable and our condition excludes that x acts as the identity. Furthermore reflections from \mathcal{W} can be distinguished as well by their eigenspaces of eigenvalue (-1) as well as by their reflecting hyperplanes, so that for $t, r \in \mathcal{T}$ we have

$$V^t = V^r \quad \Leftrightarrow \quad V^{-t} = V^{-r} \quad \Leftrightarrow \quad t = r.$$

For example in case $G = V^{-t} = V^{-r}$, we consider the short exact sequence $G \hookrightarrow V \twoheadrightarrow V/G$. Since both t and r act as the identity on V/G and since taking the adjoint map doesn't change the dimension of the eigenspaces we see that tr fixes a hyperplane. But since tr has determinant 1 it must by condition 2 act as the identity on V and thus by condition 2 is the identity of \mathcal{W} . We show in the next section, that the representation of a Coxeter group generalizing the representation of an affine Weyl group on the Cartan of a Kac-Moody algebra is always reflection faithful.

Definition 1.7. A representation of \mathcal{W} , in which reflections act as reflections and where different reflections have different (-1) -eigenspaces will be called reflection vector faithful.

Example 1.8. The geometric representation of an infinite dieder group is reflection vector faithful. However it is not faithful, since all elements except the neutral element have a fixed point set of codimension one, not only the reflections.

Notation 1.9. Let now for the sake of simplicity of notation k be an infinite field and let $R = R(V)$ denote the k -algebra of all regular functions on the space V underlying a reflection vector faithful representation. We equip R

with a \mathbb{Z} -grading $R = \bigoplus_{i \in \mathbb{Z}} R_i$ such, that we have $R_2 = V^*$ and $R_i = 0$ for uneven i . Again it would be more natural here to work with the usual grading, but in the long run we need these conventions. Let

$$\mathcal{R} = \mathcal{R}_V \subset R\text{-mod}_{\mathbb{Z}}\text{-}R$$

denote the category of all \mathbb{Z} -graded R -bimodules, which are finitely generated from the right as well as from the left, and where the action of k from the right and the left is the same. Tensoring \otimes_R makes

$$\langle \mathcal{R} \rangle$$

into a ring. For $s \in \mathcal{S}$ let $R^s \subset R$ denote the subring of all s -invariants.

Theorem 1.10. *Let V be a reflection vector faithful representation of the Coxeter system $(\mathcal{W}, \mathcal{S})$ over an infinite field. Let \mathcal{H} be the Hecke algebra and R the ring of polynomial functions on V . There is exactly one ring homomorphism*

$$\mathcal{E} : \mathcal{H} \rightarrow \langle \mathcal{R} \rangle$$

such that we have $\mathcal{E}(v) = \langle R[1] \rangle$ and $\mathcal{E}(T_s + 1) = \langle R \otimes_{R^s} R \rangle \quad \forall s \in \mathcal{S}$.

Remark 1.11. This theorem is a variant of theorem 1 from [Soe92]. I prefer the proof given here, since it does not assume any knowledge of Demazure operators.

Proof. Unicity is evident, since \mathcal{H} is generated by $T_s + 1$ as an algebra over $\mathbb{Z}[v, v^{-1}]$. Existence needs only to be proven in case of a dieder group, since \mathcal{H} can be described by relations involving each time only two generators. The dieder case will be discussed in section 4. \square

Definition 1.12. We consider in the Hecke algebra the elements $\tilde{T}_x = v^{l(x)}T_x$. By [KL79] there is exactly one involution $d : \mathcal{H} \rightarrow \mathcal{H}$ with $d(v) = v^{-1}$ and $d(T_x) = (T_{x^{-1}})^{-1}$ and for $x \in \mathcal{W}$ there is a unique $C'_x \in \mathcal{H}$ with $d(C'_x) = C'_x$ and

$$C'_x \in \tilde{T}_x + \sum_y v\mathbb{Z}[v]\tilde{T}_y$$

These elements form the so-called Kazhdan-Lusztig-basis of the Hecke algebra from [KL79].

Conjecture 1.13. Let V be a reflection faithful representation of \mathcal{W} over an infinite field and let R be the ring of regular functions on V . Then there is, at least in case $k = \mathbb{C}$, an indecomposable \mathbb{Z} -graded R -bimodule $B_x \in \mathcal{R}$ such that

$$\mathcal{E}(C'_x) = \langle B_x \rangle.$$

Remark 1.14. In case $k = \mathbb{C}$ and \mathcal{W} a finite Weyl group this is shown in [Soe92]. For \mathcal{W} a finite Weyl group and $\text{char } k$ at least the Coxeter number it is proven in [Soe00], that this conjecture is equivalent to a part of Lusztig’s conjecture concerning characters of irreducible representations of algebraic groups over k . For the case of the action of a Weyl group on the Cartan of an affine Kac-Moody algebra the conjecture is established in [Här99]. In 5.3 we construct a left inverse to \mathcal{E} , whose explicit description shows, how the preceding conjecture implies the positivity of all coefficients of all Kazhdan-Lusztig polynomials.

Remark 1.15. In 6.16 we show, that the “indecomposable bimodules in the image of \mathcal{E} ” can at least be parametrized by \mathcal{W} . Furthermore we determine in 5.15 the dimension of the Hom-spaces between any two “bimodules in the image of \mathcal{E} ” and show in which way our formulas are compatible with conjecture 1.13.

Remark 1.16. Still open is in particular the case of universal Coxeter groups (here the KL-polynomials were described by [Dye88]) and the case of general finite Coxeter groups, which should be accessible by computer.

Remark 1.17. Related investigations are in [BM01] and in partially unpublished work of Dyer [Dye94] and in unpublished work of Peter Fiebig.

2 A reflection faithful representation

Proposition 2.1. *Given a finite set \mathcal{S} and a Coxeter matrix of type \mathcal{S} of the form be a finite dimensional real vector space and let be given linearly independent vectors $(e_s)_{s \in \mathcal{S}}$ in V and linearly independent linear forms $(e_s^\vee)_{s \in \mathcal{S}}$ on V such that we have*

$$\langle e_t, e_s^\vee \rangle = -2 \cos(\pi/m_{s,t}) \quad \forall s, t \in \mathcal{S}.$$

If we assume as in [Kac90] that the dimension of V is smallest possible, then the formula $\rho(s)(v) = v - \langle v, e_s^\vee \rangle e_s$ defines a reflection faithful representation $\rho : \mathcal{W} \rightarrow \text{GL}(V)$ of our Coxeter group.

Proof. That we get in this way a representation, i.e. that the braid relations are satisfied can be deduced from the fact that for any pair s, t of simple reflections with st of finite order our space decomposes as the intersection of the kernels of e_s^\vee and e_t^\vee , which is not moved by s and t at all, and the space generated by e_s and e_t , where s and t act as generators of the usual dihedral group. In general the subspace $E \subset V$ spanned by the e_s is a subrepresentation isomorphic to the “natural representation” of [Bou81], V.4.3 and

is in particular faithful. Furthermore \mathcal{W} acts trivially on the simultaneous kernel K of all e_s^\vee and the natural action on $(V/K)^*$ is isomorphic again to the natural representation, which means that the action on (V/K) is isomorphic to the contragredient representation E^* of [Bou81], V.4.4. In view of our minimality assumption on the dimension of V we certainly have $K \subset E$ and thus a filtration by subrepresentations

$$0 \subset K \subset E \subset V.$$

The pairing $(e_s, e_t) = \frac{1}{2} \langle e_s, e_t^\vee \rangle$ now defines an invariant symmetric form on E and a nondegenerate invariant symmetric form on E/K . Now let $x \in \mathcal{W}$ be an element, whose fix point set $V^x \subset V$ is a hyperplane. We have to show $x \in \mathcal{T}$. Let $X \in \text{End } V$ denote the image of x . We have either $\det X = 1$ or $\det X = -1$ and start treating the case $\det X = 1$. If we choose a complement $\mathbb{R}c$ of the hyperplane V^x , then our endomorphism X necessarily has the form $X(v + \alpha c) = v + \alpha v_0 + \alpha c$ for some fixed $v_0 \in V^x$. Certainly we have $V^x \supset K$. Since our group acts faithfully on E we can choose $c \in E$ and deduce $v_0 \in E$. Since $E^* \cong V/K$ is a faithful representation as well, we get $v_0 \notin K$. Now we use our invariant bilinear form on E and deduce $(v + \alpha c, v + \alpha c) = (v + \alpha v_0 + \alpha c, v + \alpha v_0 + \alpha c)$ alias $2\alpha(v, v_0) + \alpha^2(2(c, v_0) + (v_0, v_0)) = 0$ for all $v \in E$ and $\alpha \in \mathbb{R}$. This gives us $(v, v_0) = 0$ for all $v \in E$ contradicting $v_0 \notin K$. Thus the case $\det X = 1$ is not possible and we can assume $\det X = -1$. Then necessarily x acts as a reflection on V and also on $V/K \cong E^*$. Since \mathcal{W} acts faithfully and transitively on the alcoves of the Tits cone, the fix point set of x cannot meet any alcove in the Tits cone. If we consider the fundamental dominant alcove of the Tits cone $C \subset E^*$ and its image under x , we find that C and xC are separated only by finite number of reflecting hyperplanes V^z for some $z \in \mathcal{T}$. On the other hand a certain nonempty open part of the reflection plane of x in E^* consists of points on segments joining points of C with points of xC , consider for example the image of C under the projection $(\text{id} + x)/2$. This open part then has to be covered by our finite number of reflecting hyperplanes, and this is impossible unless the reflecting hyperplane of x coincides with the reflecting hyperplane of some $z \in \mathcal{T}$. But then we form zx and come back to the impossible case $\det = 1$ least we have $x = z$. \square

3 Further notations and formulas

Notation 3.1. Given any finite dimensional representation V of a group \mathcal{W} we consider for any $x \in \mathcal{W}$ the (reversed) graph

$$\text{Gr}(x) = \{(x\lambda, \lambda) \mid \lambda \in V\} \subset V \times V$$

and form for any finite subset $A \subset \mathcal{W}$ in $V \times V$ the Zariski closed subset

$$\mathrm{Gr}(A) = \bigcup_{x \in A} \mathrm{Gr}(x).$$

Remark 3.2. For any $y, z \in \mathcal{W}$ we get obviously an isomorphic

$$\mathrm{Gr}(y) \cap \mathrm{Gr}(z) \xrightarrow{\sim} V^{yz^{-1}}$$

by the projection on the first coordinate and we have

$$(\mathrm{Gr}(y) + \mathrm{Gr}(z)) \cap (V \times 0) = \mathrm{im}(yz^{-1} - \mathrm{id}) \times 0.$$

Notation 3.3. Let now k be an infinite field and let as before denote R the k -algebra of all regular functions on V . We always abbreviate $\otimes_k = \otimes$. If we identify $R \otimes R$ with the regular functions on $V \times V$ via the rule $(f \otimes g)(\lambda, \mu) = f(\lambda)g(\mu)$, the regular functions on $\mathrm{Gr}(A)$ as a quotient of $R \otimes R$ from naturally a \mathbb{Z} -graded R -bimodul. This \mathbb{Z} -graded R -bimodul we denote

$$R(A) = R(\mathrm{Gr}(A)) \in R\text{-mod}_{\mathbb{Z}}\text{-}R$$

and it is easy to check that it is finitely generated as a left module and as a right module over R . For $A = \{x_1, \dots, x_n\}$ we also put $\mathrm{Gr}(A) = \mathrm{Gr}(x_1, \dots, x_n)$ and $R(A) = R(x_1, \dots, x_n)$ or even shorter $R(x) = R_x$, $R(x, y) = R_{x,y}$. In the case $A = \{y \mid y \leq x\}$ we put $R(A) = R(\leq x)$. For the right action of \mathcal{W} on R we use exponential notation $Sr^y(\lambda) = r(y\lambda) \forall y \in \mathcal{W}$, $\lambda \in V$, $r \in R$. If $1_y \in R_y$ denotes the constant function 1 on $\mathrm{Gr}(y)$ alias the image of $1 \otimes 1$ in R_y , we have $r1_y = 1_yr^y$. The notations are chosen to get $R(x) \otimes_R R(y) \cong R(xy)$ for all $x, y \in \mathcal{W}$.

4 The case of a dieder group

Remark 4.1. If \mathcal{W} is a dieder group, by [Hum90, 7.12(a)] the elementes of the selfdual basis are of the form $C'_x = v^{l(x)} \sum_{y \leq x} T_y$. We prove this as Remark 4.4.

Theorem 4.2. *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system with twoe generators $|\mathcal{S}| = 2$ and let V be a reflection vector faithful representation over an infinite field. Then the homomorphism of additive groups*

$$\begin{aligned} \mathcal{E} : \quad \mathcal{H} &\rightarrow \langle \mathcal{R} \rangle \\ v^n C'_x &\mapsto \langle R(\leq x) \rangle [n + l(x)] \end{aligned}$$

is a ring homomorphism.

Remark 4.3. For any reflection $s : V \rightarrow V$ the obvious map $R \otimes R \rightarrow R(e, s)$ induces an isomorphism $R \otimes_{R^s} R \xrightarrow{\sim} R(e, s)$. To see this, Remark, that this map certainly is a surjection and compare the dimensions of homogeneous parts, where one may use the eigenspace decomposition $R = R^s \oplus R^s \alpha$ for $\alpha \in V^*$ an equation of the reflection plane of s and a short exact sequence $R_s[-2] \hookrightarrow R(e, s) \rightarrow R_e$. Thus the theorem indeed gives the existence in Theorem 1.10. the proof biven here extends arguments of I. Herrmann [Her99].

Proof. Certainly it suffices, to show for all simple reflections s and all $x \in \mathcal{W}$ the equation

$$\mathcal{E} \left((T_s + 1) \sum_{y \leq x} T_y \right) = \mathcal{E}(T_s + 1) \mathcal{E} \left(\sum_{y \leq x} T_y \right).$$

If we put $A = \{y \in \mathcal{W} \mid y \leq x\}$, then $A \cup sA$ and $A \cap sA$ are both of the form $\{y \in \mathcal{W} \mid y \leq z\}$ for suitable z in our dieder group \mathcal{W} , unless we are in the second case and the intersection is empty. An explicite computation in the Hecke algebra now shows

$$(T_s + 1) \sum_{y \in A} T_y = \sum_{y \in A \cup sA} T_y + v^{-2} \sum_{y \in A \cap sA} T_y.$$

If we insert this results on the left hand side, our equation transforms into claiming an isomorphism of graded bimodules

$$R(A \cup sA) \oplus R(A \cap sA)[-2] \cong R \otimes_{R^s} R \otimes_R R(A),$$

whic will be shown after a preliminary Lemma as Proposition 4.6. □

Remark 4.4. To prove the formulas claimed in 4.1 for the C'_x in the dieder case we only have to show the $\gamma_x = v^{l(x)} \sum_{y \leq x} T_y$ are selfdual. For any simple reflection s with $sx > x$ however we can rewrite one of the equations of our proof to read

$$v(T_s + 1)\gamma_x = \begin{cases} \gamma_{sx} + \gamma_z \text{ with } z < x & \text{in case } l(x) > 1; \\ \gamma_{sx} & \text{in case } l(x) \leq 1, \end{cases}$$

and thus the selfduality of $v(T_s + 1)$ implies inductively the selfduality of all γ_x .

Lemma 4.5. *Let V be a finite dimensional representation of a group \mathcal{W} over an infinite field k of charakteristic $\neq 2$. Let $A \subset \mathcal{W}$ be a finite subset and $s \in \mathcal{W}$ an element with $sA = A$, acting on V as a reflection. Then we have:*

1. There is an isomorphism of graded bimodules

$$R \otimes_{R^s} R(A) \cong R(A) \oplus R(A)[-2].$$

2. For $R(A)^+ \subset R(A)$ the invariants under the action of $s \times \text{id}$ multiplication induces an isomorphism

$$R \otimes_{R^s} R(A)^+ \xrightarrow{\sim} R(A).$$

Proof. Let first W be an arbitrary finite dimensional vector space over k . Any reflection $t : W \rightarrow W$ defines an involution $t : R(W) \rightarrow R(W)$, and if we choose an equation $\beta \in W^*$ of the reflecting hyperplane W^t , we can consider

$$\partial_t = \partial_t^\beta : R(W) \rightarrow R(W), \quad f \mapsto \frac{f - tf}{2\beta}.$$

If now $X \subset W$ is a Zariski closed t -stable subset, then t induces an involution on $R(X)$ and we get an eigenspace decomposition of the form $R(X) = R(X)^+ \oplus R(X)^-$. If no irreducible component of X is contained in W^t , our operator ∂_t stabilizes the kernel of the surjection $R(W) \twoheadrightarrow R(X)$ and thus induces a map $\partial_t : R(X) \rightarrow R(X)$. It is easy to see that ∂_t and multiplication by β are mutually inverse isomorphisms $R(X)^+ \cong R(X)^-[-2]$ of \mathbb{Z} -graded R^t -modules.

Now let us take $W = V \times V$ and $t = s \times \text{id}$ for our reflection $s \in \mathcal{W}$. We apply our results to $X = \text{Gr}(A)$ and get a decomposition $R(A) = R(A)^+ \oplus R(A)^-$ as well as an isomorphism $R(A)^+ \cong R(A)^-[-2]$ by multiplication with $\alpha \otimes 1 = \beta$ for $\alpha \in V^*$ an equation of the reflecting hyperplane of s . With $R = R^s \oplus \alpha R^s$ we get part 2, and with our decomposition $R(A) = R(A)^+ \oplus R(A)^-$ and part 2 we also get part 1. \square

Proposition 4.6. *Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system with two generators $|\mathcal{S}| = 2$ and let V be a reflection faithful representation of \mathcal{W} over an infinite field. Let $s \in \mathcal{S}$ and $x \in \mathcal{W}$ be given and let $A = \{y \in \mathcal{W} \mid y \leq x\}$ denote the set of all elements below x . Then in $R\text{-Mod}_{\mathbb{Z}}\text{-}R$ we have an isomorphism*

$$R \otimes_{R^s} R(A) \cong R(A \cup sA) \oplus R(A \cap sA)[-2].$$

Proof. The case $A = sA$ is already done by the preceding Lemma. In case $A = \{e\}$ our proposition just claims the isomorphism $R(\leq s) \cong R \otimes_{R^s} R$ of 4.3. Thus only the case $A \neq \{e\}$, $A \neq sA$ remains to be treated, thus $A = \{y \in \mathcal{W} \mid y \leq x\}$ with $x \neq e$ and $sx > x$. By explicit calculation we find $A - sA = \{x, rx\}$ for some reflection $r \in \mathcal{T}$ different from s . By our assumptions the (-1) -eigenspaces of reflections of \mathcal{W} are pairwise different

and span a twodimensional subrepresentation $U \subset V$. By 3.2 there exists a form $\beta \in V^* \times V^*$, which vanishes on $\text{Gr}(x) + \text{Gr}(rx)$ but not on $U \times 0$. We consider the submodules generated by the cosets $\bar{\beta}$ and $\bar{1}$ of β and 1 in $R(A)$ over $R^s \otimes R$, call them M, N and claim

1. $M \cong R(A \cap sA)^+[-2]$ in R^s -mod $_{\mathbb{Z}}$ - R ;
2. $N \cong R(A \cup sA)^+$ in R^s -mod $_{\mathbb{Z}}$ - R ;
3. $R(A) = M \oplus N$,

where the upper index $+$ means $(s \times \text{id})$ invariants. Once this is shown, the proposition follows from the preceding lemma 4.5. Thus we only need to show our three claims.

1. To show the first claim we Remark, that for three pairwise different elements $x, y, z \in \mathcal{W}$, whose lengths do not all have the same parity, we always have

$$\text{Gr}(x) + \text{Gr}(y) + \text{Gr}(z) \supset U \times 0.$$

To show this we may without restriction assume $z = e$ and $x, y \in \mathcal{T}$. By section 3 we have $(\text{Gr}(x) + \text{Gr}(e)) \cap (U \times 0) = V^{-x} \times 0$ and the same for y . Since we assumed our representation reflection vector faithful, we have $V^{-x} \neq V^{-y}$ and therefore the inclusion. For $y \notin \{x, rx\}$ we thus have $\text{Gr}(y) + \text{Gr}(x) + \text{Gr}(rx) \supset U \times 0$, in particular our function β does not vanish on $\text{Gr}(y)$ for $y \in A \cap sA$. Thus an element of $R(A)$ annihilates $\bar{\beta}$ iff it vanishes on $\text{Gr}(A \cap sA)$. We deduce that multiplication by $\bar{\beta}$ defines an isomorphism $R(A \cap sA)[-2] \xrightarrow{\sim} R(A)\bar{\beta}$. But the image of $R^s \otimes R$ in $R(A \cap sA)$ consists precisely on the $(s \times \text{id})$ invariants, thus our isomorphism restricts to an isomorphism $R(A \cap sA)^+[-2] \xrightarrow{\sim} M$.

2. The $R^s \otimes R$ -submodule generated by $\bar{1}$ in $R(A \cup sA)$ is certainly just $R(A \cup sA)^+$ and the restriction onto $\text{Gr}(A)$ gives an injection

$$R(A \cup sA)^+ \hookrightarrow R(A).$$

This proves the second claim.

3. We first show $R(A) = M + N$. If $\alpha \in V^*$ denotes an equation of the reflecting plane V^s , then we have $R = R^s \oplus \alpha R^s$, thus $R \otimes R$ is generated as an $R^s \otimes R$ -modul by $1 \otimes 1$ und $\alpha \otimes 1$. It follows that it is also generated by $1 \otimes 1$ and an arbitrary not $(s \times \text{id})$ invariants $\beta \in V^* \times V^*$. Our β however cannot be invariant under $(s \times \text{id})$, because otherwise it would vanish on $\text{Gr}(sx)$ and thus by the inclusion of the proof of claim 1 on all of $U \times 0$. This shows $R(A) = M + N$. We only need in addition $M \cap N = 0$. This can be seen

as follows: For $y \in A \cap sA$ the restriction on $\text{Gr}(y, sy) = \text{Gr}(y) \cup \text{Gr}(sy)$ of any element of N is invariant under $s \times \text{id}$. Thus it suffices to show, that the restriction of an element of M on such a $\text{Gr}(y, sy)$ can only be invariant under $s \times \text{id}$, if it vanishes on $\text{Gr}(y, sy)$. It will be sufficient for that to show, that the restriction of β to $\text{Gr}(y, sy)$ is not invariant under $s \times \text{id}$ or also, that the restriction of β to $\text{Gr}(y) + \text{Gr}(sy)$ is not invariant under $s \times \text{id}$. But the $s \times \text{id}$ invariant forms on $\text{Gr}(y) + \text{Gr}(sy)$ have to vanish on the subspace $V^{-s} \times 0$ and $(\text{Gr}(x) + \text{Gr}(rx))$ meets $U \times 0$ in the different subspace $V^{-r} \times 0$. \square

5 Back from bimodules to the Hecke algebra

Notation 5.1. We work from now on always with a fixed reflection faithful representation of a Coxeter system over an infinite field and want in this section give an explicit left inverse for our ring homomorphism $\mathcal{E} : \mathcal{H} \rightarrow \langle \mathcal{R} \rangle$. We start with some notations. Given $B, B' \in \mathcal{R}$ we set

$$\text{Hom}(B, B') = \text{Hom}_{R \otimes R}(B, B') \in \mathcal{R}.$$

It is understood, that the left resp. right action on R on our Hom-space comes from the left resp. right action on B or equivalently on B' , in formulas $(rf)(b) = f(rb) = r(f(b))$, $(fr)(b) = f(br) = (f(b))r$, $\forall r \in R, b \in B, f \in \text{Hom}(B, B')$.

Notation 5.2. Given a finite dimensional \mathbb{Z} -graded vector space $V = \bigoplus V_i$ we define its graded dimension by

$$\underline{\dim} V = \sum (\dim V_i) v^{-i} \in \mathbb{Z}[v, v^{-1}]$$

and for a finitely generated \mathbb{Z} -graded right R -module M we define its graded rank by the rule

$$\underline{\text{rk}} M = \underline{\dim}(M/MR_{>0}) \in \mathbb{Z}[v, v^{-1}].$$

We thus have $\underline{\dim}(V[1]) = v(\underline{\dim} V)$ and $\underline{\text{rk}}(M[1]) = v(\underline{\text{rk}} M)$. Certainly our graded rank is only a reasonable notation for free modules, and we will only use it for such. By $\overline{\text{rk}} M$ we denote the image of $\underline{\text{rk}} M$ under the substitution $v \mapsto v^{-1}$. The following theorem motivates this whole section.

Theorem 5.3. *Our map $\mathcal{E} : \mathcal{H} \rightarrow \langle \mathcal{R} \rangle$ has as a left inverse the map $\langle \mathcal{R} \rangle \rightarrow \mathcal{H}$ given by the rule*

$$\langle B \rangle \mapsto \sum_{x \in \mathcal{W}} \overline{\text{rk}} \text{Hom}(B, R_x) T_x.$$

Proof. This theorem is a direct consequence of 5.7 and 5.16, which we will prove in the sequel without using it. \square

Definition 5.4. For any R -bimodule B and any subset $A \subset \mathcal{W}$ we define the subbimodule

$$\Gamma_A B = \{b \in B \mid \text{supp } b \subset \text{Gr}(A)\}$$

of all elements with support in $\text{Gr}(A)$. We put $\Gamma_{\geq i} B = \Gamma_{\{x \in \mathcal{W} \mid l(x) \geq i\}} B$ and define the category

$$\mathcal{F}_\Delta \subset \mathcal{R}$$

as the full subcategory of all graded bimodules $B \in \mathcal{R}$ such that B has its support in a set of the form $\text{Gr}(A)$ for some finite $A \subset \mathcal{W}$ and tht the quotient $\Gamma_{\geq i} B / \Gamma_{\geq i+1} B$ is for all i isomorphic to a finite direct sum of graded bimodules of the form $R_x[\nu]$ with $l(x) = i$ and $\nu \in \mathbb{Z}$.

Remark 5.5. Certainly $B \mapsto \Gamma_{\geq i} B / \Gamma_{\geq i+1} B$ is additive in B , thus \mathcal{F}_Δ is stable under forming direct sums and because of Krull-Schmid 1.3 also stable under forming direct summands.

Notation 5.6. It turns out to be convenient to introduce the graded bimodules

$$\Delta_x = R_x[-l(x)]$$

and to work in the Hecke algebra with $\tilde{T}_x = v^{l(x)} T_x$. for the multiplicity of a summand $\Delta_x[\nu]$ in one and every decomposition of $\Gamma_{\geq i} B / \Gamma_{\geq i+1} B$ for $i = l(x)$ we introduce the notation $(B : \Delta_x[\nu])$. Also we introduce the abbreviation $R[1] \otimes_{R^s} M = \theta_s M$.

Proposition 5.7. *Let $s \in \mathcal{S}$ be a simple reflection.*

1. *With B also $R \otimes_{R^s} B$ belongs to \mathcal{F}_Δ .*
2. *If we define the maps $h_\Delta : \mathcal{F}_\Delta \rightarrow \mathcal{H}$ by the rule $B \mapsto \sum_{x,\nu} (B : \Delta_x[\nu]) v^\nu \tilde{T}_x$, for all $s \in \mathcal{S}$ we get two commutative diagrams*

$$\begin{array}{ccc} \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \\ \theta_s \downarrow & & \downarrow (\tilde{T}_s + v) \\ \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \\ [1] \downarrow & & \downarrow v \\ \mathcal{F}_\Delta & \longrightarrow & \mathcal{H} \end{array}$$

3. *Our map \mathcal{E} factors via a map $\mathcal{E} : \mathcal{H} \rightarrow \langle \mathcal{F}_\Delta \rangle$ and our $h_\Delta : \langle \mathcal{F}_\Delta \rangle \rightarrow \mathcal{H}$ is left inverse to this \mathcal{E}*

Proof. The proof of this proposition needs

Lemma 5.8. *Let W be a finite dimensional vector space and $U, V \subset W$ two affine subspaces. Then $\text{Ext}_{R(W)}^1(R(U), R(V))$ is nonzero only if we have $V \cap U = V$ or if $V \cap U$ is of codimension one in V . In the later case $\text{Ext}_{R(W)}^1(R(U), R(V))$ is a free $R(U \cap V)$ -modul of rank one, generated by the class of an arbitray short exact sequence of the form*

$$R(V)[-2] \xrightarrow{\alpha} R(U \cup V) \rightarrow R(U)$$

for $\alpha \in W^*$ with $\alpha|_U = 0$, $\alpha|_V \neq 0$.

Proof. If F and G are free modules of finite rank over the k -algebras A and B and if M or N are arbitray modules over A and B , then we obviously (see [Bou70], I §4 and II §11) have

$$\text{Hom}_A(F, M) \otimes \text{Hom}_B(G, N) \xrightarrow{\sim} \text{Hom}_{A \otimes B}(F \otimes G, M \otimes N).$$

If our algebras are noetherian and M' resp. N' are finitely generated over A resp. B , we find resolutions $F^\bullet \rightarrow M'$ resp. $G^\bullet \rightarrow N'$ with F^i resp. G^j free of finite rank over A resp. B . Then $F^\bullet \otimes G^\bullet$ will be a free resolution of $M' \otimes N'$ and we deduce

$$\begin{aligned} \text{Ext}_{A \otimes B}^n(M' \otimes N', M \otimes N) &= H^n \text{Hom}_{A \otimes B}(F^\bullet \otimes G^\bullet, M \otimes N) \\ &= H^n(\text{Hom}_A(F^\bullet, M) \otimes \text{Hom}_B(G^\bullet, N)) \\ &= \bigoplus_{i+j=n} \text{Ext}_A^i(M', M) \otimes \text{Ext}_B^j(N', N) \end{aligned}$$

Thus we can restrict in our Lemma to the case with $\dim_k W = 1$, and those can easily be done in an explicit way. More precisely we may without restriction assume U and V linear and find a decomposition by $W = S \oplus U' \oplus V' \oplus W'$ mit $U = S \oplus U'$ and $V = S \oplus V'$. If we denote the dimension by $\dim W = s + u + v + w$, we will get $\text{Ext}_{R(W)}^\bullet(R(U), R(V)) \cong \text{Ext}_{k[X]}^\bullet(k[X], k[X])^s \otimes \text{Ext}_{k[X]}^\bullet(k[X], k)^u \otimes \text{Ext}_{k[X]}^\bullet(k, k[X])^v \otimes \text{Ext}_{k[X]}^\bullet(k, k)^w$. Thus Ext^1 is possible only for $v \leq 1$ and in case $v = 1$ we get $\text{Ext}^1 = k[X]^s$. \square

Now we prove the proposition. Let $s \in \mathcal{S}$ be our fixed simple reflection. Certainly we can refine our filtration $\Gamma_{\geq i}$ of B by certain $\Gamma_{\geq j} B$ with $j \in \mathbb{Z} + 0, 5$ such that for all $i \in \mathbb{Z}$ the quotients $\Gamma_{\geq i} B / \Gamma_{\geq i+0,5} B$ resp. $\Gamma_{\geq i-0,5} B / \Gamma_{\geq i} B$ are sums of $R(x)[\nu]$ with $x > sx$ resp. $x < sx$. With these choices the parameters x, y of two possible subquotients of $\Gamma_{\geq i-0,5} B / \Gamma_{\geq i+0,5} B$ only differ by a reflection in case $y = sx$: Indeed for an arbitray reflection t we always have $tx > x$ or $tx < x$, and $sy > y < x > sx$ implies $y \leq sx$ after a result of Deodhar, the so-called property Z of Coxeter groups. However all extensions in $\text{Ext}_{R \otimes R}^1(R_x, R_{sx})$ die after restriction to $R^s \otimes R$. Indeed the restriction $R_{x,sx} \rightarrow R_x$ splits over $R^s \otimes R$, since in the notiations of 4.5 it

gives an isomorphism $R_{x,sx}^+ \xrightarrow{\sim} R_x$. By 5.8 thus the restriction onto $R^s \otimes R$ of $\Gamma_{\geq i-0,5} B / \Gamma_{\geq i+0,5} B$ is isomorphic to a direct sum of copies of certain $R_x[\nu]$ with $x > sx$ and $l(x) = i$, and these occur with multiplicity $(B : R_x[\nu]) + (B : R_{sx}[\nu])$. If we now tensor with $R \otimes_{R^s} R_x \rightarrow R_{sx}$, we get the first claim and (always assuming $x > sx$) the formulas

$$\begin{aligned} (R \otimes_{R^s} B : R_x[\nu]) &= (B : R_x[\nu + 2]) + (B : R_{sx}[\nu + 2]) \\ (R \otimes_{R^s} B : R_{sx}[\nu]) &= (B : R_x[\nu]) + (B : R_{sx}[\nu]) \end{aligned}$$

This we can rewrite to

$$\begin{aligned} (\theta_s B : \Delta_x[\nu]) &= (B : \Delta_x[\nu + 1]) + (B : \Delta_{sx}[\nu]) \\ (R[1] \otimes_{R^s} B : \Delta_{sx}[\nu]) &= (B : \Delta_x[\nu]) + (B : \Delta_{sx}[\nu - 1]) \end{aligned}$$

If we now only for this proof write an element $H \in \mathcal{H}$ in the form $H = \sum_{x,\nu} (H : v^\nu \tilde{T}_x) v^\nu \tilde{T}_x$, then in the Hecke algebra we get similarly

$$\begin{aligned} ((\tilde{T}_s + v)H : v^\nu \tilde{T}_x) &= (H : v^{\nu+1} \tilde{T}_x) + (H : v^\nu \tilde{T}_{sx}) \\ ((\tilde{T}_s + v)H : v^\nu \tilde{T}_{sx}) &= (H : v^\nu \tilde{T}_x) + (H : v^{\nu-1} \tilde{T}_{sx}) \end{aligned}$$

This shows the second claim. To prove the last claim, we Remark first, that we may consider $\langle \mathcal{R} \rangle$ via the ring homomorphism \mathcal{E} of 1.10 also as a left \mathcal{H} -modul. The first part of the proposition then means, that $\langle \mathcal{F}_\Delta \rangle \subset \langle \mathcal{R} \rangle$ is an \mathcal{H} -submodul and the second part, that the map $h_\Delta : \langle \mathcal{F}_\Delta \rangle \rightarrow \mathcal{H}$ is a homomorphism of \mathcal{H} -modules. Since $\mathcal{E}(1) = \langle R_e \rangle$ belongs to \mathcal{F}_Δ , the map \mathcal{E} factorizes over $\langle \mathcal{F}_\Delta \rangle$, and since $h_\Delta \circ \mathcal{E} : \mathcal{H} \rightarrow \mathcal{H}$ is a homomorphism of \mathcal{H} -modules mapping one to one, this composition has to be the identity. \square

To show theorem 5.3 we also need a “dual” approach. More precisely we consider the filtration of our bimodules by the $\Gamma_{\leq i} B = \Gamma_{\{x \in \mathcal{W} | l(x) \leq i\}} B$ and define the category

$$\mathcal{F}_\nabla \subset \mathcal{R}$$

as the full subcategory of all graded bimodules $B \in \mathcal{R}$ such, that the support is in $\text{Gr}(A)$ for some finite $A \subset \mathcal{W}$ and that the quotients are for all i isomorphic to finite direct sum of graded bimodules of the form $R_x[\nu]$ with $l(x) = i$ and $\nu \in \mathbb{Z}$. In this context it is convenient and natural, to introduce

$$\nabla_x = R_x[l(x)].$$

For the multiplicitz of $\nabla_x[\nu]$ in a direct sum decomposition of $\Gamma_{\leq l(x)} B / \Gamma_{\leq l(x)-1} B$ we use the notation $(B : \nabla_x[\nu])$.

Remark that the multiplicities $(B : \nabla_x[\nu])$ and $(B : \Delta_x[\nu])$ concern sub-quotients for different filtrations. Even for $B \in \mathcal{F}_\Delta \cap \mathcal{F}_\nabla$ the multiplicities $(B : \Delta_x[l(x) + \nu])$ and $(B : \nabla_x[-l(x) + \nu])$ are thus different in general. Analogously we have now

Proposition 5.9. *Let $s \in \mathcal{S}$ be a simple reflection.*

1. *With B also $R \otimes_{R^s} B$ belongs to \mathcal{F}_∇ .*
2. *When we define the map $h_\nabla : \mathcal{F}_\nabla \rightarrow \mathcal{H}$ by the rule $B \mapsto \sum_{x,\mu} (B : \nabla_x[\mu]) v^{-\mu} \tilde{T}_x$, then for all $s \in \mathcal{S}$ the following diagrams are commutative*

$$\begin{array}{ccc}
\mathcal{F}_\nabla & \longrightarrow & \mathcal{H} \\
\theta_s \downarrow & & \downarrow (\tilde{T}_s + v) \\
\mathcal{F}_\nabla & \longrightarrow & \mathcal{H}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}_\nabla & \longrightarrow & \mathcal{H} \\
[1] \downarrow & & \downarrow v^{-1} \\
\mathcal{F}_\nabla & \longrightarrow & \mathcal{H}
\end{array}$$

3. *The composition $d \circ h_\nabla$ is a left inverse of $\mathcal{E} : \mathcal{H} \rightarrow \langle \mathcal{F}_\nabla \rangle$.*

Proof. We consider the functor

$$D = \text{Hom}_{-R}(\ , R) : \mathcal{R} \rightarrow R\text{-mod}_{\mathbb{Z}}\text{-}R,$$

where we provide our space of homomorphisms of right R -modules with the obvious \mathbb{Z} -grading and let the right resp. left action on DB be defined via the right resp. left action on the bimodule B , in formulas $(rf)(b) = f(rb)$ and $(fr)(b) = f(br)$ for all $b \in B$, $r \in R$, $f \in DB$. I want to avoid a general discussion whether an object of \mathcal{R} lands in \mathcal{R} again under D . In any case we have $DR_x \cong R_x$ and $D(M[\nu]) = (DM)[- \nu]$, our functor induces an equivalence of categories $D : \mathcal{F}_\nabla \xrightarrow{\sim} \mathcal{F}_\Delta^{\text{opp}}$ and evidently we have $h_\nabla = h_\Delta \circ D$. Thus we only need to establish for all $M \in \mathcal{F}_\nabla$ an isomorphism $\theta_s DM \cong D\theta_s M$. For that we first show:

Proposition 5.10. *1. The functors from $R^s\text{-mod}_{\mathbb{Z}}$ to $R\text{-mod}_{\mathbb{Z}}$ with $M \mapsto R[2] \otimes_{R^s} M$ and $M \mapsto \text{Hom}_{R^s}(R, M)$ are naturally equivalent.*

2. *The functor $R[1] \otimes_{R^s} : R\text{-mod}_{\mathbb{Z}} \rightarrow R\text{-mod}_{\mathbb{Z}}$ is selfadjoint.*

Proof. Since R is free of finite rank over R^s , our Hom – functor can also be written in the form

$$\text{Hom}_{R^s}(R, -) = \text{Hom}_{R^s}(R, R^s) \otimes_{R^s} -.$$

Now more precisely R is free over R^s with basis $1, \alpha$ for α an equation of the reflecting hyperplane $s(\alpha) = -\alpha$. The dual basis of $\text{Hom}_{R^s}(R, R^s)$ over R^s

will be denoted $1^*, \alpha^*$. The multiplication by $\alpha \in R$ is given in this basis by $\alpha\alpha^* = 1^*, \alpha 1^* = \alpha^2\alpha^*$. The choice of α therefore defines an isomorphism of R -modules

$$R[2] \xrightarrow{\sim} \text{Hom}_{R^s}(R, R^s), 1 \mapsto \alpha^*, \alpha \mapsto 1^*,$$

and the first claim is established. The second claim follows, since our two functors from the first claim are up to a change of grading just the two adjoints of the restriction of R^s . \square

Thus we indeed get isomorphisms

$$\begin{aligned} \theta_s(DM) &\cong R[1] \otimes_{R^s} DM \\ &\cong \text{Hom}_{R^s}(R[1], \text{Hom}_{-R}(M, R)) \\ &\cong \text{Hom}_{-R}(\theta_s M, R) \\ &\cong D(\theta_s M) \end{aligned}$$

The third part of Proposition 5.9 now follows, since we have $d(\tilde{T}_s + v) = \tilde{T}_s + v$ and thus the composition $d \circ h_{\nabla} \circ \mathcal{E} : \mathcal{H} \rightarrow \mathcal{H}$ is a homomorphism of \mathcal{H} -leftmodules with $1 \mapsto 1$. \square

Definition 5.11. We now define the category $\mathcal{B} \subset \mathcal{R}$ as the category of all graded bimodules $B \in \mathcal{R}$, whose class $\langle B \rangle$ lies in the image of our morphism $\mathcal{E} : \mathcal{H} \rightarrow \langle \mathcal{R} \rangle$, and call the objects of \mathcal{B} our **special bimodules**.

Remark 5.12. Certainly \mathcal{B} is stable under direct sums and shifts of grading. Much later in 6.16 we will be able to show that \mathcal{B} is stable under direct summands. To obtain a criterium for when a bimodul B belongs to \mathcal{B} , we consider for an arbitrary finite sequence $\underline{s} = (r, \dots, t)$ of simple reflections in the Hecke algebra the element $b(\underline{s}) = (T_r + 1) \dots (T_t + 1)$ and form the bimodule

$$B(\underline{s}) = R \otimes_{R^r} \dots R \otimes_{R^t} R.$$

Lemma 5.13. *A graded bimodule $B \in \mathcal{R}$ is special iff there exist objects $C, D \in \mathcal{R}$, which each are finite direct sums of objects of the form $B(\underline{s})[n]$ and such, that we have*

$$B \oplus C \cong D$$

Proof. Since $\langle B(\underline{s})[n] \rangle = \mathcal{E}(v^n b(\underline{s}))$ the $B(\underline{s})[n]$ are special. This shows our criterium to be sufficient. Since the $v^n b(\underline{s})$ generate \mathcal{H} as an abelian group, it is necessary. \square

Remark 5.14. In particular $\mathcal{E} : \mathcal{H} \rightarrow \langle \mathcal{R} \rangle$ factorizes over the split Grothendieck group $\langle \mathcal{B} \rangle$ of the additive category \mathcal{B} and 5.7 with 5.9 imply $\mathcal{B} \subset \mathcal{F}_{\Delta} \cap \mathcal{F}_{\nabla}$.

Theorem 5.15. For $M \in \mathcal{F}_\Delta$, $N \in \mathcal{B}$ and also for $M \in \mathcal{B}$, $N \in \mathcal{F}_\nabla$ the space $\text{Hom}(M, N)$ is graded free as a right R -module with rank

$$\underline{\text{rk}} \text{Hom}(M, N) = \sum_{x, \nu, \mu} (M : \Delta_x[\nu])(N : \nabla_x[\mu])v^{\mu-\nu}.$$

Proof. We only treat the case $M \in \mathcal{F}_\Delta$, $N \in \mathcal{B}$, the other case is similar. Let $i : \mathcal{H} \rightarrow \mathcal{H}$ denote the antiinvolution $i(v) = v$, $i(\tilde{T}_x) = \tilde{T}_{x^{-1}}$. We consider the symmetric $\mathbb{Z}[v, v^{-1}]$ -bilinear pairing $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{Z}[v, v^{-1}]$, which to each pair (F, G) associates the coefficient of $\tilde{T}_e = T_e$ in the representation of $i(F)G$ as a linear combination of the T or equivalently the \tilde{T} . It can be described explicitly by $\langle \tilde{T}_x, \tilde{T}_y \rangle = \delta_{xy}$, see [Lus85], §1.4. The looked-for formula thus gets the form

$$\overline{\text{rk}} \text{Hom}(M, N) = \langle h_\Delta M, h_\nabla N \rangle.$$

By 5.13 we can restrict to the case $N = B(\underline{s})$. With 5.10 we can see, that our formula is correct for the pair $(R \otimes_{R^s} M, N)$ iff it is correct for the pair $(M, R \otimes_{R^s} N)$. Without any difficulties we also see, that our formula is correct for $(M[1], N)$ if and only if it is correct for $(M, N[-1])$. Using this we can even restrict to the case $N = R_e$ which is obvious. \square

Korollar 5.16. Let $B \in \mathcal{B}$ be a special bimodule. Then we have

$$\begin{aligned} h_\Delta(B) &= \sum_{x \in \mathcal{W}} \overline{\text{rk}} \text{Hom}(B, R_x) T_x, \\ h_\nabla(B) &= \sum_{x \in \mathcal{W}} \overline{\text{rk}} \text{Hom}(R_x, B) T_x. \end{aligned}$$

Proof. By the preceding 5.15 we have

$$\begin{aligned} \underline{\text{rk}} \text{Hom}(B, R_x) &= \underline{\text{rk}} \text{Hom}(B, \nabla_x[-l(x)]) \\ &= \sum_\nu (B : \Delta_x[\nu])v^{-l(x)-\nu} \\ \underline{\text{rk}} \text{Hom}(R_x, B) &= \underline{\text{rk}} \text{Hom}(\Delta_x[l(x)], B) \\ &= \sum_\mu (B : \nabla_x[\mu])v^{-l(x)+\mu} \end{aligned}$$

and the corollary follows from the definitions. \square

6 Classification of indecomposable special bimodules

Notation 6.1. We work with a reflection faithful representation over an infinite field. For $B \in \mathcal{B}$ and $y \in \mathcal{W}$ we abbreviate $\Gamma_{\leq y} B / \Gamma_{< y} B = \Gamma_y^{\leq} B$, and also $\Gamma_{\geq y} B / \Gamma_{> y} B = \Gamma_y^{\geq} B$ and $B / \Gamma_{\neq y} B = \Gamma^y B$.

Remark 6.2. Given a bimodule with a finite filtration by subbimodules with subquotient R_x the support of any section $s \in B$ is an union of graphs of elements of our Coxete group. To show this we may without restriction assume $\text{supp } s$ irreducible. But any nonzero element $s \in B$ gives a nonzero element \bar{s} in a suitable subquotient of the Δ -flag and we deduce $\text{first supp } s \supset \text{supp } \bar{s} = \text{Gr}(x)$ for suitable $x \in W$ and then by the irreducibility of $\text{supp } s$ even $\text{supp } s = \text{Gr}(x)$. In particular we have $\Gamma_y B \cap \Gamma_{\neq y} B = 0$ and the embeddings $\Gamma_y B \hookrightarrow \Gamma_y^{\leq} B$ and $\Gamma_y^{\geq} B \hookrightarrow \Gamma^y B$. Under the same assumptions on B also the multiplication from the right with a nonzero element $p \in R$ doesn't change the support. We thus have $(\Gamma_A B)p = \Gamma_A(Bp)$ and then $(\Gamma_y^{\geq} B)p \cong \Gamma_y^{\geq}(Bp)$ and the like. We use these identifications to omit all brackets in this kind of expression.

Lemma 6.3. *Let $B \in \mathcal{F}_{\nabla}$ be given and let y_0, y_1, y_2, \dots be an enumeration of the elements of \mathcal{W} such that elements bigger in the Bruhat order have always a bigger index. Let $C(k) = \{y_0, \dots, y_k\}$ denote the set of k elements of our list and let $y_k = y$. Then the obvious map induces an isomorphism*

$$\Gamma_{\leq y} B / \Gamma_{< y} B \xrightarrow{\sim} \Gamma_{C(k)} B / \Gamma_{C(k-1)} B,$$

both sides are direct sums of objects of the form $\nabla_y[\nu]$, and a $\nabla_y[\nu]$ appears in this quotient with multiplicity $(B : \nabla_y[\nu])$ as a direct summand. Analogous results hold for \mathcal{F}_{Δ} .

Proof. If z_0, z_1, z_2, \dots is an enumeration of the elements of \mathcal{W} such, that the sequence $l(z_0), l(z_1), l(z_2), \dots$ of the lengths is increasing, and if we put $A(j) = \{z_0, z_1, \dots, z_j\}$, then the $\Gamma_{A(j)} B$ are a refinement of our filtration $\Gamma_{\leq i} B$ and $\Gamma_{A(j)} B / \Gamma_{A(j-1)} B$ is a sum of $(B : \nabla_{z_j}[\nu])$ copies of certain $\nabla_{z_j}[\nu]$. Between any two enumerations of elements of \mathcal{W} such, that Bruhat-bigger elements come later, we always can go back and forth in a finite number of steps each of which just exchanges two neighbouring incomparable elements. Since incomparable elements cannot differ by a reflection, in any of these steps there is not Ext^1 between the corresponding subquotients, thus the filtration before and after each of these steps lead to the same subquotients. The subquotient $\Gamma_{C(k)} B / \Gamma_{C(k-1)} B$ with $y_k = z_j$ thus is isomorphic to $\Gamma_{A(j)} B / \Gamma_{A(j-1)} B$, and thus $\Gamma_{\leq y} B / \Gamma_{< y} B$ is a sum of translated copies of ∇_y , and $\nabla_y[\nu]$ appear $(B : \nabla_y[\nu])$ times. \square

Proposition 6.4. *Given $B \in \mathcal{B}$ and $y \in \mathcal{W}$ our $\Gamma_y^{\leq} B$, $\Gamma_y^{\geq} B$ and $\Gamma^y B$ and $\Gamma_y B$ are graded free right R -modules, on which the $R \otimes R$ -action factors through R_y .*

Proof. For the first two modules of our list this easily follows from 6.3. For the last we then deduce from the embedding $\Gamma_y B \hookrightarrow \Gamma_y^{\leq} B$ from 6.2, that the action of $R \otimes R$ factors over R_y , so that the evaluation at 1_y gives an isomorphism $\text{Hom}(R_y, B) \xrightarrow{\sim} \Gamma_y B$, which by 5.15 shows $\Gamma_y B$ is free. The claim concerning $\Gamma^y B$ finally for $y = e$ follows from $B \in \mathcal{F}_\Delta$. In general we Remark, the by definition of the category \mathcal{B} and 5.7 we even get $B \otimes_R R_z \in \mathcal{F}_\Delta$ for all $z \in \mathcal{W}$. If we apply this to $z = y^{-1}$, we get the general case. \square

Notation 6.5. Let $\mathcal{T} \subset \mathcal{W}$ the set of all reflections, and for $t \in \mathcal{T}$ let $\alpha_t \in V^*$ be an equation of the reflecting hyperplane $\ker \alpha_t = V^t$. The α_t are unique up to scalar, we fix them for the rest of this section arbitrarily. For $y \in \mathcal{W}$ we now consider in R the element

$$p_y = \prod_{t \in \mathcal{T}, yt < y} \alpha_t.$$

Satz 6.6. *Let $y \in \mathcal{W}$ and $B \in \mathcal{B}$. Then the obvious morphisms $\Gamma_y B \rightarrow \Gamma_y^{\leq} B$ and $\Gamma_y^{\geq} B \rightarrow \Gamma^y B$ induce isomorphisms*

$$\begin{aligned} \Gamma_y B &\xrightarrow{\sim} \Gamma_y^{\leq} B p_y, \\ \Gamma_y^{\geq} B &\xrightarrow{\sim} \Gamma^y B p_y. \end{aligned}$$

Remark 6.7. The proof will only be given following 6.10. In [Soe92] it is shown in case $|\mathcal{W}| < \infty$ that there exists $c_y \in R \otimes R$ of degree $2l(y)$, vanishing on $\text{Gr}(x)$ for $x < y$, but not vanishing on $\text{Gr}(y)$ itself. Then necessarily c_y coincides on $\text{Gr}(y)$ up to a scalar with $1 \otimes p_y$ and we get an up to this scalar commutative diagram of isomorphisms

$$\begin{array}{ccc} \Gamma_y^{\leq} B & \xrightarrow{c_y} & \Gamma_y B \\ \cdot p_y \downarrow & & \parallel \\ \Gamma_y^{\leq} B p_y & \leftarrow & \Gamma_y B \end{array}$$

describing the inverse of the first isomorphism claimed in the theorem somewhat more explicitly. Similarly we also get an up to this scalar commutative diagram with isomorphisms in the leftmost square

$$\begin{array}{ccccccc} \Gamma_{\neq y} B & \hookrightarrow & B & \twoheadrightarrow & \Gamma^y B & \xrightarrow{\sim} & \Gamma^y B p_y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Gamma_{\leq y} B & \hookrightarrow & \Gamma_{\neq y} B & \twoheadrightarrow & \text{coker} & \cong & \Gamma_y^{\geq} B \end{array}$$

where all verticals up to the rightmost vertical are multiplications by c_y , the right upper horizontal is multiplication with p_y and for the last lower

horizontal we use an analogon of 6.3 to get a short exact sequence $\Gamma_{\leq y} B \hookrightarrow \Gamma_{\neq y} B \rightarrow \Gamma_y^{\geq} B$. In this way the inverse of the second isomorphism claimed in the theorem can be described somewhat more explicitly.

Remark 6.8. I would like to know, whether for every y of an infinite Coxeter group the function $c_y : \text{Gr}(\leq y) \rightarrow \mathbb{C}$ is regular, which vanishes on $\text{Gr}(x)$ for $x < y$ and on $\text{Gr}(y)$ is given by $1 \otimes p_y$.

Notation 6.9. For a reflection $t \in \mathcal{W}$ let $R_{(t)}$ denote the localization of R at all homogenous function on V , which do not vanish identically on the reflecting hyperplane V^t .

Lemma 6.10. *For any special bimodule $B \in \mathcal{B}$ its localization $B \otimes_R R_{(t)}$ in $R\text{-mod}_{\mathbb{Z}}\text{-}R_{(t)}$ is isomorphic to a direct summand in a finite direct sum of copies of $R_{y,yt} \otimes_R R_{(t)}$ and $R_y \otimes_R R_{(t)}$ for $y \in \mathcal{W}$ mit $y < yt$.*

Remark 6.11. The screwed-up formulation helps to avoid a discussion of Krull-Schmid in this context.

Proof. We prove this by induction using the inductive definition of the objects of \mathcal{B} . It is sufficient to show that for any simple reflection $s \in \mathcal{S}$ both $R\text{-}R_{(t)}$ -bimodules $R \otimes_{R^s} R_y \otimes_R R_{(t)}$ and $R \otimes_{R^s} R_{y,yt} \otimes_R R_{(t)}$ admit a decomposition into bimodules of the same type. But $R_y \otimes_R R_{(t)}$ and $R_x \otimes_R R_{(t)}$ can only extend in $R\text{-mod}_{\mathbb{Z}}\text{-}R_{(t)}$, if we have $y = xt$. Indeed after 5.8 in $R\text{-mod}_{\mathbb{Z}}\text{-}R$ there are only extensions if we have $y = xr$ for a reflection r , and these extensions die under multiplication from the right with any equation of the reflecting hyperplane of r .

To finish the first case we thus only have to show, that $sy \neq yt$ implies $syt > sy$. This however follows from Deodhar's property Z or alternatively from the geometrix insight, that neighbouring alcoves are separated just by one wall. In the second case there is nothing to do in case $sy = yt$, in this case we even have $R \otimes_{R^s} R_{y,yt} \cong R_{y,yt} \oplus R_{y,yt}[-2]$ by 4.5.

If finally we have $sy \neq yt$, the short exact sequence of bimodules $R[-2] \hookrightarrow R \otimes_{R^s} R \rightarrow R_s$ leads to a short exact sequence

$$R_{y,yt}[-2] \hookrightarrow R \otimes_{R^s} R_{y,yt} \rightarrow R_{sy,syt}$$

and this sequences has to split after applying $\otimes_R R_{(t)}$, since all extensions between the bimodules R_y, R_{yt} as submodules and R_{sy}, R_{syt} as quotients split after we apply $\otimes_R R_{(t)}$. \square

Proof of 6.6. The cokernel of $\Gamma_y B \hookrightarrow \Gamma_{\leq y} B$ contains no elements with support $\text{Gr}(y)$. Thus the restriction gives an embedding

$$\text{Hom}(\Gamma_y^{\leq} B, R_y) = \text{Hom}(\Gamma_{\leq y} B, R_y) \hookrightarrow \text{Hom}(\Gamma_y B, R_y).$$

We first show, that this embedding even lands in $\text{Hom}(\Gamma_y B, R_y p_y)$. Indeed, if for a reflection $t \in \mathcal{T}$ with $yt < y$ we pass by $\otimes_R R_{(t)}$ to the localizations and decompose B as in lemma 6.10, then only the summands of the form $R_{y, yt} \otimes_R R_{(t)}$ contribute to the two Hom-spaces we want to compare and we deduce, that our embedding lands in $\text{Hom}(\Gamma_y B, R_y \alpha_t \otimes_R R_{(t)})$, since this is true when we replace B by $R_{y, yt}$. Since this holds for all t with $yt < y$, and since the intersection of all these $R_y \alpha_t \otimes_R R_{(t)}$ with R_y is just $R_y p_y$, we get the looked-for factorization

$$\text{Hom}(\Gamma_y^{\leq} B, R_y) = \text{Hom}(\Gamma_{\leq y} B, R_y) \hookrightarrow \text{Hom}(\Gamma_y B, R_y p_y).$$

Since $\Gamma_y^{\leq} B$ is by 6.4 a free R_y -module, we further see, that the embedding $\Gamma_y B \hookrightarrow \Gamma_y^{\leq} B$ lands in $\Gamma_y^{\leq} B p_y$. Now we show by comparing dimensions, that this embedding is in fact an isomorphism. By Lemma 6.3 we get already

$$\text{rk} \Gamma_y^{\leq} B = \sum (B : \nabla_y[\nu]) v^{\nu+l(y)}.$$

On the other hand by 6.2 the action of $R \otimes R$ on $\Gamma_y B$ factorizes over R_y and we thus get $\Gamma_y B = \text{Hom}(R_y, B) = \text{Hom}(\Delta_y[l(y)], B)$. This however is by 5.15 a free right R -module of rank $\text{rk} \Gamma_y B = \sum (B : \nabla_y[\mu]) v^{\mu-l(y)}$. But since p_y lives in degree $2l(y)$, this coincides with the rank of $\Gamma_y^{\leq} B p_y$ and we get the first isomorphism of our theorem.

To deduce the second isomorphism we Remark, that dually the cokernel of the embedding $\Gamma_{\geq y} B \hookrightarrow B$ contains no elements with support in $\text{Gr}(y)$. Thus the restriction induces an embedding

$$\text{Hom}(B, R_y) \hookrightarrow \text{Hom}(\Gamma_{\geq y} B, R_y) = \text{Hom}(\Gamma_y^{\geq} B, R_y).$$

We first show now as before with Lemma 6.10 on the local structure of the bimodules from \mathcal{B} , that this embedding even lands in the subspace $\text{Hom}(\Gamma_{\geq y} B, R_y p_y)$. Then comparing the dimensions of the homogeneous parts we see that this embedding is in fact an isomorphism

$$\text{Hom}(B, R_y) \xrightarrow{\sim} \text{Hom}(\Gamma_y^{\geq} B, R_y p_y).$$

From there we get either comparing dimensions or dualizing with 6.4 the second claim of the theorem. \square

comment 6.12.

Remark 6.13. Für $A \subset \mathcal{W}$ eine Teilmenge und M einen R -Bimodul setzen wir ganz allgemein $\Gamma^A M = M / \Gamma_{\mathcal{W}-A} M$. Für M torsionsfrei als R -Rechtsmodul sind auch die $\Gamma^A M$ torsionsfreie R -Rechtsmoduln, in der Tat gilt dann nämlich

$$\Gamma_{\mathcal{W}-A} M = M \cap \Gamma_{\mathcal{W}-A}(M \otimes_R \text{Quot } R)$$

Remark 6.14. We will soon be able to show that \mathcal{B} is stable under the formation of direct summands. However as long as we don't know this yet, we set $\text{add } \mathcal{B}$ to be the category of all \mathbb{Z} -graded R -bimodules occurring as direct summands of special bimodules. Now we generalize 5.15 to $\text{add } \mathcal{B}$ and show:

Lemma 6.15. *For $M \in \mathcal{F}_\Delta$, $N \in \text{add } \mathcal{B}$ and also for $M \in \text{add } \mathcal{B}$, $N \in \mathcal{F}_\nabla$ the space $\text{Hom}(M, N)$ is graded free as a right R -module and has the rank*

$$\underline{\text{rk}} \text{Hom}(M, N) = \sum_{x, \nu, \mu} (M : \Delta_x[\nu]) (N : \nabla_x[\mu]) v^{\mu - \nu}.$$

Proof. We start with the first case. To begin with we Remark that the isomorphism $\Gamma_y N \xrightarrow{\sim} \Gamma_y^\leq N p_y$ of Satz 6.6 also holds for all $N \in \text{add } \mathcal{B}$. From there we get the claim for $M = \Delta_y$ and $N \in \text{add } \mathcal{B}$, since we have

$$\begin{aligned} \underline{\text{rk}} \text{Hom}(\Delta_y, N) &= \underline{\text{rk}} \Gamma_y N[l(y)] \\ &= \sum_{\mu} (\Gamma_y N[l(y)] : \nabla_y[\mu]) v^{\mu + l(y)} \\ &= \sum_{\mu} (\Gamma_y^\leq N[-l(y)] : \nabla_y[\mu]) v^{\mu + l(y)} \\ &= \sum_{\mu} (N : \nabla_y[\mu]) v^{\mu}. \end{aligned}$$

In the general case we argue by induction on the length of a Δ -flag of M . Indeed let $x \in \mathcal{W}$ be an element of maximal length with $(M : \Delta_x[\nu]) \neq 0$ for some $\nu \in \mathbb{Z}$. By maximality of x we have a short exact sequence

$$\Gamma_x M \hookrightarrow M \twoheadrightarrow \text{coker}$$

with $\Gamma_x M$ and coker in \mathcal{F}_Δ and

$$(M : \Delta_y[\nu]) = (\Gamma_x M : \Delta_y[\nu]) + (\text{coker} : \Delta_y[\nu])$$

for all $y \in \mathcal{W}$ and $\nu \in \mathbb{Z}$. Certainly on the right side of this equation the first summand is zero in case $y \neq x$ and the second in case $y = x$. But in any case our dimension formulas 5.15 given for all $N \in \mathcal{B}$ a short exact sequence

$$\text{Hom}(\text{coker}, N) \hookrightarrow \text{Hom}(M, N) \twoheadrightarrow \text{Hom}(\Gamma_x M, N).$$

This sequences then has to be exact for all $N \in \text{add } \mathcal{B}$ as well and thus induction on the length of a Δ -flag of M reversly shows the dimension formulas claimed for all $N \in \text{add } \mathcal{B}$.

The second case is treated analogously. That 6.6 also holds for all $B \in \text{add } \mathcal{B}$ gives the cases $N = \nabla_y$. In general we argue on the length of a ∇ -flag

of N , take an element x of maximal length with $(N : \nabla_x[\nu]) \neq 0$ for at least one $\nu \in \mathbb{Z}$, get by maximality of x a short exact sequence

$$\ker \hookrightarrow N \rightarrow \Gamma^x N$$

with $\Gamma^x N$ and \ker in \mathcal{F}_∇ and

$$(N : \nabla_y[\nu]) = (\ker : \nabla_y[\nu]) + (\Gamma^x N : \nabla_y[\nu])$$

for all $y \in \mathcal{W}$ and $\nu \in \mathbb{Z}$, and again our dimension formulas 6.15 for all $M \in \mathcal{B}$ a short exact sequence

$$\mathrm{Hom}(M, \ker) \hookrightarrow \mathrm{Hom}(M, N) \rightarrow \mathrm{Hom}(M, \Gamma^x N),$$

which has to be exact for all $M \in \mathrm{add} \mathcal{B}$ and permits the looked-for inductive argument. \square

Satz 6.16. *1. For all $x \in \mathcal{W}$ there is up to isomorphism a unique indecomposable special bimodule $B_x \in \mathcal{B}$ with support in $\mathrm{Gr}(\leq x)$ and $(B_x : \Delta_x[\nu]) = 1$ for $\nu = 0$ and zero for $\nu \neq 0$.*

2. The map $(x, \nu) \mapsto B_x[\nu]$ defines a bijection

$$\mathcal{W} \times \mathbb{Z} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{indecomposable objects of } \mathcal{B}, \\ \text{up to isomorphism} \end{array} \right\}$$

3. The bimodules B_x are selfdual, in formulas $DB_x \cong B_x$.

4. The category \mathcal{B} is stable under the formation of direct summands, in formulas $\mathcal{B} = \mathrm{add} \mathcal{B}$.

Proof. We first show the first three points for $\mathrm{add} \mathcal{B}$ instead of \mathcal{B} . Let $x = st \dots r$ be a reduced word. In $R \otimes_{R^s} R \otimes_{R^t} \dots R \otimes_{R^r} R[l(x)]$, which by 5.7 up to indecomposability possesses all properties we ask of B_x , we choose the indecomposable summand B_x with $(B_x : \Delta_x) = 1$ and thus already get a possible $B_x \in \mathrm{add} \mathcal{B}$, which even has to be selfdual as the unique direct summand of a selfdual bimodule, whose support contains $\mathrm{Gr}(x)$.

Now it only remains to show that except these B_x and their translates $B_x[\nu]$ there are no indecomposables in $\mathrm{add} \mathcal{B}$. But let $M \in \mathrm{add} \mathcal{B}$ and let $x \in \mathcal{W}$ be an element of maximal length $(M : \Delta_x[\nu]) \neq 0$ for some $\nu \in \mathbb{Z}$. We show that $B_x[\nu]$ is a direct summand of M . By maximality of x first by 6.15 or more precisely its proof the obvious maps give surjections

$$\begin{aligned} \mathrm{Hom}(M, B_x) &\rightarrow \mathrm{Hom}(\Gamma_x M, B_x) = \mathrm{Hom}(\Gamma_x M, \Gamma_x B_x), \\ \mathrm{Hom}(B_x, M) &\rightarrow \mathrm{Hom}(B_x, \Gamma^x M) = \mathrm{Hom}(\Gamma^x B_x, \Gamma^x M). \end{aligned}$$

The we Remark, that by maximality of x with 6.3 we even get $\Gamma_x^{\leq} M \xrightarrow{\sim} \Gamma^x M$, so that by 6.6 the embedding $\Gamma_x M \hookrightarrow \Gamma^x M$ gives isomorphisms

$$i : \Gamma_x M \xrightarrow{\sim} \Gamma^x M p_x \quad \text{and} \quad i : \Gamma_x B_x \xrightarrow{\sim} \Gamma^x B_x p_x$$

. Given $m \in \Gamma^x M$ homogeneous we now find $f \in \text{Hom}(\Gamma^x B_x, \Gamma^x M)$ homogeneous with $f(b) = m$ for $b \in \Gamma^x B_x$ a homogeneous generator. If we choose m even so, that it is part of a homogeneous basis of the graded free R -module $\Gamma^x M$, then we also get $g \in \text{Hom}(\Gamma_x M, \Gamma_x B_x)$ with $g(i^{-1}(m p_x)) = i^{-1}(b p_x)$. For homogeneous lifts $\tilde{f} : B_x \rightarrow M$ and $\tilde{g} : M \rightarrow B_x$ of f and g we deduce, that $\tilde{g} \circ \tilde{f} : B_x \rightarrow B_x$ induces the identity on $\Gamma^x B_x$. Thus $\tilde{g} \circ \tilde{f}$ can not be nilpotent, thus it is an isomorphism since B_x is indecomposable. Finally by induction on the length of x , that all B_x already belong to \mathcal{B} : Indeed our long tensor product from above decomposes into a summand B_x along with other summands, which by their support necessarily are of the form $B_y[\nu]$ with $l(y) < l(x)$. This shows the other claim. \square

Remark 6.17. If we find $B \in \mathcal{B}$ with $\langle B \rangle = \mathcal{E}(C'_x)$, then we can deduce $B \cong B_x$ and thus $\langle B_x \rangle = \mathcal{E}(C'_x)$. Indeed such a B is necessarily indecomposable, because calculating the dimension of the homogeneous components of its endomorphism ring we get nothing in negative degrees and but the ground field in degree zero.

Remark 6.18. For all $x \in \mathcal{W}$ at least $h_{\Delta}(B_x) = h_{\nabla}(B_x)$ is a selfdual element of \mathcal{H} of the form $C'_x + \sum_{y < x} h_y C'_y$ for suitable selfdual $h_y \in \mathbb{Z}[v, v^{-1}]$. Let indeed $x = st \dots r$ be a reduced word. Then

$$R \otimes_{R^s} R \otimes_{R^t} \dots R \otimes_{R^r} R[l(x)]$$

is selfdual and thus decomposes into B_x along with summands of the form B_y and $B_y[\nu] \oplus B_y[-\nu]$ for $y < x$. We deduce $h_{\Delta}(B_x) = h_{\nabla}(B_x)$ is selfdual by induction on the Bruhat order, and then we obviously get

$$h_{\Delta}(B_x) = h_{\nabla}(B_x) = C'_x + \sum_{y < x} h_y C'_y$$

for suitable selfdual $h_y \in \mathbb{Z}[v, v^{-1}]$. However I cannot show that these h_y have nonnegative coefficients.

7 Discussion of the main conjecture

A natural approach towards the proofs of conjecture 1.13 seems to me induction on the length of x . Here I formulate some variants of this approach as an omnibus Lemma. Let $\underline{\text{Hom}}(B, B')$ denote the $(R \otimes R)$ -homomorphism of \mathbb{Z} -degree zero between two \mathbb{Z} -graded R -bimodules B, B' .

Lemma 7.1. *Let $x \in \mathcal{W}$ and $s \in \mathcal{S}$ be given with $sx > x$. We further assume that $\langle B_y \rangle = \mathcal{E}(C'_y)$ for all $y \in \mathcal{W}$ with $l(y) \leq l(x)$. Then the following are equivalent:*

1. We have $\langle B_{sx} \rangle = \mathcal{E}(C'_{sx})$;
2. For all y with $l(y) \leq l(x)$ the composition of morphisms $\underline{\text{Hom}}(B_y, \theta_s B_x) \times \underline{\text{Hom}}(\theta_s B_x, B_y) \rightarrow \underline{\text{End}} B_y = k$ defines a nondegenerate pairing of k -vectorspaces.
3. For all y with $l(y) \leq l(x)$ the map $\Gamma_y \theta_s B_x \hookrightarrow \Gamma^y \theta_s B_x$ defines an inclusion $(\Gamma_y \theta_s B_x)_{l(y)} \hookrightarrow \Gamma^y \theta_s B_x p_y \otimes_R k$.
4. For all y with $l(y) \leq l(x)$ definiert die offensichtliche Abbildung eine Inklusion $(\Gamma_y \theta_s B_x)_{l(y)} \hookrightarrow \Gamma_y^{\geq} \theta_s B_x \otimes_R k$.
5. For all y with $l(y) \leq l(x)$ the obvious map defines an inclusion $(\Gamma_y^{\leq} \theta_s B_x)_{-l(y)} \hookrightarrow \Gamma^y \theta_s B_x \otimes_R k$.

Proof. $1 \Rightarrow 2$: Left to the reader. $2 \Rightarrow 1$: In the notation of [Soe97] we can write

$$C'_s C'_x = C'_{sx} + \sum_{y < sx} m_y C'_y$$

for certain $m_y \in \mathbb{Z}$ and we have $\mathcal{E}(C'_s C'_x) = \langle \theta_s B_x \rangle$. By 5.15 and the induction hypothesis we thus get

$$\dim \underline{\text{Hom}}(B_y, \theta_s B_x) = m_y = \dim \underline{\text{Hom}}(\theta_s B_x, B_y).$$

If our pairing is not degenerate, we can for every y split off just m_y copies of B_y from $\theta_s B_x$ and thus get $B \in \mathcal{B}$ with $\langle B \rangle = \mathcal{E}(C'_{sx})$. By Remark 6.17 we then have $B \cong B_{sx}$.

$3 \Leftrightarrow 2$: Comparing dimensions we see, that on the homomorphisms of degree zero the obvious maps are isomorphisms

$$\begin{aligned} \underline{\text{Hom}}(B_y, \theta_s B_x) &\xrightarrow{\sim} \underline{\text{Hom}}(\Delta_y, \theta_s B_x) \\ \underline{\text{Hom}}(\theta_s B_x, B_y) &\xrightarrow{\sim} \underline{\text{Hom}}(\theta_s B_x, \nabla_y). \end{aligned}$$

By 6.6 we know, that the composition of morphisms of degree zero $\Delta_y \rightarrow B_y \rightarrow \nabla_y$ is up to a scalar right multiplication by p_y and that for all $B \in \mathcal{B}$ any composition $\Delta_y \rightarrow B \rightarrow \nabla_y$ lands in $\nabla_y p_y$. Thus we can rewrite our pairing of part 2 as a pairing

$$\underline{\text{Hom}}(\Delta_y, \theta_s B_x) \times \underline{\text{Hom}}(\theta_s B_x, \nabla_y) \rightarrow p_y k \subset \underline{\text{Hom}}(\Delta_y, \nabla_y).$$

But now such a pairing between two spaces is nothing else but a map from one to the

$$\begin{aligned}
\underline{\mathrm{Hom}}(\Delta_y, \theta_s B_x) &= (\Gamma_y \theta_s B_x)_{l(y)} \\
\underline{\mathrm{Hom}}(\theta_s B_x, \nabla_y) &= \underline{\mathrm{Hom}}(\Gamma^y \theta_s B_x, \nabla_y) \\
&= \underline{\mathrm{Hom}}(\Gamma^y \theta_s B_x, k[l(y)]) \\
&= \underline{\mathrm{Hom}}((\Gamma^y \theta_s B_x) \otimes_R k, k[l(y)]) \\
\underline{\mathrm{Hom}}(\theta_s B_x, \nabla_y)^* &= ((\Gamma^y \theta_s B_x) \otimes_R k)_{l(y)}
\end{aligned}$$

and thus get $3 \Leftrightarrow 2$. the variants 4 and 5 are reformulations of 3 using 6.6. \square

Remark 7.2. It seems natural, to investigate the embeddings

$$\Gamma_y B_x \hookrightarrow \Gamma^y B_x$$

more closely. If we define a surjection $R \rightarrow k[T]$ as the restriction on the line $k\rho$, then I expect, that the cokernel of $(\Gamma_y B_x) \otimes_R k[T] \hookrightarrow (\Gamma^y B_x) \otimes_R k[T]$ is a direct sum of copies of $(k[T]/(T^{i+1}))[i]$, where we have again set $\deg T = 2$. This should even hold more generally for all lines, which are not contained in a reflecting hyperplane.

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