ROOTS OF UNITY AND POSITIVE CHARACTERISTIC

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Abstract. This article gives an overview of [AJS94]. I outline our results to the effect that representation categories of both quantum groups at a root of one and semisimple algebraic groups in positive characteristic admit a common combinatorial model over ¡b¿Z¡/b¿. This allows in particular to deduce Lusztig's old modular conjectures from their newer quantum analogues (which are a theorem).

1. Introduction

In [AJS94] we proved that for a fixed root system Lusztig's quantum conjecture [Lus89, Lus90a] implies his modular conjecture [Lus80] in almost every characteristic, i.e. up to finitely many exceptions. For any given characteristic however we still do not know whether it is an exception or not. This article is meant as a guide to [AJS94]. The idea is to just cite the results of some heavy computations there thus allowing the main stream of arguments to flow without obstructions. We start explaining the problem in the easiest example.

Let k be an algebraically closed field of characteristic $\mathrm{char} k = p > 0$ and consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, k)$ for some $n \geq 2$. A representation of $\mathfrak{g}$ is a pair (ρ, V) where V is a k-vectorspace an $\rho : \mathfrak{g} \rightarrow \mathrm{End}_k V$ a Lie algebra homomorphism. Such a representation is called restricted if and only if $\rho(D^p) = \rho(D)^p$ for all $D \in \mathfrak{g}$. (Remark that $D \in \mathfrak{g}$ implies $D^p \in \mathfrak{g}$, since $tr(D^p) = (trD)^p$. We will from now on write $D^{[p]}$ instead of D^p in order to avoid confusion with the p-th power in the enveloping algebra later on.) The problem is to describe the simple restricted representations of $\mathfrak{g}$ in some detail, for example to compute their dimensions.

More generally let $G \supset B \supset T$ be a semisimple simply connected algebraic group over k, a Borel subgroup and a maximal torus. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be the corresponding Lie algebras. By definition $\mathfrak{g}$ is the Lie algebra of all left invariant vector fields D on G, i.e. the Lie algebra of all k-linear derivations D of the ring $k[G]$ of regular functions on the group that commute with left translations. Now obviously $D \in \mathfrak{g}$

implies $D^p \in \mathfrak{g}$ where we just take the p-th power in $\mathrm{End}_k k[G]$. In order to avoid confusion later we write here again $D^{[p]}$ instead of D^p.

A restricted representation of $\mathfrak{g}$ again is a pair (ρ, V) such that $\rho(D^{[p]}) = \rho(D)^p$ for all $D \in \mathfrak{g}$. Every rational representation of G differentiates to a restricted representation of $\mathfrak{g}$. It is not true in general that simple representations of G differentiate to simple representations of $\mathfrak{g}$. Nevertheless a detailed knowledge of the simple restricted representations of $\mathfrak{g}$ leads via known results ([Jan87], II.3) to a detailed knowledge of all simple rational representations of G, and those in turn lead to a knowledge of all simple representations of finite Chevalley groups in their own characteristic, see e.g. [Car63].

Thus it is an important problem to describe the simple restricted representations of $\mathfrak{g}$. In our work we deduce such a description (for almost every characteristic) from similar results in the theory of quantum groups at a root of unity. This is the last step of a bigger program proposed by Lusztig to establish his conjectures describing the simple restricted $\mathfrak{g}$-modules. The previous steps from affine flag manifolds to Kac-Moody algebras to quantum groups at a root of unity have been accomplished by Kashiwara-Tanisaki [KT94] and Kazhdan-Lusztig [KL93, KL94, Lus94], so that by now the whole program is sucessfully completed in the simply laced cases, see [And94, Soe]. To settle the non simply laced case it remains only to extend [KT94] to non-integral weights.

2. Review of classical results

2.1. **Verma modules and simple modules.** We begin with an exposition of the classical theory of restricted representations of $\mathfrak{g}$. More details can be found in [Jan87], II.9.

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Define the restricted enveloping algebra $U^{\mathrm{res}}(\mathfrak{g}) = U(\mathfrak{g})/(D^{[p]} - D^p)$ to be its quotient by the twosided ideal generated by the $D^{[p]} - D^p$ for $D \in \mathfrak{g}$. Here we have $D^{[p]} \in \mathfrak{g}$ whereas D^p is the p-th power in $U(\mathfrak{g})$. By definition a restricted representation of $\mathfrak{g}$ is nothing else but a $U^{\mathrm{res}}(\mathfrak{g})$-module, and $U^{\mathrm{res}}(\mathfrak{g})$ obviously is a finite dimensional k-algebra (whose dimension can be seen to be $p^{\dim \mathfrak{g}}$).

Let $X = X(T) = \mathrm{Hom}(T, k^\times)$ be the weight lattice. Any $\nu \in X$ differentiates to give $\bar{\nu} \in \mathfrak{h}^*$. Remark that $\bar{\nu} = \bar{\mu}$ if and only if $\nu - \mu \in pX$. Let $k_{\bar{\nu}}$ be the onedimensional representation of $\mathfrak{h}$ corresponding to $\bar{\nu}$. Now there is unique Lie algebra homomorphism $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$ left inverse to the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{b}$, thus we can consider the $k_{\bar{\nu}}$ also as representations of $\mathfrak{b}$.

One then induces to obtain the so-called "Baby-Verma-modules" $Z(\bar{\nu}) = U^{\mathrm{res}}(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_{\bar{\nu}}$. (The terminology is due to Cline, Parshall, and Scott. Indeed these modules are quite little, namely of finite dimension. In their behaviour however they turn out to be more difficult to control than ordinary Verma modules.) The Baby-Vermas have unique simple quotients $L(\bar{\nu})$, and the map $\nu \mapsto L(\bar{\nu})$ induces a bijection from X/pX to the isomorphism classes of simple restricted representations of $\mathfrak{g}$. The Baby-Verma-modules are well understood, so one might think the whole problem is to compute the Jordan-Hölder multiplicities $[Z(\bar{\lambda}) : L(\bar{\mu})]$ for $\bar{\lambda}, \bar{\mu} \in X/pX$.

However the square matrix of these multiplicities has determinant zero in general, so it does not allow us to translate our knowledge about the Baby-Verma-modules into knowledge about their simple quotients. To overcome this difficulty we will consider G_1T-modules rather than restricted $\mathfrak{g}$-modules. Namely take the adjoint action of T on $\mathfrak{g}$. It induces an X-grading on $\mathfrak{b}, \mathfrak{g}, U(\mathfrak{b}), U(\mathfrak{g}), U^{\mathrm{res}}(\mathfrak{g})$ and it seems natural to consider the category $U^{\mathrm{res}}(\mathfrak{g})\text{-mod}_X$ of all X-graded $U^{\mathrm{res}}(\mathfrak{g})$-modules $M = \bigoplus_{\nu \in X} M_\nu$. For any X-graded object M and any $\mu \in X$ define the shifted X-graded object $M[\mu]$ to have homogeneous parts $(M[\mu])_\nu = M_{\nu-\mu}$.

It turns out that it is even more natural to consider the full subcategory

$$\mathcal{C} = \mathcal{C}_k = \left\{ M \in U^{\mathrm{res}}(\mathfrak{g})\text{-mod}_X \;\middle|\; \begin{array}{l} M \text{ is finitely generated and} \\ Hm = \bar{\nu}(H)m \\ \forall H \in \mathfrak{h},\ \nu \in X,\ m \in M_\nu. \end{array} \right\}$$

One can show that the category of all finitely generated X-graded $U^{\mathrm{res}}(\mathfrak{g})$-modules breaks up into a direct sum

$$U^{\mathrm{res}}(\mathfrak{g})\text{-mod}_X^{fg} = \bigoplus_\mu \mathcal{C}[\mu]$$

with μ running over (a system of representatives of) the classes X/pX. The category $\mathcal{C}$ can also be identified with the category of finite dimensional representations of the subgroup scheme $G_1T \subset G$ generated by T and the kernel G_1 of the Frobenius. This will have no importance for us. However remark that every finite dimensional rational G-module E restricts to an object of $\mathcal{C}$: We just differentiate the G-action to get a $\mathfrak{g}$-action and take the X-grading coming from the action of T.

If M resp. N are X-graded right resp. left modules over an X-graded ring R, then $M \otimes_R N$ admits a natural X-grading with $(M \otimes_R N)_\lambda$ the image of $\bigoplus_{\nu+\mu=\lambda}(M_\nu \otimes_{\mathbb{Z}} N_\mu)$ in $M \otimes_R N$. For $\nu \in X$ we denote by k_ν the $U(\mathfrak{b})$-module $k_{\bar{\nu}}$ considered as an X-graded object concentrated in degree ν and induce to get the X-graded $U(\mathfrak{g})$-module

$$Z(\nu) = Z_k(\nu) = U^{\text{res}}(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\nu.$$

In this way we associate to every $\nu \in X$ the Baby-Verma-module $Z(\nu) = Z_k(\nu) \in \mathcal{C}$. This comes with a distinguished basis vector $v_\nu = 1 \otimes 1$ of $Z(\nu)_\nu$. Our $Z(\nu)$ has a unique simple quotient $L(\nu) = L_k(\nu) \in \mathcal{C}$. If we forget the X-grading on $L(\nu)$, $Z(\nu)$ we find our old $L(\bar{\nu})$, $Z(\bar{\nu})$.

The map $\nu \mapsto L(\nu)$ determines a bijection between X and the simple isomorphism classes in $\mathcal{C}$. (This is the real reason why it is better to consider $\mathcal{C}$ rather than all of $U^{\text{res}}(\mathfrak{g})\text{-mod}_X$.) The basic problem we want to solve can now be formulated precisely.

Problem 2.1.1. *Compute* $\dim_k L(\lambda)_\nu$ *for any* $\lambda, \nu \in X$.

Consider the group ring $\mathbb{Z}[X]$ of the weight lattice. Each $\nu \in X$ gives an element $e^\nu \in \mathbb{Z}[X]$. For $M \in \mathcal{C}$ we form its character $\text{ch}M = \sum_\nu (\dim_k M_\nu) e^\nu \in \mathbb{Z}[X]$. In this language our problem is to compute the characters $\text{ch}L(\lambda)$ of the simple objects.

The characters of our Baby-Verma-modules are easy to compute. One gets

$$\text{ch}Z(\lambda) = ue^\lambda$$

with

$$u = \prod_{\alpha \in R^+} (e^0 + e^{-\alpha} + e^{-2\alpha} + \ldots + e^{-(p-1)\alpha}),$$

where $R^+ \subset X$ are the roots of B. Let us denote for $M \in \mathcal{C}$ by $[M : L(\lambda)] \in \mathbb{N}$ the Jordan-Hölder multiplicity in $\mathcal{C}$. Then we have $[Z(\lambda) : L(\lambda)] = 1$ and $[Z(\lambda) : L(\mu)] \neq 0$ implies $\mu \in \lambda - \mathbb{N}R^+$. Now the obvious equality $\text{ch}Z(\lambda) = \sum_\mu [Z(\lambda) : L(\mu)] \text{ch}L(\mu)$ tells us that our problem can be reformulated as

Problem 2.1.2. *Compute* $[Z(\lambda) : L(\mu)]$ *for any* $\lambda, \mu \in X$.

2.2. **The linkage principle.** ([Jan87], II.9.12.) The so-called linkage principle says that lots of these multiplicities are zero a priori. More precisely let us recall the affine Weyl group $\mathcal{W}$. It is an affine reflection group acting on $X \otimes_{\mathbb{Z}} \mathbb{R}$, the semidirect product of the finite Weyl group W with the translations by the root lattice $\mathbb{Z}R$. For any $l \in \mathbb{N}_{\geq 1}$ we define a new action of $\mathcal{W}$ on X by conjugating the old action first with a homothety by l and then with a translation by the halfsum of positive roots $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. In formulas, we set

$$w \cdot_l \lambda = lw(l^{-1}(\lambda + \rho)) - \rho.$$

Recall p was the characteristic of k.

Theorem 2.2.1. *[Linkage principle]* $[Z(\lambda) : L(\mu)] \neq 0$ *implies* $\mathcal{W} \cdot_p \lambda = \mathcal{W} \cdot_p \mu$, *i.e.* λ *and* μ *are in the same orbit under the* $(\cdot_p)$*-action of the affine Weyl group* $\mathcal{W}$ *on* X.

This already greatly reduces the number of multiplicities we actually have to compute. Using the linkage principle we arrive at the translation principle which further reduces the number of multiplicities we need to compute. This however needs some preparation.

2.3. **Duality.** First we discuss the duality on our category $\mathcal{C} = \mathcal{C}_k$. For any object $M \in \mathcal{C}$ the contragredient representation $M^* = \mathrm{Hom}_k(M, k)$ with its X-grading given by $(M^*)_\nu = (M_{-\nu})^*$ certainly is again an object of $\mathcal{C}$. This duality has the drawback not to fix the isomorphism classes of simple objects. To remedy this inconvenience, we consider a Chevalley antiautomorphism $\tau : \mathfrak{g} \to \mathfrak{g}$, i.e. an involutive antiautomorphism that is the identity on $\mathfrak{h}$ and exchanges the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ for any root α. Then define $M^\tau \in \mathcal{C}$ to be the vector space M^* with the $\mathfrak{g}$-action given by $(Df)(m) = f(\tau(D)m)$ for all $D \in \mathfrak{g}$, $f \in M^*$, $m \in M$ and equip M^τ with the X-grading given by $(M^\tau)_\nu = (M_\nu)^*$. Now $M \mapsto M^\tau$ is a contravariant exact involutive functor $\mathcal{C} \to \mathcal{C}$, and $L(\lambda)^\tau \cong L(\lambda)$ for all $\lambda \in X$. We often write $Z(\lambda)^\tau = Z^\tau(\lambda)$.

It is a very useful fact in the theory that $Z^\tau(\lambda)$ is isomorphic to a "Z-object for the opposite Borel". In particular $Z^\tau(\lambda)$ has a unique simple quotient, $Z(\lambda)$ has a unique simple submodule, and these two are isomorphic to the same simple object, call it $L(\check{\lambda})$.

Lemma 2.3.1. *We have*

$$\mathrm{Ext}^i_{\mathcal{C}}(Z(\lambda), Z^\tau(\mu)) \cong \begin{cases} k & i = 0,\ \lambda = \mu; \\ 0 & \textit{else}, \end{cases}$$

$$\mathrm{Ext}^i_{\mathcal{C}}(Z^\tau(\lambda), Z(\mu)) \cong \begin{cases} k & i = 0,\ \lambda = \mu; \\ 0 & \textit{else}. \end{cases}$$

Proof. We only prove and use the most easy cases $i = 0, 1$. For the general case see for example [Jan87], II.11.1. Note that the second statement is the first for the opposite Borel. So we only treat the first statement. Suppose we are given $M \in \mathcal{C}$ such that $M_\nu = 0$ if $\nu \in \lambda + \mathbb{N}R^+$ and $\nu \neq \lambda$. Then by definition of $Z(\lambda)$ we have an isomorphism $\mathrm{Hom}_{\mathcal{C}}(Z(\lambda), M) \to M_\lambda$ given by $f \mapsto f(v_\lambda)$, and in particular $\mathrm{Ext}^1_{\mathcal{C}}(Z(\lambda), M) = 0$. Thus the first statement is proven for $\mu \notin \lambda + \mathbb{N}R^+$ and for $\mu = \lambda$. In the remaining case $\mu \in \lambda + \mathbb{N}R^+$, $\mu \neq \lambda$ we use duality to identify $\mathrm{Ext}^i_{\mathcal{C}}(Z(\lambda), Z^\tau(\mu)) = \mathrm{Ext}^i_{\mathcal{C}}(Z(\mu), Z^\tau(\lambda))$ and thus reduce to a known case. $\square$

Remark 2.3.2. This proof also shows that $\mathrm{Ext}^1_{\mathcal{C}}(Z(\lambda), Z(\mu)) \neq 0$ implies $\mu \in \lambda + \mathbb{N}R^+$ and $\mu \neq \lambda$.

2.4. **The reciprocity principle.** Every simple module for a finite dimensional algebra admits a projective cover, i.e. an indecomposable projective surjecting onto it, and this projective cover is unique up to nonunique isomorphism. Copying the arguments one sees that the same holds for our category $\mathcal{C}$. Let me describe the known properties of the projective cover $Q(\lambda)$ of a simple object $L(\lambda)$.

First of all $Q(\lambda)$ admits a so-called Z-filtration, i.e. a filtration where all sucessive quotients are isomorphic to some $Z(\mu)$. The multiplicity $(Q(\lambda) : Z(\mu))$ of $Z(\mu)$ occuring as a subquotient is independent of the particular Z-filtration. This is deduced most easily from the fact that the $\mathrm{ch} Z(\lambda)$ are linearly independent. Humphreys [Hum71] showed in this situation the following reciprocity principle:

Theorem 2.4.1. $(Q(\lambda) : Z(\mu)) = [Z(\mu) : L(\lambda)]$.

Proof. For any module $Q \in \mathcal{C}$ with a Z-filtration we have $(Q : Z(\mu)) = \dim_k \mathrm{Hom}_{\mathcal{C}}(Q, Z^\tau(\mu))$ as is easily deduced from Lemma 2.3.1 by induction on the length of the Z-filtration. On the other hand we have for any module $Z \in \mathcal{C}$ the equaltities $[Z : L(\lambda)] = [Z^\tau : L(\lambda)] = \dim_k \mathrm{Hom}_{\mathcal{C}}(Q(\lambda), Z^\tau)$, as is easily deduced by induction on the length of Z. □

The $Q(\lambda)$ are known [Jan87], II.11.5(3) to be selfdual, $Q(\lambda) \cong Q(\lambda)^\tau$. It follows that they are not only the projective covers of the $L(\lambda)$, but their injective hulls as well. It follows furthermore that the $Q(\lambda)$ admit a Z^τ-filtration. For later use we cite from [Jan87], II.11.8 a specially easy case.

Lemma 2.4.2. *Suppose* $\lambda \in pX - \rho$. *Then* $Q(\lambda) = Z(\lambda) = Z^\tau(\lambda) = L(\lambda)$.

2.5. **Alcove geometry.** We want next to explain the translation principle and have to repeat some alcove geometry, see [Bou81]. Let E be a real euclidean space. A group $\mathcal{W}$ of orthogonal transformations of E is called a "reflection group" if and only if (1) $\mathcal{W}$ is generated by the reflections it contains and (2) the family $\mathcal{H} = \{E^s \mid s \in \mathcal{W} \text{ is a reflection}\}$ of reflecting hyperplanes is locally finite.

Since $\mathcal{H}$ is locally finite, the complement of the union of all reflecting hyperplanes $E - \bigcup_{H \in \mathcal{H}} H$ is open. The connected components of this complement are called alcoves. Let $\mathcal{A}$ denote the set of all alcoves. Then for any alcove $A \in \mathcal{A}$ the map $\mathcal{W} \to \mathcal{A}$, $w \mapsto wA$ is a bijection. In other words, the alcoves form a principal homogeneous

$\mathcal{W}$-space. Furthermore if A is an alcove, any $x \in E$ is conjugate under $\mathcal{W}$ to exactly one point y in the closure of A. In other words, $\bar{A}$ is a fundamental domain for the action of $\mathcal{W}$ on E.

More generally remark that each hyperplane $H \subset E$ defines a partition of E into three subsets, namely H itself and two open halfspaces. Consider the smallest partition of E that refines these partitions for every reflecting hyperplane $H \in \mathcal{H}$. The parts of this partition are called "facets". All facets are convex. The open facets are just the alcoves. The closure of any facet is a union of facets.

The facets which are open in some reflecting hyperplane are called walls. Let $\mathcal{S}$ denote the set of $\mathcal{W}$-orbits of walls. The walls in the closure of any given alcove form a system of representatives for $\mathcal{S}$. Any wall lies in the closure of precisely two alcoves. For $A \in \mathcal{A}$ and $s \in \mathcal{S}$ we denote by $As \in \mathcal{A}$ the unique alcove $\neq A$ sharing with A a wall from s.

A point $v \in E$ is called special if (1) it is a facet by itself and (2) for every $H \in \mathcal{H}$ there exists $H' \in \mathcal{H}$ which is parallel to H and contains v. We let $\mathcal{P}$ denote the set of special points.

All the notations introduced above will be applied to the reflection groups $(\mathcal{W}\cdot_p)$ acting on the affine space $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ (equipped with any W-invariant metric). Its reflecting hyperplanes are given by the equations $H_{\alpha,m} = \{\lambda \mid \langle \lambda + \rho, \alpha^\vee \rangle = pm\}$ for $\alpha \in R,\ m \in \mathbb{Z}$. The special points are $\mathcal{P} = pX - \rho$.

To be very precise we should now enrich our notations $\mathcal{A}, \mathcal{S}, \mathcal{H}, \mathcal{P}$ etc. with an index p to indicate their dependence on p. However for any two p the corresponding sets of alcoves etc. are in natural bijection under a suitable homothety of E with center in $(-\rho)$. Thus we hope it will not lead to confusion if we continue to write $\mathcal{A}, \mathcal{S}, \mathcal{H}, \mathcal{P}$ etc. without an index. We will however introduce a notation for the identifications $\mathcal{P} \to pX - \rho$ and call them λ_p. The action of $\mathbb{Z}R \subset \mathcal{W}$ on $\mathcal{A}$ extends to an action of X by translations, given by $A \mapsto A + p\nu$ for $A \in \mathcal{A},\ \nu \in X$.

The choice of positive roots determines a choice of positive halfspace for each reflecting hyperplane $H \in \mathcal{H}$. Namely for $H = H_{\alpha,m}$ with $\alpha \in R^+,\ m \in \mathbb{Z}$ put $H^+ = \{\lambda \mid \langle \lambda + \rho, \alpha^\vee \rangle > pm\}$. This is the halfspace meeting all translates of the dominant Weyl chamber. Call the opposite open halfspace H^-. For a facet F one defines its upper closure

$$\hat{F} = \{\lambda \in \overline{F} \mid \lambda \in H \Rightarrow F \subset \overline{H^-} \text{ for all } H \in \mathcal{H}\}.$$

For every special point $v \in \mathcal{P}$ there is exactly one alcove A such that v lies in its upper closure, $v \in \hat{A}$. Call this alcove $A = v^-$.

Lemma 2.5.1. *Let $A^- \in \mathcal{A}$ be the unique alcove containing $(-\rho)$ in its upper closure and let A be any alcove contained in the "negative Weyl chamber", i.e. $A \subset H^-_{\alpha,0}$ for all $\alpha \in R^+$. Then there exists a sequence of alcoves $A = A_0, A_1, \ldots, A_r = A^-$ such that for suitable walls $s_1, \ldots, s_r \in \mathcal{S}$ we have $A_i = A_{i-1}s_i$ and A_i lies in the positive halfspace of the unique $H \in \mathcal{H}$ separating A_i and A_{i-1}.*

Proof. Certainly we can find a point $x \in A$ such that the line segment $[x, -\rho]$ does not meet any facets except alcoves, walls, and the point $-\rho$. The alcoves met by this line segment obviously form the sequence needed. □

2.6. **The translation principle.** ([Jan87], II.9.19.) The linkage principle together with the reciprocity principle tell us that $[Q(\lambda) : L(\mu)] \neq 0$ implies $\mathcal{W} \cdot_p \lambda = \mathcal{W} \cdot_p \mu$. Thus any $M \in \mathcal{C}$ admits a unique direct sum decomposition $M = \bigoplus_\Omega M_\Omega$ with Ω running over the $(\cdot_p)$-orbits in X, such that $[M_\Omega : L(\lambda)] \neq 0$ implies $\lambda \in \Omega$. Namely M_Ω is the sum of all images of morphisms $Q(\lambda) \to M$ with $\lambda \in \Omega$. We put $\mathcal{C}(\Omega) = \{M \in \mathcal{C} \mid [M : L(\lambda)] \neq 0 \Rightarrow \lambda \in \Omega\}$ and get even a direct sum decomposition of categories $\mathcal{C} = \bigoplus_\Omega \mathcal{C}(\Omega)$. The corresponding projection and injection functors will be denoted by $i_\Omega : \mathcal{C}(\Omega) \to \mathcal{C}$ and $\mathrm{pr}_\Omega : \mathcal{C} \to \mathcal{C}(\Omega),\ M \mapsto M_\Omega$.

After these preparations we come to translation. Let $\Omega, \Gamma \subset X$ be two $(\cdot_p)$-orbits. The translation functor

$$T_\Omega^\Gamma : \mathcal{C}(\Omega) \to \mathcal{C}(\Gamma)$$

is defined as the composition

$$T_\Omega^\Gamma = \mathrm{pr}_\Gamma \circ (E_\Omega^\Gamma \otimes) \circ i_\Omega;$$

Here $E_\Omega^\Gamma \in \mathcal{C}$ is obtained as follows: Let λ and μ be the representatives of Ω and Γ in the closure $\bar{A}$ of a fixed alcove $A \in \mathcal{A}$. Then E_Ω^Γ is the simple G-module with extremal weight $\mu - \lambda$, i.e. with highest weight the dominant member of the Weyl group orbit $(\mu - \lambda)$. We may consider E_Ω^Γ as an object of $\mathcal{C}$. It is clear that for $M, N \in \mathcal{C}$ the tensor product $M \otimes_k N$ with its obvious $\mathfrak{g}$-action and X-grading again lies in $\mathcal{C}$.

The translation functors are exact and commute with our duality. The right (and left) adjoint of T_Ω^Γ is just T_Γ^Ω. From this it follows that if we apply a translation functor to a projective object it stays projective. We say that Γ is on more walls than Ω and write $\Omega \to \Gamma$ if and only if for each $\lambda \in \Omega$ there is some $\mu \in \Gamma$ which lies in the closure of the facet of λ. This μ then is unique and we denote it by λ_Γ.

Proposition 2.6.1. *Suppose Γ is on more walls than Ω.*

1.

$$T_\Omega^\Gamma Z(\lambda) \cong Z(\lambda_\Gamma) \quad \forall \lambda \in \Omega;$$

$$T_\Omega^\Gamma L(\lambda) = \begin{cases} L(\lambda_\Gamma) & \textit{if } \lambda_\Gamma \textit{ is in the upper} \\ & \textit{closure of the facet of } \lambda; \\ 0 & \textit{else.} \end{cases}$$

2. *$T_\Gamma^\Omega Z(\mu)$ has a Z-filtration with multiplicities*

$$(T_\Gamma^\Omega Z(\mu) : Z(\lambda)) = \begin{cases} 1 & \textit{if } \lambda \in \Omega,\ \lambda_\Gamma = \mu; \\ 0 & \textit{else.} \end{cases}$$

If $\Omega \to \Gamma \to \Omega$, then $\lambda \mapsto \lambda_\Gamma$ is in fact a bijection from Ω to Γ and $T_\Omega^\Gamma : \mathcal{C}(\Omega) \to \mathcal{C}(\Gamma)$ is an equivalence of categories.

A $(\cdot_p)$-orbit Ω is called regular if it does not meet any reflecting hyperplane. A regular orbit meets any alcove in exactly one point. The following are equivalent for $l \in \mathbb{N}_{\geq 1}$:

1. There exists a regular $(\cdot_l)$-orbit $\Omega \subset X$.
2. The $(\cdot_l)$-orbit of $0 \in X$ is regular.
3. l is at least the Coxeter number h of our root system.

This fact is the reason for the importance of the Coxeter number in our approach. Remark that the formulas for translation functors tell us how to determine all multiplicities $[Z(\lambda) : L(\mu)]$ with $\lambda, \mu \in X$ once we know them for all λ, μ in a given regular orbit.

2.7. **Wall-crossing.** Let now $\Omega \subset X$ be a regular orbit. For every $s \in \mathcal{S}$ we define a wall-crossing functor

$$\Theta_s : \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$$

as follows: Choose a $(\cdot_p)$-orbit $\Gamma = \Gamma(s) \subset X$ which lies in s. This is always possible when there exist regular orbits, see [Jan87], II.6.3(1). Then put $\Theta_s = T_\Gamma^\Omega \circ T_\Omega^\Gamma$. One can show that this functor does not depend on the choice of Γ up to natural equivalence. However we won't use this fact. From the properties of translation functors recalled above we deduce that Θ_s is exact, selfadjoint, commutes with the duality and maps projectives to projectives. We deduce in addition from Proposition 2.6.1 that for $\lambda \in \Omega$ the module $\Theta_s Z(\lambda)$ has a two-step filtration with subquotients $Z(\lambda)$ and $Z(\lambda')$, where $\lambda' \in \Omega$ is determined by the condition $\{\lambda, \lambda'\} = \{\mu \in \Omega \mid \mu_\Gamma = \lambda_\Gamma\}$.

Let us reformulate this result in other notations. If $p = \mathrm{char} k \geq h$ we associate to every alcove $A \in \mathcal{A}$ the Baby-Verma $Z_k(A) = Z_k(\lambda)$, where $\lambda \in X$ is given by $\{\lambda\} = (\mathcal{W} \cdot_p 0) \cap A$. We form furthermore its unique simple quotient $L_k(A) = L_k(\lambda)$ and denote its projective cover by $Q_k(A) = Q_k(\lambda)$. Then $\Theta_s Z_k(A) \cong \Theta_s Z_k(As)$ has a twostep filtration with subquotients $Z_k(A), Z_k(As)$, and if A lies below the hyperplane $H \in \mathcal{H}$ which separates it from As there is a short exact sequence

$$Z_k(As) \hookrightarrow \Theta_s Z_k(A) \twoheadrightarrow Z_k(A),$$

as is easily deduced from remark 2.3.2. In addition we claim that in this setup $\Theta_s L_k(As) = 0$ and $[\Theta_s L_k(A) : L_k(As)] \neq 0$. Here the first statement follows directly from Proposition 2.6.1, whereas the second one can be deduced from

$$\mathrm{Hom}_{\mathcal{C}}(Z_k(As), \Theta_s L_k(A)) = \mathrm{Hom}_{\mathcal{C}}(T_\Omega^\Gamma Z_k(As), T_\Omega^\Gamma L_k(A)) = k \neq 0.$$

We continue to assume $p \geq h$. We then write T^{out} for the translation functor from any orbit to the regular orbit $\mathcal{W} \cdot_p 0$. For any special point $v \in \mathcal{P}$ let us abbreviate $Z_k(\lambda_p(v)) = Z_k(v)$.

Lemma 2.7.1. *We have $Q_k(v^-) \cong T^{\mathrm{out}} Z_k(v)$. In particular we find $(Q_k(v^-) : Z_k(A)) = 1$ if $v \in \bar{A}$, zero else.*

Proof. The second statement follows from the first (using Proposition 2.6.1 (ii)). Let Ω, Γ be the $(\cdot_p)$-orbits of $0, v$. By Lemma 2.4.2 the Baby-Verma $Z_k(v)$ is projective, hence so is $T^{\mathrm{out}} Z_k(v) = T_\Gamma^\Omega Z_k(\lambda_p(v))$. From Proposition 2.6.1 (i) we deduce $T_\Omega^\Gamma L_k(A) \cong L_k(\lambda_p(v))$ if $v \in \hat{A}$, i.e. $A = v^-$, and zero else. We deduce

$$\mathrm{Hom}_{\mathcal{C}}(T_\Gamma^\Omega Z_k(v), L_k(A)) = \mathrm{Hom}_{\mathcal{C}}(Z_k(v), T_\Omega^\Gamma L_k(A)) = \begin{cases} k & \text{if } A = v^-; \\ 0 & \text{else.} \end{cases}$$

Thus indeed $T^{\mathrm{out}} Z_k(v) \cong Q_k(v^-)$. $\square$

2.8. **The quantum analogue.** ([AJS94], 2–7 with $A = k$.) The theory exposed in the preceding sections has a complete analogue in characteristic zero. Namely suppose given a root system R; a system of a simple roots $\Delta \subset R$; an integer $p > 1$ prime to all entries of the Cartan matrix, hence odd; and a primitive p-th root of unity ζ. To these data Lusztig [Lus90b], 8.2 associates the "restricted quantum group", a finite dimensional $\mathbb{Z}R$-graded algebra $\mathbf{u}$ over $k = \mathbb{Q}(\zeta)$.

Its degree-zero part contains the group ring of $\mathbb{Z}R/2p\mathbb{Z}R$ and for $\alpha \in R$ we denote by $K_\alpha \in \mathbf{u}$ the corresponding element. In this case we put

$$\mathcal{C} = \mathcal{C}_k = \left\{ M \in \mathbf{u}\text{-mod}_X \;\middle|\; \begin{array}{l} M \text{ is finitely generated and} \\ K_\alpha m = \zeta^{d_\alpha \langle \nu, \alpha^\vee \rangle} m \\ \forall\, \alpha \in \Delta, \nu \in X, m \in M_\nu \end{array} \right\}$$

where as usual $\{d_\alpha\}_{\alpha \in \Delta}$ are the unique integers in $\{1, 2, 3\}$ such that the matrix $(d_\alpha \langle \beta, \alpha^\vee \rangle)_{\alpha, \beta \in \Delta}$ is symmetric and each irreducible component of R contains an $\alpha \in \Delta$ with $d_\alpha = 1$. The theory explained in the previous sections has a complete analogue in in this setting, just some proofs and constructions become opaque. For example one can construct for any $\lambda \in X$ an object $Z(\lambda) = Z_k(\lambda) \in \mathcal{C}_k$ such that

1. $\dim Z(\lambda)_\lambda = 1$
2. $Z(\lambda)_\nu \neq 0 \Rightarrow \nu \in \lambda - \mathbb{N}R^+$ and
3. if $M \in \mathcal{C}$ is such that $M_\nu = 0$ whenever $\nu \in \lambda + \mathbb{N}R^+$ and $\nu \neq \lambda$, then the map $\mathrm{Hom}_{\mathcal{C}}(Z(\lambda), M) \to M_\lambda$, $f \mapsto f(v_\lambda)$ is a bijection, for any basis v_λ of $Z(\lambda)_\lambda$.

Clearly the pair $(Z(\lambda), v_\lambda)$ is characterized by these conditions up to unique isomorphism. Again the $Z(\lambda)$ have unique simple quotients $L(\lambda)$, these $L(\lambda)$ are just the simples of $\mathcal{C}$, the linkage principle and reciprocity principle hold, one can construct a duality and translation functors with the same properties: The only change is that we have to consider the $(\cdot_p)$-action of $\mathcal{W}$ on X, where p is the integer involved in the construction of $\mathbf{u}$. For $p \geq h$ we again associate to every alcove A the objects $Z_k(A), L_k(A)$ and $Q_k(A)$ to be just $Z_k(\lambda), L_k(\lambda), Q_k(\lambda)$ with λ the unique representative of $\mathcal{W} \cdot_p 0$ in A.

3. The main results

3.1. **Independence of the characteristic.** Let us start to explain independence of the characteristic in the simplest context. If A is a finite dimensional (associative unitary) algebra over a field, it may decompose into a product of subalgebras, $A = \prod_B B$. There is a unique maximal such decomposition and the corresponding subalgebras B are called the blocks of A. Any indecomposable A-module M is annihilated by all blocks except one: the block of M. In a representation theoretic context the block of the trivial onedimensional representation is called the "principal block".

Let us also recall the notion of Morita equivalence. Let k be a field. Two finite dimensional k-algebras are said to be Morita-equivalent if and only if their respective categories of finite dimensional modules are equivalent k-categories. For example the matrix algebras of various sizes are all Morita equivalent to the ground field.

Fix now a root system R with Coxeter number h. Put $\mathbb{Z}' = \mathbb{Z}[n^{-1} \mid 1 < n < h]$. The following is [AJS94], 16.18.

Theorem 3.1.1. *There exists a $\mathbb{Z}'$-algebra $\mathcal{E}$, free of finite rank as a $\mathbb{Z}'$-module, such that*

1. *for any algebraically closed field k of characteristic* $\mathrm{char} k > h$ *the k-algebra $\mathcal{E} \otimes_{\mathbb{Z}} k$ is Morita equivalent to the principal block of the restricted enveloping algebra $U^{\mathrm{res}}(\mathfrak{g})$ over k corresponding to R,*

2. *for any integer $p > h$ prime to all entries of the Cartan matrix of R and any primitive p-th root of unity $\zeta \in \mathbb{C}$, the $\mathbb{Q}(\zeta)$-algebra $\mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Q}(\zeta)$ is Morita equivalent to the principal block of the restricted quantum group $\mathbf{u}$ over $\mathbb{Q}(\zeta)$ corresponding to R.*

This is not easy to prove, so let us start with a simpler task in form of a lemma. Although this lemma is not needed for the logic of our argument, we hope that its proof will clarify our strategy.

Lemma 3.1.2. *We can associate to each algebraically closed field k of characteristic $\operatorname{char} k > h$ a k-algebra $\mathcal{E}_k$ which is Morita equivalent to the principal block of the corresponding $U^{\mathrm{res}}(\mathfrak{g})$, in such a way that $\dim_k \mathcal{E}_k$ is independent of k.*

Proof. Consider our category $\mathcal{C} = \mathcal{C}_k$ and the $(\cdot_p)$-orbit $\Omega = \Omega_k = \mathcal{W} \cdot_p 0 \subset X$. Since we assumed $p > h$ it is regular ($p \geq h$ would be enough). Consider $Q_k = Q_k(A^-) = T^{\mathrm{out}} Z_k(-\rho)$. For any sequence $\sigma : [1, n] \to \mathcal{S}$ consider the iterated wall crossing functor $\Theta_\sigma = \Theta_{\sigma(1)} \dots \Theta_{\sigma(n)}$. Let $|\sigma|$ denote the length n of σ. We claim that for sufficiently big N the k-algebras $\mathcal{E}_k = \mathcal{E}_{k,N} = \operatorname{End}_{\mathfrak{g}}(\bigoplus_{|\sigma| \leq N} \Theta_\sigma Q_k)^{\mathrm{opp}}$ meet our requirements. To establish this we need to prove

1. There exists $N \gg 0$ such that for all k with $\operatorname{char} k > h$ the k-algebra $\mathcal{E}_{k,N}$ is Morita equivalent to the principal block of $U^{\mathrm{res}}(\mathfrak{g})$.
2. The dimension $\dim_k \mathcal{E}_{k,N}$ is independent of k.

We start with (i). It is known that the simple modules of the principal block of $U^{\mathrm{res}}(\mathfrak{g})$ are the $L(\overline{x \cdot 0})$ for $x \in W$. In particular, any $M \in \mathcal{C}(\Omega)$ belongs to the principal block of $U^{\mathrm{res}}(\mathfrak{g})$ when we forget the X-grading on M. It is also clear that any projective object in $\mathcal{C}$ is projective as an $U^{\mathrm{res}}(\mathfrak{g})$-module. Since $Q_k = Q_k(A^-)$ is projective so are all $\Theta_\sigma Q_k$. Thus it only remains to show that for big N our sum $\bigoplus_{|\sigma| \leq N} \Theta_\sigma Q_k$ is a generator of the principal block of $U^{\mathrm{res}}(\mathfrak{g})$, i.e. that for each $x \in W$ the simple object $L(\overline{x \cdot 0})$ is a quotient of some $\Theta_\sigma Q_k$. We now apply Lemma 2.5.1. Choose an alcove A in the negative Weyl chamber such that $x \cdot 0 + p\mathbb{Z}R$ meets A. Choose a sequence $A = A_0, A_1, \dots , A_r = A^-$ of alcoves as in the lemma, with $A_i = A_{i-1} s_i$ for some $s_i \in \mathcal{S}$. Section 2.7 tells us that

$$[\Theta_{s_r} \dots \Theta_{s_1} L_k(A) : L_k(A^-)] \neq 0,$$

thus

$$0 \neq \operatorname{Hom}_{\mathcal{C}}(Q_k, \Theta_{s_r} \dots \Theta_{s_1} L_k(A)) = \operatorname{Hom}_{\mathcal{C}}(\Theta_{s_1} \dots \Theta_{s_r} Q_k, L_k(A))$$

and thus indeed $\operatorname{Hom}_{\mathfrak{g}}(\Theta_\sigma Q_k,\ L(\overline{x \cdot 0})) \neq 0$ for a suitable sequence σ.

Next comes (ii). First we claim that for any $M, N \in \mathcal{C}(\Omega)$ we have

$$\mathrm{Hom}_{\mathfrak{g}}(M,N) = \bigoplus_{\nu \in \mathbb{Z}R} \mathrm{Hom}_{\mathcal{C}}(M, N[p\nu]).$$

From the breakup $U^{\mathrm{res}}(\mathfrak{g})\text{-mod}_X^{fg} = \bigoplus \mathcal{C}[\mu]$ with μ running over X/pX it is clear that we have $\mathrm{Hom}_{\mathfrak{g}}(M,N) = \bigoplus_{\nu \in X} \mathrm{Hom}_{\mathcal{C}}(M, N[p\nu])$. For $\nu \in X$ one verifies easily that $Z(\lambda)[p\nu] \cong Z(\lambda + p\nu), L(\lambda)[p\nu] \cong L(\lambda + p\nu)$ and thus $\mathcal{C}(\Omega)[p\nu] = \mathcal{C}(\Omega + p\nu)$. Since we required $p > h$ we know that $\Omega + p\nu = \Omega$ implies $\nu \in \mathbb{Z}R$. So indeed $\mathrm{Hom}_{\mathcal{C}}(M, N[p\nu]) = 0$ for $\nu \notin \mathbb{Z}R$.

It remains to be shown that for any two sequences σ, τ and $\nu \in \mathbb{Z}R$ the dimension of $\mathrm{Hom}_{\mathcal{C}}(\Theta_\sigma Q_k, \Theta_\tau Q_k[p\nu])$ is independent of k. But remark that for any two projectives $P, Q \in \mathcal{C}$ we have $\dim \mathrm{Hom}_{\mathcal{C}}(P,Q) = \sum_\lambda (Q : Z(\lambda))(P : Z(\lambda))$, as can be seen from the reciprocity principle: Namely it is enough to show this equality for indecomposable P, e.g. $P = Q(\nu)$, and then we calculate

$$\begin{array}{rcl} \dim \mathrm{Hom}_{\mathcal{C}}(Q(\nu), Q) & = & [Q : L(\nu)] \\ & = & \sum_\lambda (Q : Z(\lambda))[Z(\lambda) : L(\nu)] \\ & = & \sum_\lambda (Q : Z(\lambda))(Q(\nu) : Z(\lambda)). \end{array}$$

(Another way would be to use selfduality $P^\tau \cong P$ along with Lemma 2.3.1.) But it is clear from the properties of translation functors that the Verma-flag multiplicities $(\Theta_\sigma Q_k : Z_k(A))$ depend only on σ and A, not on p or k. So indeed $\dim_k \mathcal{E}_{k,N}$ is independent of k. □

We are now in position to explain precisely what independence of the characteristic we can prove. For our root system R with Coxeter number h let us introduce the class of fields $\mathrm{Bon}(R)$ consisting of

1. all algebraically closed fields k of characteristic $p = \mathrm{char} k \geq h$ and
2. all fields $k = \mathbb{Q}(\zeta)$ where ζ is a primitive p-th root of unity in $\mathbb{C}$ such that $p \geq h$ and p is prime to all entries of the Cartan matrix of R.

To any $k \in \mathrm{Bon}(R)$ is associated a number $p = p(k)$. Let I be a finite set consisting of pairs (σ, v) where σ is a finite sequence in $\mathcal{S}$ and $v \in \mathcal{P}$ is a special point. For any $k \in \mathrm{Bon}(R)$ we can construct the k^I-algebra

$$\mathcal{B}_k = \mathrm{End}_{\mathcal{C}}(\bigoplus \Theta_\sigma Q_k(v^-))$$

where we sum over all $(\sigma, v) \in I$.

Theorem 3.1.3. *There exists a $(\mathbb{Z}')^I$-algebra $\mathcal{B}$, free of finite rank as a $\mathbb{Z}'$-module, such that $\mathcal{B}_k \cong \mathcal{B} \otimes_{\mathbb{Z}} k$ for all $k \in \mathrm{Bon}(R)$.*

We also give a more precise variant, so that the reader may deduce Theorem 3.1.1. Define for $M \in \mathcal{C}$ the $\mathbb{Z}R$-graded ring $\mathrm{End}^\sharp_{\mathcal{C}} M = \bigoplus_{\nu \in \mathbb{Z}R} \mathrm{Hom}_{\mathcal{C}}(M, M[p\nu])$. Consider the $\mathbb{Z}R$-graded k^I-algebra

$$\mathcal{E}_k = \mathrm{End}^\sharp_{\mathcal{C}}(\bigoplus \Theta_\sigma Q_k(v^-)).$$

Theorem 3.1.4. *There exists a $\mathbb{Z}R$-graded $(\mathbb{Z}')^I$-algebra $\mathcal{E}$, free of finite rank as a $\mathbb{Z}'$-module, such that $\mathcal{E}_k \cong \mathcal{E} \otimes_{\mathbb{Z}} k$ for all $k \in \mathrm{Bon}(R)$.*

The remainder of this article splits into two independent parts. In the following subsection we will explore the consequences of our theorems. After that we will start a new section to explain the proof.

3.2. **Composition series of Verma modules.** The following theorem is what we were really after.

Theorem 3.2.1. *For any root system R there exists a bound $b(R)$ such that over all $k \in \mathrm{Bon}(R)$ with $\mathrm{char} k = 0$ or $\mathrm{char} k \geq b(R)$ the multiplicities $[Z_k(B) : L_k(A)]$ are constant in k and depend only on the alcoves $A, B \in \mathcal{A}$.*

The point is that these multiplicities are known for $\mathrm{char} k = 0$ in the simply laced case. Using the theorem we can get them for $\mathrm{char} k \geq b(R)$ as well.

Proof. Recall that $Z_k(\lambda)[p\nu] = Z_k(\lambda + p\nu)$, $L_k(\lambda)[p\nu] = L_k(\lambda + p\nu)$ for all $\nu \in X$. Thus it is sufficient to compute $[Z_k(\lambda) : L_k(\mu)]$ for all μ in a system of representatives of X/pX. Recall furthermore that by the translation principle if we know the multiplicities $[Z_k(\lambda) : L_k(\mu)]$ for some regular μ, we know them for all μ in the closure of its alcove. Thus it suffices to compute the $[Z_k(B) : L_k(A)]$ with A running over a system of representatives of the X-orbits in $\mathcal{A}$.

As shown in [AJS94], 16.20 it is possible to find alcoves $A_1, \dots, A_r \in \mathcal{A}$ and special points $v_1 \dots, v_r \in \mathcal{P}$ and sequences $\sigma_1, \dots, \sigma_r$ in $\mathcal{S}$ such that

$$\Theta_{\sigma_i} Q_k(v_i^-) \cong Q_k(A_i) \oplus \bigoplus_{j<i} Q_k(A_j)^{m_k(i,j)}$$

for suitable $m_k(i,j)$, and such that each X-orbit in $\mathcal{A}$ contains at least one of the A_i. Now we claim that it suffices to show that the $m_k(i,j)$ are independent of k, for $\mathrm{char} k = 0$ or $\mathrm{char} k \gg 0$. Indeed, since by subsection 2.7 the Verma flag multiplicities on the left hand side are independent of k, this would mean that by induction on i the Verma flag multipicities $(Q_k(A_i) : Z_k(B))$ are independent of k as well, for $\mathrm{char} k = 0$ or $\mathrm{char} k \gg 0$, and the theorem would follow via the reciprocity principle.

So why are the $m_k(i,j)$ independent of k, for $\mathrm{char} k = 0$ or $\mathrm{char} k \gg 0$? We will deduce this from Theorem 3.1.3 and have to recall some generalities from ring theory. For a finite dimensional algebra A over a field k the following statements are equivalent:

1. The only idempotents are $0, 1$ and $0 \neq 1$.
2. A is indecomposable when considered as a right A-module.
3. A is indecomposable when considered as a left A-module.
4. A is local, i.e. all noninvertible elements form an ideal (and $1 \neq 0$).

As the basic example, note that a finite length object M of some abelian category is indecomposable if and only if its endomorphism ring is local [Ben91], 1.4.5. Let us call A "absolutely local" if it stays local under any field extension k'/k.

Suppose now $\mathfrak{o} \supset \mathfrak{m}$ is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and A is an $\mathfrak{o}$-algebra, free of finite rank as an $\mathfrak{o}$-module.

Lemma 3.2.2. *If $A \otimes_{\mathfrak{o}} \mathrm{Quot}\mathfrak{o}$ is absolutely local, then $A \otimes_{\mathfrak{o}} (\mathfrak{o}/\mathfrak{m})$ is local.*

Proof. This is a direct consequence of "lifting of idempotents". Namely consider the $\mathfrak{m}$-adic completion $\hat{\mathfrak{o}}$ of $\mathfrak{o}$. It is known ([Ben91], 1.9.4) that each idempotent e of $A \otimes_{\mathfrak{o}} (\mathfrak{o}/\mathfrak{m})$ lifts to an idempotent of $A \otimes_{\mathfrak{o}} \hat{\mathfrak{o}}$ and thus leads to an idempotent in $A \otimes_{\mathfrak{o}} \mathrm{Quot}\hat{\mathfrak{o}}$. This has to be zero or one, by assumption. Thus e itself is zero or one. □

An idempotent e of a ring is called "primitive" if it is nonzero and does not admit a decomposition $e = e_1 + e_2$ into two nonzero idempotents that are orthogonal, i.e. satisfy $e_1 e_2 = e_2 e_1 = 0$. As the basic example, if M is a finite length object in some abelian category, then $e \in \mathrm{End} M$ is a primitive idempotent if and only if it is a projection onto an indecomposable summand of M. If our ring is an algebra over a field, call e "absolutely primitive" if it stays primitive under any field extension. Let now $\mathfrak{o} \supset \mathfrak{m}$ and A be as in the previous lemma.

Lemma 3.2.3. *Let $e \in A$ be an idempotent. If the image of e in $A \otimes_{\mathfrak{o}} \mathrm{Quot}\mathfrak{o}$ is absolutely primitive, then the image of e in $A \otimes_{\mathfrak{o}} (\mathfrak{o}/\mathfrak{m})$ is primitive.*

Proof. By our definitions, an idempotent e of a finite dimensional algebra B over a field is (absolutely) primitive if and only if the algebra eBe is (absolutely) local. The lemma now follows from the previous one. □

To finish the preparations, let M be any object of an additive category with a decomposition $M = \bigoplus_{i=1}^n N_i$. Let $\pi_i \in \mathrm{End} M$ be the corresponding projections. Then we have $N_i \cong N_j$ if and only if π_i is conjugate to π_j by a unit in $\mathrm{End}M$.

Now we can explain why the $m_k(i,j)$ do not depend on k for $\mathrm{char}k = 0$ or $\mathrm{char}k \gg 0$. Namely consider the k-algebra $k^n = k \times \ldots \times k$. We write 1_i for the idempotent $(0, \ldots, 1, \ldots, 0)$ with a one at the i-th place. Consider the k^n-algebra $\mathcal{B}_k = \mathrm{End}_{\mathcal{C}}(\bigoplus_{i=1}^n \Theta_{\sigma_i} Q_k(v_i^-))$, where $1_i \in k^n$ acts as projection onto the i-th factor. The $m_k(i,j)$ can be read off from the k^n-algebra $\mathcal{B}_k$ as follows: Write each $1_i \in \mathcal{B}_k$ as a sum $1_i = e_i^0 + \ldots + e_i^{r_i}$ of pairwise orthogonal primitive idempotents. Exactly one of the e_i^s, say e_i^0, won't be conjugate to any e_j^t with $j < i$. (One should think of e_i^0 as the projection of $\Theta_{\sigma_i} Q_k(v_i^-)$ onto $Q_k(A_i)$.) Now $m_k(i,j)$ is the number of idempotents among the e_i^s which are conjugate to e_j^0.

By Theorem 3.1.3 we have a $(\mathbb{Z}')^n$-algebra $\mathcal{B}$, free of finite rank as a $\mathbb{Z}'$-module, such that $\mathcal{B} \otimes_{\mathbb{Z}} k \cong \mathcal{B}_k$ for all our k. Choose now for some $k = \mathbb{Q}(\zeta)$ a system of e_i^s. For suitable $b \in \mathbb{Z}$, all e_i^s and all conjugacy relations among them come from elements $\tilde{e}_i^s \in \mathcal{B} \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta, b^{-1}]$ and conjugacy relations in this ring. The $e_i^s \in \mathcal{B}_{\mathbb{Q}(\zeta)}$ are absolutely primitive, since the endomorphisms of simple objects of $\mathcal{C}_{\mathbb{Q}(\zeta)}$ are reduced to scalars. We can then deduce from Lemma 3.2.3 that the $\tilde{e}_i^s$ give primitive idempotents $\bar{e}_i^s$ in $\mathcal{B}_k$ whenever b is invertible in k. Clearly the $\bar{e}_i^s$ are conjugate if the e_i^s are. But there can be no additional conjugacy relations among the $\bar{e}_i^s$, since the number n of conjugacy classes of primitive idempotents in $\mathcal{B}_k$ is independent of k. From the description of the $m_k(i,j)$ in terms of the structure of the k^n-algebra $\mathcal{B}_k$ given above it is now clear that $m_k(i,j)$ is independent of k over all $k \in \mathrm{Bon}(R)$ such that b is invertible in k. $\square$

4. Combinatorics

4.1. **The deformed representation categories.** At the basis of our whole work is the idea to "deform" the category $\mathcal{C}$, thus to think of the $Z(\lambda)$ as embedded in a whole family of objects $Z(\lambda + \varepsilon)$ for "small" ε in $\mathfrak{h}^*$. These $Z(\lambda + \varepsilon)$ however will no longer be restricted $\mathfrak{g}$-modules resp. $\mathbf{u}$-modules. In fact, the whole category $\mathcal{C}$ can be viewed as the special fibre at $\varepsilon = 0$ of a flat family $\mathcal{C}_\varepsilon$. For generic ε this category is semisimple with simple objects the $Z(\lambda + \varepsilon)$, so everything is very easy. For subgeneric ε (fixed by just one reflection of W) the category $\mathcal{C}_\varepsilon$ is still tractable and looks like an $\mathfrak{sl}_2$-category. Since we have a flat family, once we know it for all ε on at most one reflecting hyperplane, we know it everywhere and can recover the most singular fibre at the point $\varepsilon = 0$, which is just our category $\mathcal{C}$.

We now put this idea into formulas. Let us treat the case of positive characteristic first. Here instead of the restricted enveloping algebra $U^{\mathrm{res}}(\mathfrak{g})$ we consider the bigger algebra $U = U(\mathfrak{g})/I$ where $I \subset U(\mathfrak{g})$ is the twosided ideal generated by all $D^{[p]} - D^p$ for $D \in \mathfrak{g}$ root vectors, i.e. $D \in \mathfrak{g}_\alpha$ for some $\alpha \in R$. The algebra U inherits an X-grading from $U(\mathfrak{g})$. Let $S(\mathfrak{h}) = S$ be the symmetric algebra over $\mathfrak{h}$. The injection $S \hookrightarrow U(\mathfrak{g})$ gives us an injection $S \hookrightarrow U$, and U is a free S-module of rank $p^{\dim \mathfrak{g} - \dim \mathfrak{h}}$. For later use remark also that the map $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$ leads to a morphism $U(\mathfrak{b}) \to S = U(\mathfrak{h})$.

Let A be any commutative noetherian S-algebra with structural homomorphism $\pi : S \to A$. We consider A as an X-graded algebra concentrated in degree zero and define the deformation category $\mathcal{C}_A$ to consist of all finitely generated X-graded U-A-bimodules M such that

$$Hm = m(\pi(H) + \bar{\nu}(H))$$

for all $H \in \mathfrak{h},\ \nu \in X,\ m \in M_\nu$. As before we denote here by $\bar{\nu} \in \mathfrak{h}^*$ the differential of $\nu \in X$. It is clear that for $A = k$ with π the augmentation we get back our category $\mathcal{C}_k$. In $\mathcal{C}_A$ we define for any $\lambda \in X$ the "deformed Baby-Verma with highest weight λ" by the formula

$$Z_A(\lambda) = U \otimes_{U(\mathfrak{b})} (k_\lambda \otimes_k A).$$

Here $U(\mathfrak{b})$ acts on k_λ as explained before, on A via the composition $U(\mathfrak{b}) \to S \to A$, and $(k_\lambda \otimes_k A)$ is understood to be the tensor product representation. The right action of A is understood to just affect the third factor of this tensor product, and the X-grading is determined by giving $(k_\lambda \otimes_k A)$ the degree λ.

The quantum case is similar and we refer to [AJS94], 1.3 for the details. Let us just note that one constructs in analogy with the case of positive characteristic a $\mathbb{Z}R$-graded $\mathbb{Q}(\zeta)$-algebra U along with a homogeneous surjection $U \twoheadrightarrow \mathbf{u}$. The degree-zero part of U contains the group ring of $\mathbb{Z}R$ and we denote by $K_\alpha \in U$ the element corrresponding to $\alpha \in R$. Consider the ring of Laurent polynomials $S = \mathbb{Q}(\zeta)[K_\alpha, K_\alpha^{-1}] \subset U$. One might view S as the ring of regular functions on a torus. For any commutative noetherian S-algebra A with structural morphism $\pi : S \to A$ we define $\mathcal{C}_A$ to be the category of all finitely generated X-graded U-A-bimodules M such that $K_\alpha m = m(\pi(K_\alpha)\zeta^{d_\alpha \langle \nu, \alpha^\vee \rangle})$ for all $\alpha \in R,\ \nu \in X,\ m \in M_\nu$. For $A = \mathbb{Q}(\zeta) = k$ with π the "evaluation at the identity", i.e. $\pi(K_\alpha) = 1$ for all α, we just get back our old $\mathcal{C}_k$.

Homomorphisms in $\mathcal{C}_A$ will just be written $\mathrm{Hom}_{\mathcal{C}}$. If A' is a commutative noetherian A-algebra, we have a base change functor $\otimes_A A' :$

$\mathcal{C}_A \to \mathcal{C}_{A'}$ that maps $Z_A(\lambda)$ to $Z_{A'}(\lambda)$. The following proposition collects some basic properties of our deformation categories. For the proof we refer to [AJS94], 2.13 and section 3.

Proposition 4.1.1. *Let A, A' be as above.*

1. *There are enough projectives in $\mathcal{C}_A$.*
2. *If $P, Q \in \mathcal{C}_A$ are projective, then $\mathrm{Hom}_{\mathcal{C}}(P,Q)$ is a projective A-module.*
3. *If $P \in \mathcal{C}_A$ is projective, so is $P \otimes_A A' \in \mathcal{C}_{A'}$.*
4. *If $P, Q \in \mathcal{C}_A$ are projective, the natural map*
$$\mathrm{Hom}_{\mathcal{C}}(P,Q) \otimes_A A' \to \mathrm{Hom}_{\mathcal{C}}(P \otimes_A A', Q \otimes_A A')$$
is an isomorphism.

Any A determines a partition of X into the so called A-blocks (see [AJS94], section 6). We define this to be the finest possible partition such that λ and μ belong to the same A-block if $\mathrm{Hom}_{\mathcal{C}}(Z_A(\lambda), Z_A(\mu))$ or $\mathrm{Ext}^1_{\mathcal{C}}(Z_A(\lambda), Z_A(\mu))$ do not vanish. We can then prove that we have a decomposition of categories $\mathcal{C}_A = \bigoplus \mathcal{C}_A(\Omega)$ with Ω running over the A-blocks, where $\mathcal{C}_A(\Omega)$ denotes the category of all quotients of objects with a Z_A-filtration such that all subquotients $Z_A(\lambda)$ have parameter $\lambda \in \Omega$. We still use the notation $\mathcal{C}_A(\Omega)$ if Ω is a union of A-blocks. If A' is an A-algebra, each A-block is a union of A'-blocks. Remark that the partition of X into k-blocks refines the partition into $(\cdot_p)$-orbits. For example each special point is a k-block by itself.

Let us consider now the special case where $A = \hat{S}$ is the completion of S at the kernel of our map $S \to k$ from above. We prove that the $\hat{S}$-blocks coincide with the k-blocks. So for any two $(\cdot_p)$-orbits $\Omega, \Gamma \subset X$ we can define as in subsection 2.6 (see [AJS94], section 7) the translation functor $T_\Omega^\Gamma : \mathcal{C}_{\hat{S}}(\Omega) \to \mathcal{C}_{\hat{S}}(\Gamma)$, as the composition $T_\Omega^\Gamma = \mathrm{pr}_\Gamma \circ (E_\Omega^\Gamma \otimes) \circ i_\Omega$ (where we should think a bit why for $E \in \mathcal{C}$, $M \in \mathcal{C}_A$ the tensor product $E \otimes_k M$ is in a natural way an object of $\mathcal{C}_A$) and the functorial diagrams

$$\begin{array}{ccc} \mathcal{C}_{\hat{S}}(\Omega) & \xrightarrow{T_\Omega^\Gamma} & \mathcal{C}_{\hat{S}}(\Gamma) \\ \downarrow{\scriptstyle \otimes_{\hat{S}} k} & & \downarrow{\scriptstyle \otimes_{\hat{S}} k} \\ \mathcal{C}_k(\Omega) & \xrightarrow{T_\Omega^\Gamma} & \mathcal{C}_k(\Gamma) \end{array}$$

commute. Again the left (and right) adjoint of T_Ω^Γ is T_Γ^Ω. Again our translation functors are exact and (by having an exact adjoint functor) transform projectives to projectives. The formulas from Proposition 2.6.1 for translation of Z-objects generalize to our deformed situation without essential changes. Again we can define the wall-crossings $\Theta_s : \mathcal{C}_{\hat{S}}(\Omega) \to \mathcal{C}_{\hat{S}}(\Omega)$ for any regular $(\cdot_p)$-orbit $\Omega \subset X$. For any special point $v \in pX - \rho$ the deformed Verma $Z_{\hat{S}}(v) = Z_{\hat{S}}(\lambda_p(v))$ is projective in $\mathcal{C}_{\hat{S}}$.

Assume now $p \geq h$ and recall our k^I-algebra

$$\mathcal{B}_k = \mathrm{End}_{\mathcal{C}}(\bigoplus \Theta_\sigma Q_k(v^-)) = \mathrm{End}_{\mathcal{C}}(\bigoplus \Theta_\sigma T^{\mathrm{out}} Z_k(v))$$

from the end of subsection 3.1. By Proposition 4.1.1 (iv) we may view it as a special fibre of the $\hat{S}^I$-algebra $\mathcal{B}_{\hat{S}} = \mathrm{End}_{\mathcal{C}}(\bigoplus \Theta_\sigma T^{\mathrm{out}} Z_{\hat{S}}(v))$, in other words $\mathcal{B}_k \cong \mathcal{B}_{\hat{S}} \otimes_{\hat{S}} k$. Remark that by Proposition 4.1.1 (ii) the algebra $\mathcal{B}_{\hat{S}}$ is free over $\hat{S}$, so we really deformed $\mathcal{B}_k$ into a flat family.

Now we have to work with variable k, so we write $\hat{S} = \hat{S}(k)$ to indicate its dependence on k. Let $S(\mathbb{Z})$ denote the symmetric algebra over $\mathbb{Z}R$ and put $S(\mathbb{Z}') = S(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}'$. For $\alpha \in R$ denote by h_α the corresponding element of $S(\mathbb{Z}')$. We define morphisms $S(\mathbb{Z}') \to \hat{S}(k)$ for all $k \in \mathrm{Bon}(R)$: In case $\mathrm{char} k > 0$ map h_α to $d_\alpha H_\alpha$, where $H_\alpha \in \mathfrak{h}$ is the image of the coroot $\alpha^\vee$. In case $\mathrm{char} k = 0$ we map h_α to $\log K_\alpha$ (viewed as power series in $(K_\alpha - 1)$, thus as an element of $\hat{S}(k)$). We deduce Theorem 3.1.3 from the following more detailed

Theorem 4.1.2. *There exists a $S(\mathbb{Z}')^I$-algebra $\mathcal{B}_{S(\mathbb{Z}')}$, free of finite rank as an $S(\mathbb{Z}')$-module, such that*

$$\mathcal{B}_{\hat{S}(k)} \cong \mathcal{B}_{S(\mathbb{Z}')} \otimes_{S(\mathbb{Z}')} \hat{S}(k)$$

for every $k \in \mathrm{Bon}(R)$.

From this we obtain Theorem 3.1.3 just putting $\mathcal{B} = \mathcal{B}_{S(\mathbb{Z}')} \otimes_{S(\mathbb{Z}')} \mathbb{Z}'$. So we should explain how to construct $\mathcal{B}_{S(\mathbb{Z}')}$. The point is to give some combinatorial description for the algebras $\mathcal{B}_{\hat{S}(k)}$. This description will make sense also over $\mathbb{Z}'$ and construct for us the algebra $\mathcal{B}_{S(\mathbb{Z}')}$.

4.2. **The functor into combinatorics.** Let us fix again k and write $\hat{S}(k) = \hat{S}$. The images of the h_α in $\hat{S}$ will be denoted h_α as well. To describe the algebras $\mathcal{B}_{\hat{S}}$ we have to describe homomorphisms between projective objects $P, Q \in \mathcal{C}_{\hat{S}}$. This task we attack as follows. For any $\beta \in R^+$ put $\hat{S}^\beta = \hat{S}[h_\alpha^{-1} \mid \alpha \in R^+, \alpha \neq \beta]$ and define $\hat{S}^\emptyset = \hat{S}[h_\alpha^{-1} \mid \alpha \in R^+]$. If our characteristic is prime to all entries of the Cartan matrix, we have $\hat{S} = \bigcap_{\beta \in R^+} \hat{S}^\beta$ inside $\hat{S}^\emptyset$. For any $\hat{S}$-module M put $M^\beta = M \otimes_{\hat{S}} \hat{S}^\beta$, $M^\emptyset = M \otimes_{\hat{S}} \hat{S}^\emptyset$. Using Proposition 4.1.1 we deduce for any two projectives $P, Q \in \mathcal{C}_{\hat{S}}$

$$\mathrm{Hom}_{\mathcal{C}}(P, Q) = \bigcap_{\beta \in R^+} \mathrm{Hom}_{\mathcal{C}}(P^\beta, Q^\beta) \subset \mathrm{Hom}_{\mathcal{C}}(P^\emptyset, Q^\emptyset).$$

Let us abbreviate $\mathcal{C}_{\hat{S}^\beta} = \mathcal{C}^\beta$ and $Z_{\hat{S}^\beta}(\lambda) = Z^\beta(\lambda)$. Remark that Spec$\hat{S}^\beta$ is just Spec$\hat{S}$ with all walls except the β-wall deleted, so it is not surprising that $\mathcal{C}^\beta$ behaves like a category of representations of $\mathfrak{sl}_2$. More precisely, for every $\lambda \in X$ define $\beta \uparrow \lambda = s_H\lambda$ where H is the lowest reflecting hyperplane orthogonal to β such that λ lies below or on H, i.e. $H = H_{\beta,m}$ where m is minimal with $\langle \lambda + \rho, \beta^\vee \rangle \leq pm$. Then $\lambda \mapsto \beta \uparrow \lambda$ is a bijection. We denote its inverse by $\lambda \mapsto \beta \downarrow \lambda$. In [AJS94], section 8 we prove:

Proposition 4.2.1. 1. *If $\beta \uparrow \lambda = \lambda$ then $Z^\beta(\lambda)$ is projective in $\mathcal{C}^\beta$.*

2. *If $\beta \uparrow \lambda \neq \lambda$ then $\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda))$ is isomorphic to $\hat{S}^\beta / h_\beta \hat{S}^\beta$, and if we represent an extension by a short exact sequence*
$$Z^\beta(\beta \uparrow \lambda) \hookrightarrow Q \twoheadrightarrow Z^\beta(\lambda),$$
then Q is projective in $\mathcal{C}^\beta$ if and only if our extension generates Ext^1 as an $\hat{S}^\beta$-module.
3. *The projectives $Z^\beta(\lambda)$ and Q from (i) and (ii) generate $\mathcal{C}^\beta$.*
4. *The $\hat{S}^\beta$-blocks are just the smallest sets stable under $\beta \uparrow$ and $\beta \downarrow$. In particular they all lie entirely inside a line with direction β.*

The category $\mathcal{C}_{\hat{S}^\emptyset} = \mathcal{C}^\emptyset$ is even simpler. Let us abbreviate $Z_{\hat{S}^\emptyset}(\lambda) = Z^\emptyset(\lambda) \in \mathcal{C}^\emptyset$. As proved in [AJS94], the remark closing 6.13 we have

Proposition 4.2.2. *The $Z^\emptyset(\lambda)$ are all projective and generate $\mathcal{C}^\emptyset$. The $\hat{S}^\emptyset$-blocks are just points.*

Now we want to glue together all the $\mathcal{C}^\beta$ to obtain a description of $\mathcal{C}$. To put this intuition into formulas we introduce for any $(\cdot_p)$-orbit Ω the "combinatorial category" $\mathcal{K}(\Omega, \hat{S})$. An object $\mathcal{M} \in \mathcal{K}(\Omega, \hat{S})$ consists by definition of the following data:

1. A collection $\mathcal{M}(\lambda)$, $\lambda \in \Omega$ of finitely generated $\hat{S}^\emptyset$-modules such that $\mathcal{M}(\lambda) = 0$ for all but finitely many $\lambda \in \Omega$.
2. For each $\beta \in R^+$, $\lambda \in \Omega$ a finitely generated $\hat{S}^\beta$-submodule $\mathcal{M}(\lambda, \beta)$ of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ resp. $\mathcal{M}(\lambda)$ if $\beta \uparrow \lambda \neq \lambda$ resp. $\beta \uparrow \lambda = \lambda$.

A morphism $\varphi : \mathcal{M} \to \mathcal{N}$ is a collection of $\hat{S}^\emptyset$-linear maps $\varphi_\lambda : \mathcal{M}(\lambda) \to \mathcal{N}(\lambda)$ such that $\varphi_\lambda \oplus \varphi_{\beta\uparrow\lambda}$ resp. φ_λ maps $\mathcal{M}(\lambda, \beta)$ to $\mathcal{N}(\lambda, \beta)$ if $\beta \uparrow \lambda \neq \lambda$ resp. $\beta \uparrow \lambda = \lambda$. Morphisms in $\mathcal{K}(\Omega, \hat{S})$ will just be written $\mathrm{Hom}_{\mathcal{K}}$.

Next we construct a functor
$$\mathbb{V}_\Omega = \mathbb{V} : \mathcal{C}_{\hat{S}}(\Omega) \longrightarrow \mathcal{K}(\Omega, \hat{S}).$$

It will depend on the choice of a collection $e^\beta(\lambda)$ of generators of the $\hat{S}^\beta$-modules $\mathrm{Ext}^1_{\mathcal{C}}(Z^\beta(\lambda), Z^\beta(\beta \uparrow \lambda))$, one for each pair $(\lambda, \beta) \in \Omega \times$

R^+ such that $\beta \uparrow \lambda \neq \lambda$. Namely for $M \in \mathcal{C}_{\hat{S}}(\Omega)$ put $(\mathbb{V}M)(\lambda) = \operatorname{Hom}_{\mathcal{C}}(Z^{\emptyset}(\lambda), M^{\emptyset})$. If $\beta \uparrow \lambda = \lambda$ put $(\mathbb{V}M)(\lambda, \beta) = \operatorname{Hom}_{\mathcal{C}}(Z^{\beta}(\lambda), M^{\beta})$. If finally $\beta \uparrow \lambda \neq \lambda$ we choose a representative $Z^{\beta}(\beta \uparrow \lambda) \hookrightarrow Q^{\beta}(\lambda) \twoheadrightarrow Z^{\beta}(\lambda)$ of $e^{\beta}(\lambda)$. If we extend scalars to $\hat{S}^{\emptyset}$, then by Proposition 4.2.2 this sequence splits canonically to determine an isomorphism

$$Q^{\beta}(\lambda)^{\emptyset} = Z^{\emptyset}(\beta \uparrow \lambda) \oplus Z^{\emptyset}(\lambda).$$

Now let $(\mathbb{V}M)(\lambda, \beta)$ be the image of $\operatorname{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), M^{\beta})$ in

$$\operatorname{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda)^{\emptyset}, M^{\emptyset}) = (\mathbb{V}M)(\lambda) \oplus (\mathbb{V}M)(\beta \uparrow \lambda).$$

As a simple example, define for $v \in \mathcal{P}$ the object

$$\mathcal{Z}_{\hat{S}}(v) \in \mathcal{K}(\mathcal{W} \cdot_p \lambda_p(v), \hat{S})$$

by the formulas

$$(\mathcal{Z}_{\hat{S}}(v))(\mu) = \begin{cases} \hat{S}^{\emptyset} & \text{if } \mu = \lambda_p(v); \\ 0 & \text{else,} \end{cases}$$

and

$$(\mathcal{Z}_{\hat{S}}(v))(\mu, \beta) = \begin{cases} \hat{S}^{\beta} & \text{if } \mu = \lambda_p(v); \\ 0 & \text{else.} \end{cases}$$

Then we have $\mathbb{V}Z_{\hat{S}}(v) \cong \mathcal{Z}_{\hat{S}}(v)$.

The following Theorem is as easy to prove as it is central to our approach.

Theorem 4.2.3. *Suppose the characteristic is prime to all entries of the Cartan matrix. Let Ω be a $(\cdot_p)$-orbit and suppose $M, N \in \mathcal{C}_{\hat{S}}(\Omega)$ are projective as $\hat{S}$-modules. Then the functor $\mathbb{V}$ induces an isomorphism*

$$\operatorname{Hom}_{\mathcal{C}}(M, N) \tilde{\to} \operatorname{Hom}_{\mathcal{K}}(\mathbb{V}M, \mathbb{V}N).$$

Proof. Start with any $\lambda \in \Omega$. By Proposition 4.2.2 it is a $\hat{S}^{\emptyset}$-block by itself, so we can decompose $\mathcal{C}^{\emptyset}(\Omega) = \bigoplus_{\lambda \in \Omega} \mathcal{C}^{\emptyset}(\lambda)$. Furthermore $Z^{\emptyset}(\lambda)$ is a projective generator of $\mathcal{C}^{\emptyset}(\lambda)$ and $\operatorname{End}_{\mathcal{C}} Z^{\emptyset}(\lambda) = \hat{S}^{\emptyset}$, so $\operatorname{Hom}_{\mathcal{C}}(Z^{\emptyset}(\lambda), \quad) : \mathcal{C}^{\emptyset}(\lambda) \to \hat{S}^{\emptyset}$-mod is fully faithful. This means that we get a bijection

$$\operatorname{Hom}_{\mathcal{C}}(M^{\emptyset}, N^{\emptyset}) \tilde{\to} \bigoplus_{\lambda \in \Omega} \operatorname{Hom}_{\hat{S}^{\emptyset}}(\mathbb{V}M(\lambda), \mathbb{V}N(\lambda))$$

and the pretended isomorphism from the theorem already has to be an injection. Now an element $(\varphi_\lambda)_{\lambda \in \Omega}$ on the right hand side of our bijection maps $(\mathbb{V}M)(\lambda, \beta)$ to $(\mathbb{V}N)(\lambda, \beta)$ if and only if composition with the corresponding morphism $\varphi : M^{\emptyset} \to N^{\emptyset}$ on the left hand side maps $\operatorname{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), M^{\beta}) \subset \operatorname{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda)^{\emptyset}, M^{\emptyset})$ to $\operatorname{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda), N^{\beta}) \subset \operatorname{Hom}_{\mathcal{C}}(Q^{\beta}(\lambda)^{\emptyset}, N^{\emptyset})$, where we put $Q^{\beta}(\lambda) = Z^{\beta}(\lambda)$ in case $\beta \uparrow \lambda = \lambda$.

Since the $Q^\beta(\lambda)$ generate $\mathcal{C}^\beta(\Omega)$, we deduce that $\varphi(M^\beta) \subset N^\beta$ for all β and hence $\varphi(M) \subset N$. $\square$

4.3. **Combinatorics and Translations.** Using the theorem, we can now rewrite the algebras at the center of our interest as

$$\begin{array}{rcl} \mathcal{B}_{\hat{S}} & = & \mathrm{End}_{\mathcal{C}}(\bigoplus \Theta_\sigma T^{\mathrm{out}} Z_{\hat{S}}(v)) \\ & = & \mathrm{End}_{\mathcal{K}}(\bigoplus \mathbb{V}\Theta_\sigma T^{\mathrm{out}} Z_{\hat{S}}(v)). \end{array}$$

To describe them in a combinatorial way, we should describe $\mathbb{V}_\Gamma T_\Omega^\Gamma M$ in terms of $\mathbb{V}_\Omega M$. Certainly this will depend on the choices of extensions used to define $\mathbb{V}_\Omega$ and $\mathbb{V}_\Gamma$. For simplicity we only treat translation from or to a regular orbit.

Let us start defining the combinatorial translations. Let Ω, Γ be two $(\cdot_p)$-orbits and suppose Ω is regular. For any $H \in \mathcal{H}$ let $\alpha(H) \in R^+$ be the orthogonal root, i.e. the root $\alpha \in R^+$ such that $H = H_{\alpha,m}$ for some m. For $\beta \in R^+$ define $\mathcal{H}(\beta) = \{H \in \mathcal{H} \mid s_\beta \alpha(H) \notin R^+\}$. For any $\lambda \in \Omega$, $\beta \in R^+$ define $a_\lambda^\beta \in \hat{S}^\emptyset$ by the formula

$$a_\lambda^\beta = \prod_{\substack{H\in\mathcal{H}(\beta) \\ \lambda_\Gamma \in H,\, \lambda \in H^+}} h_{-\alpha(H)} \prod_{\substack{H\in\mathcal{H}(\beta) \\ \lambda_\Gamma \in H,\, \lambda \in H^-}} h_{\alpha(H)}^{-1}$$

and put $b_\lambda^\beta = (a_\lambda^\beta)^{-1}$. Then define the "combinatorial translation to the walls"

$$\mathcal{T} = \mathcal{T}_\Omega^\Gamma : \mathcal{K}(\Omega, \hat{S}) \to \mathcal{K}(\Gamma, \hat{S})$$

by the prescriptions

$$\begin{array}{rcl} (\mathcal{T}\mathcal{M})(\mu) & = & \bigoplus_{\lambda_\Gamma = \mu} \mathcal{M}(\lambda) \\ (\mathcal{T}\mathcal{M})(\mu, \beta) & = & \left\{ \begin{array}{ll} \bigoplus_{\lambda_\Gamma = \mu = (\beta\uparrow\lambda)_\Gamma} (b_\lambda^\beta \oplus \mathrm{id})\mathcal{M}(\lambda, \beta) & \text{if } \beta \uparrow \mu = \mu; \\ \bigoplus_{\lambda_\Gamma = \mu} (b_\lambda^\beta \oplus \mathrm{id})\mathcal{M}(\lambda, \beta) & \text{if } \beta \uparrow \mu \neq \mu. \end{array} \right. \end{array}$$

Also we define the "combinatorial translation from the walls"

$$\mathcal{T}' = \mathcal{T}_\Gamma^\Omega : \mathcal{K}(\Gamma, \hat{S}) \to \mathcal{K}(\Omega, \hat{S})$$

by the prescriptions

$$\begin{array}{rcl} (\mathcal{T}'\mathcal{N})(\lambda) & = & \mathcal{N}(\lambda_\Gamma) \\ (\mathcal{T}'\mathcal{N})(\lambda, \beta) & = & \left\{ \begin{array}{ll} (a_\lambda^\beta \oplus 1)\mathcal{N}(\lambda_\Gamma, \beta), & \text{if } \beta \uparrow \lambda_\Gamma \neq \lambda_\Gamma; \\ \mathcal{N}(\lambda_\Gamma, \beta) \oplus \mathcal{N}(\lambda_\Gamma + p\beta, \beta), & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma + p\beta; \\ \{(x + a_\lambda^\beta y, y) \mid x, y \in \mathcal{N}(\lambda_\Gamma, \beta)\}, & \text{if } (\beta \uparrow \lambda)_\Gamma = \lambda_\Gamma \\ & \text{and } \beta \uparrow \lambda \neq \lambda. \end{array} \right. \end{array}$$

That these are the correct combinatorial translations is the content of the following

Theorem 4.3.1. *Suppose p is at least the Coxeter number. Then it is possible to simultaneously choose extensions*

$$e^{\beta}(\lambda) \in \mathrm{Ext}^1_{\mathcal{C}}(Z^{\beta}(\lambda), Z^{\beta}(\beta \uparrow \lambda))$$

for all $\lambda \in X$, $\beta \in R^+$ with $\beta \uparrow \lambda \neq \lambda$ such that for the functors $\mathbb{V}$ defined by these choices we have natural equivalences

$$\mathcal{T}_{\Gamma}^{\Omega} \mathbb{V}_{\Gamma} \cong \mathbb{V}_{\Omega} T_{\Gamma}^{\Omega},\ \mathcal{T}_{\Omega}^{\Gamma} \mathbb{V}_{\Omega} \cong \mathbb{V}_{\Gamma} T_{\Omega}^{\Gamma}$$

whenever Ω is regular and Γ is subregular (i.e. each element lies on only one wall) or Γ consists of special points.

The proof is a big calculation contained in [AJS94], 10–13. We will not repeat it here. Let now $\mathcal{T}^{\mathrm{out}}$ denote the combinatorial translation functors from anything to the orbit $\mathcal{W} \cdot_p 0$. We denote the combinatorial wall-crossings and their concatenations also by Θ_s, Θ_σ. From the theorem we deduce

$$\mathcal{B}_{\hat{S}} = \mathrm{End}_{\mathcal{K}}(\bigoplus \mathbb{V}\Theta_\sigma T^{\mathrm{out}} Z_{\hat{S}}(v)) = \mathrm{End}_{\mathcal{K}}(\bigoplus \Theta_\sigma \mathcal{T}^{\mathrm{out}} \mathcal{Z}_{\hat{S}}(v))$$

and this is indeed a purely combinatorial description of $\mathcal{B}_{\hat{S}}$.

4.4. **Independence of the characteristic.** Let us now explain how one constructs the algebra whose existence is claimed in Theorem 4.1.2. To begin with let D be any integral domain with a morphism $S(\mathbb{Z}') \to D$ that doesn't annihilate any h_α. We denote the image of h_α in D by h_α as well, form $D^{\emptyset} = D[h_\alpha^{-1} \mid \alpha \in R^+]$ and for any $\beta \in R^+$ also $D^{\beta} = D[h_\alpha^{-1} \mid \alpha \in R^+, \alpha \neq \beta]$. Let in addition $\mathcal{O}$ be any orbit of facets under the affine Weyl group. Then we define the combinatorial category $\mathcal{K}(\mathcal{O}, D)$ in analogy with our previous definition. So an object of $\mathcal{K}(\mathcal{O}, D)$ consists of the following data:

1. A collection $\mathcal{M}(F)$, $F \in \mathcal{O}$ of finitely generated $D^{\emptyset}$-modules, almost all of them zero.
2. For each $\beta \in R^+$, $F \in \mathcal{O}$ a finitely generated D^{β}-submodule $\mathcal{M}(F, \beta)$ of $\mathcal{M}(F) \oplus \mathcal{M}(\beta \uparrow F)$ resp. $\mathcal{M}(F)$ if $\beta \uparrow F \neq F$ resp. $\beta \uparrow F = F$.

Here we put $\beta \uparrow F = s_H F$ with $H = H_{\beta,m}$ for the minimal m such that $F \not\subset H^+_{\beta,m}$. It is clear that for $\Omega \subset X$ a $(\cdot_p)$-orbit contained in $\mathcal{O}$ we have $\mathcal{K}(\mathcal{O}, \hat{S}(k)) = \mathcal{K}(\Omega, \hat{S}(k))$ canonically. Let us for every special point $v \in \mathcal{P}$ denote by $\mathcal{O}(v)$ its orbit and define

$$\mathcal{Z}_D(v) \in \mathcal{K}(\mathcal{O}(v), D)$$

by the formulas

$$(\mathcal{Z}_D(v))(F) = \begin{cases} D^{\emptyset} & \text{if } F = v; \\ 0 & \text{else,} \end{cases}$$
$$(\mathcal{Z}_D(v))(F,\beta) = \begin{cases} D^{\beta} & \text{if } F = v; \\ 0 & \text{else.} \end{cases}$$

Remark in addition that a_λ^β is a well defined element of any $D^\emptyset$ and depends only on the alcove of λ and the orbit $\mathcal{O}$ of facets containing Γ, so given an orbit of facets $\mathcal{O}$ we can associate in the same way to any $A \in \mathcal{A}$ and $\beta \in R^+$ the constant $a_A^\beta \in D^\emptyset$. Thus our formulas for combinatorial translations give us functors

$$\mathcal{T}_{\mathcal{A}}^{\mathcal{O}} : \mathcal{K}(\mathcal{A}, D) \to \mathcal{K}(\mathcal{O}, D)$$

and

$$\mathcal{T}_{\mathcal{O}}^{\mathcal{A}} : \mathcal{K}(\mathcal{O}, D) \to \mathcal{K}(\mathcal{A}, D).$$

Again we define the wall-crossings

$$\Theta_s = \mathcal{T}_s^{\mathcal{A}} \mathcal{T}_{\mathcal{A}}^s : \mathcal{K}(\mathcal{A}, D) \to \mathcal{K}(\mathcal{A}, D),$$

their concatenations Θ_σ and the $\mathcal{T}^{\text{out}} : \mathcal{K}(\mathcal{O}, D) \to \mathcal{K}(\mathcal{A}, D)$ for any $\mathcal{O}$. Now we simply construct the algebra $\mathcal{B}_{S(\mathbb{Z}')}$ we were after as

$$\mathcal{B}_{S(\mathbb{Z}')} = \mathrm{End}_{\mathcal{K}}(\bigoplus \Theta_\sigma \mathcal{T}^{\text{out}} \mathcal{Z}_{S(\mathbb{Z}')}(v)).$$

To prove that it has the properties asserted in Theorem 4.1.2 for $\mathrm{char} k = 0$ or $\mathrm{char} k \gg 0$ is a rather dull base change argument. Remark that this would already be enough for our application to Lusztig's conjecture. To get the properties asserted for all k with $\mathrm{char} k \geq h$ is much more subtle, but we have to omit the details contained in [AJS94], 14–16. Let us just remark that this involves putting $\mathbb{Z}$-gradings on our $\mathcal{K}$-categories and on $\mathcal{B}_{S(\mathbb{Z}')}$. These seem to us a very interesting structure which should be explored further. Remark for example that from there we get ([AJS94], 18.17) quite remarkable $\mathbb{Z}$-gradings on the restricted enveloping algebras $U^{\text{res}}(\mathfrak{g})$ themselves that are not visible to the naked eye.

5. Corrections to [**AJS94**]

Here comes a list of the errors in [AJS94] we know up to now. We are grateful to Steen Ryom-Hansen and Masaharu Kaneda for pointing out some of them.

page 20, (D): ... for all $\mu \in X, m \in \ldots$
page 20, (C): ... for all $\mu\nu \in X$.
page 20, (B): ... for all $\mu \in X$.
page 21, $\uparrow 4$**:** $\zeta^{d_\alpha \langle \mu, \alpha^\vee \rangle}$

page 21, 8.13(2): $0 \longrightarrow Z^\beta(\lambda') \xrightarrow{i} Y \xrightarrow{j} Z^\beta(\lambda) \longrightarrow 0$
page 44, $\downarrow$ 19: The existence follows from *2.16.b*
page 61, $\uparrow$ 2: ... from *5.7(3)* to construct ...
page 62, $\downarrow$ 17: ... we can take $\tilde{\pi}(H_\alpha)\dots$
page 69, $\downarrow$ 9: $\dots \mathrm{ch} Z_F(\lambda - n_\beta(\lambda)\beta)\dots$
page 69, $\downarrow$ 11: $\dots L_F(\lambda - n_\beta(\lambda)\beta)\dots$
page 70, $\downarrow$ 12: ... the definition in *6.9* and *6.10*.
page 73, $\downarrow$ 5: ... *(cf. 1.3)* ...
page 78, $\uparrow$ 12: $\dots E_{w\cdot\mu-\lambda} = E_{w'(\mu-\lambda)}$, *where* w' *is the linear part of* w.
page 100, $\uparrow$ 4,6: Z_A ersetzen durch Z_β
page 104, $\downarrow$ 15: for all $\lambda \in X$.
page 108, $\uparrow$ 5: $\mathrm{Hom}_{\mathcal{C}}(Z^\beta(\beta \uparrow \lambda), M^\beta = 0, \dots$
page 113, $\uparrow$ 12, 14: (2) wird (3) und (3) wird zu (4)
page 124, $\downarrow$ 7: by *(3)*; we have obviously
page 130, Theorem $\uparrow$ 5: ... there is exactly one $H \in \mathcal{H}(\beta)$ with $\alpha(H) = \beta$ and $\mu \in H$;
page 130, 15.3: In the part preceding the formula (1) all three $<$ should be $>$ and all three $>$ should be $<$.
page 131, $\downarrow$ 21: ... as in *7.5*.
page 189, $\downarrow$ 11: ... Set for all $\mu \in \Omega$
page 191, $\downarrow$ 6, (4): $\begin{cases} \mathrm{Hom}_{\mathfrak{g}}(M_k, N_k), \dots \\ \mathrm{Hom}_u(M_k, N_k), \dots \end{cases}$
page 192, $\downarrow$ 5, (5): ... if $(\beta \uparrow \lambda)_\Gamma = \Gamma$.
page 192, $\uparrow$ 14: change 10.12 into 10.2(1).
page 201, $\uparrow$ 8: $\dots \mathcal{FC}_{A(k)}(\Omega), \dots$
page 205, $\downarrow$ 6: $\dots (\beta \uparrow \lambda)_\Gamma \in \beta \dots$
page 211, $\downarrow$ 13: $\dots S_k^\beta(0, h_\beta g_i)\dots$
page 213, 16: In the quantum group case one should add another assumption to most results in Section 16 involving a $\sharp$. The problem is that in the proof of part two of Proposition 16.5 c) we use 14.13(4) and thus need that $p(k)$ is prime to the connection index $[X : \mathbb{Z}R]$ or, more generally, the following condition
$(*)$: $\Omega_k \cap (\Omega_k + p\nu) \neq \emptyset$ for $\nu \in X$ implies $\nu \in \mathbb{Z}R$.
Under the hypothesis $p(k) > h$ this condition is always satisfied for the choice $\lambda_0(k) = 0$, i.e. for the "principal block". Indeed, for $p(k) > h$ the equality $w\rho - \rho = p\nu$ with $w \in W$ and $\nu \in X$ implies already $\nu = 0$. The condition $(*)$ should be added to make true the statements involving an upper index $\sharp$ in Proposition 16.5 c), 16.8(4), 16.9(3), and Theorem 16.18.
page 215, $\downarrow$ 16: (*Recall from 9.7* that ...)

page 222, last line: Both λ should be ν.

page 224,16.19(1): Here we should remark that in our application we know N_1 by 16.11, so the place where we loose control for small characteristic is rather the choice of N_2 and N around 16.19(2) and (3).

page 235, $\downarrow$ 17: ... Lemma 15.17 implies that...

page 237, $\downarrow$ 1: a direct summand of $Q^{(\lambda)}(S_k)\dots$

page 254: In the last big displayed calculation of 18.15 four lower indices Γ are missing. The correct formula is

$$\begin{aligned} S_k\langle -o(\lambda,\Gamma)\rangle &\cong \mathrm{Hom}_{E_\Omega,Y}(T^*\tilde{Z}_{S_k}(\lambda_\Gamma), \tilde{Z}'_{S_k}(\lambda)) \\ &\cong \mathrm{Hom}_{E_\Gamma,Y}(\tilde{Z}_{S_k}(\lambda_\Gamma), T_*\tilde{Z}'_{S_k}(\lambda)) \\ &\cong \mathrm{Hom}_{E_\Gamma,Y}(\tilde{Z}_{S_k}(\lambda_\Gamma), T_!\tilde{Z}'_{S_k}(\lambda)\langle i\rangle) \\ &\cong \mathrm{Hom}_{E_\Gamma,Y}(\tilde{Z}_{S_k}(\lambda_\Gamma), \tilde{Z}'_{S_k}(\lambda_\Gamma)\langle -o(\lambda,\Gamma)+i\rangle) \\ &\cong S_k\langle i - o(\lambda,\Gamma)\rangle. \end{aligned}$$

page 283-296, Appendices B and C: In Appendix B we work with the version of the braid group action introduced in [Lus93]. In Appendix C we work with the old version from [Lus90b]. The two versions differ on each weight space by a sign, so the root vectors constructed with the different versions differ by a sign. So if Proposition C.5 and Corollary C.6 hold for one version, they also hold for the other.

page 309: The last M in line 11 should be barred.

page 317, $\uparrow$ 13: ... (Contemporary Mathematics ...)

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