CHARACTER FORMULAS FOR TILTING MODULES OVER QUANTUM GROUPS AT ROOTS OF ONE

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1. Motivation

Let k be an algebraically closed field, $n \geq 1$ an integer. Our motivating problem is the study of the irreducible representations of the symmetric group $\mathcal{S}_n$ over k, in other words of the irreducible modules over the group ring $k\mathcal{S}_n$. In case $\mathrm{char} k = 0$ their classification is well known, we have a bijection

$$\mathrm{Irr} k\mathcal{S}_n \leftrightarrow \{\text{Partitions of } n\}.$$

In case $\mathrm{char} k = p > 0$ the classification is less well known, we have a bijection

$$\mathrm{Irr} k\mathcal{S}_n \leftrightarrow \left\{ \begin{array}{c} \text{Partitions of } n \text{ which do not} \\ \text{have } p \text{ or more equal pieces} \end{array} \right\}.$$

The next question is, given such a partition, to determine the dimension of the corresponding irreducible representation of the symmetric group. In case $\mathrm{char} k = 0$, these dimensions are again well known. We just have to display our partition as a Young diagram and count the standard tableaux of this shape, i.e. the dimension in question is the number of possibilities we have to enumerate the boxes of our Young diagram such that each row and column is increasing. For all this see [Jam78]. In case $\mathrm{char} k = p > 0$, the dimensions are not even known for all partitions of n into three pieces.

This is the problem where tilting modules suggest a new line of attack [Don93, Erd94, Erd95, Mat96]. Namely consider a finite dimensional vector space V over k of dimension $\dim_k V \geq n$. Then "Schur-Weyl duality" (see [CL74] or refSW below for the case of arbitrary characteristic) says that

$$\mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n} = k\mathcal{S}_n$$

with the symmetric group permuting the factors of a tensor in $V^{\otimes n}$.

If $\mathrm{char} k = 0$, we know that $V^{\otimes n}$ decomposes into a direct sum of irreducible subrepresentations under $\mathrm{GL}(V)$. In general, this is no longer true. But suppose

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda} T(\lambda)^{m(\lambda)}$$

is a decomposition of the $\mathrm{GL}(V)$-module $V^{\otimes n}$ into indecomposable (not in general irreducible) direct summands, where the isomorphism classes of indecomposables $T(\lambda)$ appearing are parametrized by a suitable set Λ and occur with multiplicity $m(\lambda) > 0$. Then it is clear from abstract algebra that the quotient of $k\mathcal{S}_n$ by its Jacobson radical is just a product of matrix algebras of the form

$$k\mathcal{S}_n/\mathrm{rad}k\mathcal{S}_n \cong \Pi_{\lambda \in \Lambda} M(m(\lambda) \times m(\lambda), k)$$

and we get a bijection

$$\begin{array}{ccc} \mathrm{Irr}k\mathcal{S}_n & \longleftrightarrow & \Lambda \\ \left(\begin{array}{c} \text{some irreducible} \\ \text{of dimension } m(\lambda) \end{array} \right) & \mapsto & \lambda. \end{array}$$

Therefore it is of the utmost importance to determine the decomposition of $V^{\otimes n}$ into indecomposables. In the next section we will following [Don93] define what is a "tilting module" for $G = \mathrm{GL}(V)$ or more generally for an arbitrary reductive algebraic group G. We will see that the indecomposable tilting modules are classified by their highest weights, and that all the summands $T(\lambda)$ of $V^{\otimes n}$ above are actually tilting modules.

Since the character of $V^{\otimes n}$ is known, we just need to determine the character of the indecomposable tilting module for any given highest weight to obtain the looked-for decomposition of $V^{\otimes n}$. We can only solve the analogous problem with "algebraic groups in positive characteristic" replaced by "quantum groups at roots of one", and conjecture (see [And97]) that if the highest weight is not too big, the quantized problem has the same solution as the original one. In the hope that the reader is now sufficiently motivated let me go on to define tilting modules.

2. Tilting modules for reductive algebraic groups

Let k be an algebraically closed field, $G \supset B \supset H$ a connected reductive algebraic group over k, a Borel subroup and a maximal torus (see [Jan87] for all this foundational material). The inclusion $H \subset B$ admits a unique left inverse $B \twoheadrightarrow H$ which allows us to consider every character of H as a character of B. For λ in the character lattice

$X = X(H)$ of H we consider the induced representation

$$\begin{aligned}\nabla(\lambda) &= \operatorname{ind}_B^G k_\lambda \\ &= \left\{ f : G \to k \;\middle|\; \begin{array}{ll} f \text{ is algebraic and satisfies} & \\ f(xb) = \lambda(b)^{-1} f(x) & \forall x \in G, b \in B \end{array} \right\}\end{aligned}$$

with the action of $g \in G$ on $f \in \nabla(\lambda)$ defined by $(gf)(x) = f(g^{-1}x)$. If we let $R \subset X$ be the root system of G, take as positive roots the complement $R^+ = R - R(B)$ of the roots of B, and let $X^+ \subset X$ be the corresponding dominant weights, then we have $\nabla(\lambda) \neq 0$ iff $\lambda \in X^+$.

In case char$k = 0$ our $\nabla(\lambda)$ is precisely the simple representation of G with highest weight λ. In general, the character of $\nabla(\lambda)$ is still given by the Weyl character formula, but the $\nabla(\lambda)$ are no longer simple for all λ, as we can readily see from the example $G = \mathrm{GL}(2, k)$: In this case letting ρ be half the positive root we can identify the representation $\nabla(n\rho)$ with the obvious representation $k[X, Y]^n$ of $\mathrm{GL}(2, k)$ on polynomials in two variables homogeneous of degree n, and for char$k = p$ obviously $kX^p + kY^p \subset k[X, Y]^p$ is an invariant subspace. We can further see that this subspace has no invariant complement, so $\nabla(p\rho)$ is not completely reducible if char$k = p > 0$.

Definition 2.1. [Don93] A rational representation T of G is called a "tilting module" if and only if both T and its dual T^* admit a finite ∇-flag, i.e. a finite filtration by G-stable subspaces such that all sucessive subquotients are isomorphic to some $\nabla(\lambda)$. In particular a tilting module by definition is always of finite dimension over k.

Remark 2.2. In the terminology of Donkin (due to Ringel [Rin91]) such representations would in fact be called a "partial tilting modules". However common usage is now to just call them tilting modules.

Examples 2.3. The trivial representation $k = \nabla(0)$ is always tilting. If more generally $\nabla(\lambda)$ is simple, one can show that $\nabla(\lambda)^* \cong \nabla(-w_\circ\lambda)$ for $w_\circ$ the longest element of the Weyl group, hence every simple $\nabla(\lambda)$ is tilting. As a special case of this we see that V is a tilting module for $\mathrm{GL}(V)$.

To have a non-trivial example let us consider the tensor product $\nabla(\rho) \otimes \nabla(p\rho - \rho)$ for $G = \mathrm{GL}(2, k)$, char$k = p$. This is certainly selfdual and one may see by hand that it fits into a short exact sequence

$$\nabla(p\rho - 2\rho) \hookrightarrow \nabla(\rho) \otimes \nabla(p\rho - \rho) \twoheadrightarrow \nabla(p\rho).$$

Therefore $\nabla(\rho) \otimes \nabla(p\rho - \rho)$ is a tilting module for $G = \mathrm{GL}(2, k)$.

The following theorem collects some of the reasons, why tilting modules are interesting.

Theorem 2.4. 1. *Every summand of a tilting module is tilting.*
2. *The tensor product of two tilting modules is tilting.*
3. *The indecomposable tilting modules are classified by their highest weights, more precisely we have a bijection*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{indecomposable tilting modules,} \\ \text{up to isomorphism} \end{array} \right\} & \leftrightarrow & X^+ \\ T & \mapsto & \text{“the highest weight of } T\text{”} \end{array}$$

Notation 2.5. We denote by $T(\lambda)$ the indecomposable tilting module with highest weight λ.

Proof. Let us give some indications on where this theorem comes from. Part (2) follows easily from the following theorem due in full generality to [Mat90]:

Theorem 2.6. *For all $\lambda, \mu \in X^+$ the tensor product $\nabla(\lambda) \otimes \nabla(\mu)$ admits a ∇-flag.*

To explain part (1), let us define $\Delta(\lambda) = \nabla(-w_\circ\lambda)^*$ for every $\lambda \in X$. (The parametrization of the $\Delta(\lambda)$ is just set up in such a way that $\Delta(\lambda)$ and $\nabla(\lambda)$ have the same character.) Let Ext_G denote extensions in the category of all rational representations of G. Part (1) then follows easily from the following result of Donkin [Don81], (see also [Ja], II, 4.16).

Proposition 2.7. *For a finite dimensional rational representation T of G the following are equivalent:*
1. *T admits a ∇-flag.*
2. $\mathrm{Ext}^1_G(\Delta(\lambda), T) = 0 \quad \forall \lambda \in X^+$.

Part (3) of the theorem can be proved in a very general context, as is explained in the next section following [Rin91]. For this we use the usual partial order on X^+ given by $\lambda \geq \mu$ iff $\lambda \in \mu + \mathbb{N}R^+$ and then only need to know that $\mathrm{Hom}_G(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda \leq \mu$, $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\mu))$ vanishes unless $\lambda < \mu$, and both are finite dimensional for all λ, μ. This can be deduced from results in [Ja] without too much difficulty. □

Remark 2.8. From the theorem and the fact that V is tilting for $\mathrm{GL}(V)$ it is clear that the Krull-Schmid-decomposition of the $\mathrm{GL}(V)$-module $V^{\otimes n}$ has to be of the form

$$V^{\otimes n} = \bigoplus_{\lambda \in X^+} T(\lambda)^{m(\lambda)}$$

where the $m(\lambda)$ are suitable multiplicities $m(\lambda) \geq 0$ and the $T(\lambda)$ are our indecomposable tilting modules from 2.5. Already for $G = \mathrm{GL}(3, k)$

however the characters of the $T(\lambda)$ are not all known, and what's worse, there isn't even a general conjecture. If λ is in the "lowest p^2-alcove", then $T(\lambda)$ is conjectured to have the same character as its quantum analogue, see [And97].

Remark 2.9. Let me explain how the above results imply the fundamental formula $\mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n} = k\mathcal{S}_n$ (if k is infinite and $\dim_k V \geq n$). There is no problem in verifying explicitely that $\mathrm{End}_{k\mathcal{S}_n} V^{\otimes n}$ is the k-linear subspace of $\mathrm{End}_k V^{\otimes n}$ generated by the image of $\mathrm{GL}(V)$. If $\mathrm{char} k = 0$, then $V^{\otimes n}$ is a semisimple $k\mathcal{S}_n$-module and thus by the Jacobson density theorem $k\mathcal{S}_n$ surjects onto the bicommutant of its image in $\mathrm{End}_k V^{\otimes n}$, i.e. we have a surjection $k\mathcal{S}_n \twoheadrightarrow \mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n}$.

For $\mathrm{char} k$ is arbitrary it is still clear that $k\mathcal{S}_n \to \mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n}$ is an injection in case $\dim_k V \geq n$. The problem is surjectivity. For this we will count dimensions and just have to show that the dimension of $\mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n}$ is independent of the ground field k. For this in turn we need to know that $\dim_k \mathrm{Hom}_G(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}$ and $\mathrm{Ext}^1_G(\Delta(\lambda), \nabla(\mu)) = 0$ for all λ, $\mu \in X^+$. (The first statement can be found in say [Jan87], the second one also is a special case of 2.7 above.)

Then if T has a finite Δ-flag where $\Delta(\lambda)$ occurs with multiplicity $(T : \Delta(\lambda))$ and T' has a finite ∇-flag where $\nabla(\lambda)$ occurs with multiplicity $(T' : \nabla(\lambda))$, we get

$$\dim_k \mathrm{Hom}_G(T, T') = \sum_\lambda (T : \Delta(\lambda))(T' : \nabla(\lambda)).$$

In particular, we get

$$\dim_k \mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n} = \sum_\lambda (V^{\otimes n} : \Delta(\lambda))(V^{\otimes n} : \nabla(\lambda))$$

which is indeed independent from the ground field k. As an aside, we see from comparing characters that $(T : \Delta(\lambda)) = (T : \nabla(\lambda))$ if T is tilting.

3. Existence and unicity of tilting modules in an abstract context

Suppose k is a field, $\mathcal{C}$ an abelian k-category, and $\{\Delta(\lambda)\}_{\lambda \in \Lambda}$ a collection of indecomposable objects of $\mathcal{C}$ parametrized by a partially ordered set $(\Lambda, \leq)$, such that the following conditions are satisfied:

1. $\mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda \leq \mu$.
2. $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) = 0$ unless $\lambda < \mu$.
3. $\dim \mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) < \infty$, $\dim \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), \Delta(\mu)) < \infty \quad \forall \lambda, \mu$.
4. For $\lambda \in \Lambda$ there are only finitely many elements below it.

In this situation we have

Proposition 3.1. [Rin91] *For every $\lambda \in \Lambda$ there exists a unique (up to non-unique isomorphism) indecomposable object $T = T(\lambda) \in \mathcal{C}$, which satisfies the following two conditions:*
(a) $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\nu), T) = 0 \quad \forall \nu \in \Lambda.$
(b) *T admits a finite Δ-flag starting with $\Delta(\lambda) \subset T$.*

Remark 3.2. By a finite Δ-flag of an object $T \in \mathcal{C}$ we mean a filtration $0 = T_0 \subset T_1 \subset T_2 \subset \ldots T_r = T$ such that $T_i/T_{i-1} \cong \Delta(\lambda_i)$ for suitable $\lambda_i \in \Lambda$.

Definition 3.3. $T(\lambda)$ might be called the indecomposable tilting module with parameter λ, but in this generality the terminology is not commonly used.

Proof. Existence: We prove the existence of $T(\lambda)$ by induction on the number of elements below λ. If λ is already minimal, we can take $T(\lambda) = \Delta(\lambda)$. If not, we find by condition (4) a minimal $\mu \in \Lambda$ with $\mu < \lambda$. By induction, we know there exists $\tilde{T} \in \mathcal{C}$ such that $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\nu), \tilde{T}) = 0$ for all $\nu \neq \mu$ and that $\tilde{T}$ admits a finite Δ-flag with subquotients $\Delta(\nu)$, $\nu \neq \mu$. If $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), \tilde{T}) = 0$ we can take $\tilde{T} = T(\lambda)$ and are through. If not, we have at least $\dim_k \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), \tilde{T}) < \infty$ by condition (3). Put $\tilde{T} = T_0$, choose a nonzero element e of this Ext-group, and represent it by a short exact sequence

$$T_0 \hookrightarrow T_1 \twoheadrightarrow \Delta(\mu).$$

A segment of the corresponding long exact sequence of Ext-groups is

$$\mathrm{Hom}_{\mathcal{C}}(\Delta(\mu), \Delta(\mu)) \to \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), T_0) \to \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), T_1) \to 0,$$

the last zero since we have $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), \Delta(\mu)) = 0$ by condition (1). Now the boundary map maps $\mathrm{id} \in \mathrm{Hom}_{\mathcal{C}}(\Delta(\mu), \Delta(\mu))$ to our nonzero element e. Thus $\dim_k \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), T_1) < \dim_k \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\mu), T_0)$ and by other long exact sequences still $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\nu), T_1) = 0$ if $\nu \neq \mu$. Continuing this way, we construct inductively $T_2, T_3, \ldots$ until we arrive at some T_r that fits into a short exact sequence

$$\tilde{T} \hookrightarrow T_r \twoheadrightarrow \Delta(\mu)^r$$

and satisfies $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\nu), T_r) = 0 \quad \forall \nu \in \Lambda$. In case $\mathrm{End}_{\mathcal{C}}\Delta(\mu) = k$ it can be shown that T_r is actually indecomposable, but in our general situation this need not be true. To circumvent this problem we rather show that if m is the smallest possible integer such that there exists a short exact sequence

$$\tilde{T} \hookrightarrow T \twoheadrightarrow \Delta(\mu)^m$$

with T satisfying $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\nu), T) = 0 \quad \forall \nu$, then this T is indecomposable and hence our looked-for $T(\lambda)$. Indeed, we can describe $\tilde{T}$ as the biggest subobject of T killed by all morphisms $T \to \Delta(\mu)$, and hence any decomposition $T = T' \oplus T''$ induces a decomposition $\tilde{T} = \tilde{T}' \oplus \tilde{T}''$. Since $\tilde{T}$ is indecomposable, we may assume $\tilde{T} = \tilde{T}'$, $\tilde{T}'' = 0$ and get $T'/\tilde{T} \oplus T'' \cong \Delta(\mu)^m$. Now the Krull-Schmid theorem implies $T'/\tilde{T} \cong \Delta(\mu)^n$ with $n \leq m$, but by minimality of m we have necessarily $n = m$, hence $T'' = 0$.

Unicity: Since T is indecomposable, $\mathrm{End}_{\mathcal{C}} T$ admits no idempotents except zero and one. Since T has a finite Δ-flag, $\mathrm{End}_{\mathcal{C}} T$ is of finite dimension over k by (2). Together this implies that every element of $\mathrm{End}_{\mathcal{C}} T$ is either nilpotent or an isomorphism, by the Lemma of Fitting.

Suppose now $T, T' \in \mathcal{C}$ are two indecomposables satisfying (a) and (b). Consider the diagram

$$\begin{array}{ccccc} \Delta(\lambda) & \hookrightarrow & T & \twoheadrightarrow & \mathrm{coker} \\ || & & & & \\ \Delta(\lambda) & \hookrightarrow & T' & \twoheadrightarrow & \mathrm{coker}' \end{array}$$

with short exact horizontals. By (b) coker has a Δ-flag, thus by (a) we have $\mathrm{Ext}^1_{\mathcal{C}}(\mathrm{coker}, T') = 0$, thus the restriction $\mathrm{Hom}_{\mathcal{C}}(T, T') \to \mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), T')$ is a surjection and we can find $\alpha : T \to T'$ inducing the identity on $\Delta(\lambda)$. Similarily we also find $\beta : T' \to T$ inducing the identity on $\Delta(\lambda)$. Since $\beta \circ \alpha \in \mathrm{End}_{\mathcal{C}} T$ is not nilpotent, it has to be an isomorphism, and the same holds for $\alpha \circ \beta$. We conclude $T \cong T'$. $\square$

The multiplicity of $\Delta(\nu)$ as subquotient in a Δ-flag of $T(\lambda)$ will be denoted by $(T(\lambda) : \Delta(\nu))$. This number is independent of the Δ-flag. Indeed, if we enumerate the elements below λ as $\lambda_1, \ldots, \lambda_n$ such that $\lambda_i > \lambda_j \Rightarrow i < j$, in particular $\lambda_i = \lambda$ and λ_n is minimal, then by the vanishing of the relevant $\mathrm{Ext}^1_{\mathcal{C}}$ any Δ-flag can be transformed without changing its multiplicities to a Δ-flag where first come the subquotients $\Delta(\lambda_1)$, then the $\Delta(\lambda_2)$ etc. It is then clear from our inductive construction of $T(\lambda)$ that all multiplicities are well determined.

4. Character formulas for indecomposable tilting modules over graded Lie algebras

We will explain next how to determine the characters of indecomposable tilting modules for quantum groups at roots of unity or equivalently [KL93, KL94] how to solve the translated problem in a suitable category $\mathcal{O}$ of representations of an affine Lie algebra. First we will work with an arbitrary $\mathbb{Z}$-graded Lie-algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ over the field k subject only to the following three conditions

1. $\dim_k \mathfrak{g}_i < \infty \quad \forall i \in \mathbb{Z}$.
2. $\mathfrak{g}$ is generated by $\mathfrak{g}_{-1}, \mathfrak{g}_0, \mathfrak{g}_1$.
3. There exists a character $\gamma : \mathfrak{g}_0 \to k$ such that $\mathrm{tr}(\mathrm{ad}X \mathrm{ad}Y : \mathfrak{g}_0 \to \mathfrak{g}_0) = \gamma([X, Y])$ for all $X \in \mathfrak{g}_1$, $Y \in \mathfrak{g}_{-1}$.

Let us put $\mathfrak{g}_{<0} = \mathfrak{n}$, $\mathfrak{g}_{\geq 0} = \mathfrak{b}$ and denote the enveloping algebras of $\mathfrak{g}, \mathfrak{n}, \mathfrak{b}$ by U, N, B. Let $\mathcal{M}$ denote the category of all $\mathbb{Z}$-graded $\mathfrak{g}$-modules, which are graded free of finite rank over N. The following result is due to Archipov [Ark96] and can also be found in [Soe97a].

Theorem 4.1. *There exists an equivalence of categories $\mathcal{M} \to \mathcal{M}^{\mathrm{opp}}$ such that short exact sequences of $\mathfrak{g}$-modules on both sides correspond, and that $U \otimes_B E$ gets mapped to $U \otimes_B (k_{-\gamma} \otimes E^*)$ for every finite dimensional $\mathbb{Z}$-graded representation E of $\mathfrak{g}_0$.*

Proof. I will only give some indications on how this result may be proven, more details can be found in [Soe97a]. Denote for a $\mathbb{Z}$-graded space $M = \oplus M_i$ by $M^{\circledast}$ its graded dual, so $(M^{\circledast})_i = (M_{-i})^*$. We make $N^{\circledast}$ into a $\mathbb{Z}$-graded N-bimodule in the most obvious way, the left action of N on itself giving rise to the right action on $N^{\circledast}$ and vice versa. The key ingredient to our equivalence will be a very peculiar $\mathbb{Z}$-graded U-bimodule S, the so-called "semi-regular bimodule", whose existence is assured by the following

Proposition 4.2. *There exists a $\mathbb{Z}$-graded U-bimodule $S = S_\gamma$ along with an inclusion $\iota : N^{\circledast} \hookrightarrow S$ of $\mathbb{Z}$-graded N-bimodules such that the following holds:*

1. *The map $U \otimes_N N^{\circledast} \to S$, $u \otimes f \mapsto u\iota(f)$ is a bijection.*
2. *The map $N^{\circledast} \otimes_N U \to S$, $f \otimes u \mapsto \iota(f)u$ is a bijection.*
3. *Up to a twist by γ the inclusion $\iota : N^{\circledast} \hookrightarrow S$ is compatible with the adjoint action of $\mathfrak{g}_0$ on both spaces, more precisely $\iota(f \circ \mathrm{ad}H) + (\mathrm{ad}H)\iota(f) = \iota(f)\gamma(H)$ for all $H \in \mathfrak{g}_0$ and $f \in N^{\circledast}$.*

This Proposition can be checked by brute force, but I still don't know a good proof. Anyhow, we can now write down the equivalence of categories claimed by the theorem as the functor $M \mapsto (S \otimes_U M)^{\circledast}$, and it is not difficult to check that this has the requested properties. □

Suppose now we are working over a ground field k of characteristic zero and $\mathfrak{g}$ is semisimple under the adjoint action of $\mathfrak{g}_0$. Let $\mathcal{O}$ be the category of all $\mathbb{Z}$-graded $\mathfrak{g}$-modules which are locally finite over $\mathfrak{g}_{\geq 0}$ and semisimple as modules over $\mathfrak{g}_0$. Let Λ be the set of isomorphism classes of finite dimensional simple $\mathbb{Z}$-graded $\mathfrak{g}_0$-modules. Any $E \in \Lambda$ is necessarily concentrated in just one degree. For $E \in \Lambda$ the Verma module $\Delta(E) = U \otimes_B E$ admits a unique simple quotient $L(E)$, and in this way Λ parametizes the simple objects of $\mathcal{O}$ up to isomorphism. Adapting the arguments of section 3 to our situation, we can prove

Theorem 4.3. *For every $E \in \Lambda$ there exists a unique (up to non-unique isomorphism) indecomposable object $T = T(E) \in \mathcal{O}$ which satisfies the following two conditions:*

1. $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), T(E) = 0 \quad \forall F \in \Lambda.$
2. *T admits a Δ-flag starting with $\Delta(E)$, i.e. a filtration $0 = T_0 \subset T_1 \subset T_2 \subset \ldots$ such that $T_1 \cong \Delta(E)$, $\bigcup T_i = T$ and $T_i/T_{i-1} \cong \Delta(F_i)$ for suitable $F_i \in \Lambda$.*

Certainly we call $T(E)$ the tilting module with parameter E. The main theorem is a formula for the multiplicity $(T(E) : \Delta(F))$ of $\Delta(F)$ as subquotient in a Δ-flag of $T(E)$ as above. To formulate the theorem in this generality, we have to introduce the module

$$\nabla(E) = (U \otimes_{U(\mathfrak{g}_{\leq 0})} E^*)^{\circledast}$$

which is the correct generalization of what is known as the dual Verma in the Kac-Moody set-up. Let $[\nabla(E) : L(F)]$ denote the Jordan-Hölder-multiplicity of $L(F)$ in $\nabla(E)$. The "abstract character formula for tilting modules" then says

Theorem 4.4. $(T(E) : \Delta(F)) = [\nabla(k_{-\gamma} \otimes F^*) : L(k_{-\gamma} \otimes E^*)]$ *for all* $E, F \in \Lambda$.

Proof. We will only prove this under the additional assumption that the simple object $L = L(k_{-\gamma} \otimes E^*)$ admits an indecomposable projective cover $P = P(k_{-\gamma} \otimes E^*)$ in $\mathcal{O}$. (The general case is more or less the same, details can be found in [Soe97a].) Under our assumption one proves as usual that P admits a finite Δ-flag ending with $\Delta(k_{-\gamma} \otimes E^*)$, and a suitably general form of the reciprocity principle tells us that the multiplicities in this Δ-flag are given by $(P : \Delta(F)) = [\nabla(F) : L]$ for $F \in \Lambda$.

Now we take a second look at our functor $\mathcal{M} \xrightarrow{\sim} \mathcal{M}^{\mathrm{opp}}$ to realize that it also gives an equivalence $\mathcal{M} \cap \mathcal{O} \xrightarrow{\sim} (\mathcal{M} \cap \mathcal{O})^{\mathrm{opp}}$. This in turn means that our functor transforms P into an indecomposable object T that has a Δ-flag starting with $\Delta(E)$ and satisfies $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(F), T) = 0 \quad \forall F \in \Lambda$. In other words, $P = P(k_{-\gamma} \otimes E^*)$ gets transformed into our tilting module $T = T(E)$, and we can calculate

$$\begin{aligned} (T(E) : \Delta(F)) &= (P(k_{-\gamma} \otimes E^*) : \Delta(k_{-\gamma} \otimes F^*)) \\ &= [\nabla(k_{-\gamma} \otimes F^*) : L(k_{-\gamma} \otimes E^*)]. \end{aligned}$$

□

5. The case of quantum groups

If we run the abstract formula from Theorem 4.4 for the special case of affine Lie algebras, we see that the character formulas for tilting modules in negative level are determined by the character formulas for simple modules in positive level and vice versa. Now results of Kazhdan and Lusztig [KL93, KL94] tell us how to relate representations of quantum groups at roots of one to category $\mathcal{O}$ at negative level for affine Lie algebras. Putting all this together, we see that the character formulas of [KT95] for simple highest weight modules in positive level lead to character formulas for tilting modules for quantum groups at roots of one.

This however only works fine in simply laced cases. In general we would need the extension of the results of [KT95] to non-integral highest weights, which is still missing from the literature. The (partially conjectural) outcome is a formula expressing the characters of tilting modules in terms of Kazhdan-Lusztig polynomials, see [Soe97a] for details and [Soe97b] for motivation.

In the special case that our quantum group U is "quantized $\mathrm{GL}(V)$", we have by [Du95] 3.6 the quantized Schur-Weyl duality

$$\mathcal{H} = \mathrm{End}_U V^{\otimes n}$$

where $\mathcal{H}$ denotes the "quantization" of $k\mathcal{S}_n$, i.e. the Hecke algebra of the symmetric group. The same arguments as in section 1 now tell us in which way the character formulas for indecomposable tilting modules over U determine the dimensions of all simple modules over $\mathcal{H}$. However I do not yet see how to obtain from there the formulas for these dimensions conjectured in [LLT96] and proved by Araki [Ara96] and Grojnowski.

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