ON THE RELATION BETWEEN INTERSECTION COHOMOLOGY AND REPRESENTATION THEORY IN POSITIVE CHARACTERISTIC

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Abstract. In this article we investigate the relation between the rational representation theory of semisimple algebraic groups over a field k of finite characteristic p and intersection cohomology with coefficients in k of (complex) Schubert varieties.

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partially supported by the EC TMR network "Algebraic Lie Representations", EC-contract no ERB FMRX-CT97-0100.

1. Introduction

Let k be a field of positive characteristic p and let $G \supset B \supset T$ be a semisimple split simply connected algebraic group over k with a Borel and a maximal torus. A weight on the torus $\lambda \in X(T)$ leads to an induced representation $\mathrm{ind}_B^G k_\lambda = H^0(G/B, \mathcal{O}(\lambda)) = H^0(\lambda)$. If this induced representation is nonzero, it admits a unique simple subrepresentation $L(\lambda) \subset H^0(\lambda)$, and every simple G-module is isomorphic to precisely one such $L(\lambda)$, see e.g. [Jan87].

We would like to compute the multiplicities $[H^0(\lambda) : L(\mu)] \in \mathbb{N}$ of $L(\mu)$ occuring in a composition series of $H^0(\lambda)$ for weights λ, μ such that $H^0(\lambda) \neq 0 \neq H^0(\mu)$. For these multiplicities Lusztig [Lus80] has put forward a conjecture. It is proven up to now [AJS94] that these conjectures are valid for every given root system in sufficiently high characteristic. If however the root system is none of A_1, A_2, A_3, B_2, G_2, one doesn't know for a single characteristic whether it is sufficiently high.

The goal of this article is to forward this problem to the topologists or geometers. Certainly we know that intersection cohomology of (complex) Schubert varieties with coefficients in a field of sufficiently high charcteristic behaves as if the characteristic would be zero. But again it is not known for any characteristic whether it actually is sufficiently high. We want to explain how a solution of this topologico-geometrical problem would at least solve part of our representation theoretic problem.

To state our results more precisely, we need more notations. Let $X(T) \supset R \supset R^+$ be the weight lattice, the root system $R = R(G,T)$, and the system of positive roots $R^+ = -R(B,T)$. In particular we have $H^0(\lambda) \neq 0$ if and only if λ is dominant. Let h denote the Coxeter number of our root system. Let $\rho \in X(T)$ be the half sum of positive roots and (W, S) the Coxeter system of $G \supset B$. In particular W is the (finite) Weyl group.

Let on the other hand $\mathcal{G} \supset \mathcal{B}$ be a complex semisimple algebraic group corresponding to the dual root system $R^\vee$ with a Borel, and let us identify its Coxeter system with (W, S) in the obvious way. Let $\mathrm{st} = (p-1)\rho \in X(T)$ denote the "Steinberg weight". In the following we equip complex varieties always with their natural "metric" topology, never with the Zariski-topology. In section 2.11 we will show:

Theorem 1.1. *Let the characteristic of our field k be bigger than the Coxeter number of our root system. For fixed $y \in W$ the following are equivalent:*

1. $[H^0(\mathrm{st} + x\rho) : L(\mathrm{st} + y\rho)] = 1$ *for all* $x \in W$ *such that* $x \leq y$ *in the Bruhat order.*
2. *The cohomology ring* $H^\bullet(\overline{\mathcal{B}y\mathcal{B}/\mathcal{B}}; k)$ *with coefficients in* k *of the complex Schubert variety* $\overline{\mathcal{B}y\mathcal{B}/\mathcal{B}}$ *satisfies Poincaré duality.*

Remarks 1.0.1. Let us call a (complex) variety X "k-smooth" if and only if $H^\bullet(X; k)$ satisfies Poincaré duality. For root systems of type ADE every $\mathbb{Q}$-smooth Schubert variety is actually smooth by unpublished results of Peterson [Car94] and hence is k-smooth for all k. In general this is not true. It would be interesting to know wether every $\mathbb{Q}$-smooth Schubert variety is at least k-smooth for all k with $\mathrm{char} k >$ Coxeter number.

Consider now the bounded below derived category $D^+\mathrm{Sh}_k(\mathcal{G}/\mathcal{B})$ of the category $\mathrm{Sh}_k(\mathcal{G}/\mathcal{B})$ of all sheaves of k-vector spaces on the complex variety $\mathcal{G}/\mathcal{B}$. Let $\mathcal{K} \subset D^+\mathrm{Sh}_k(\mathcal{G}/\mathcal{B})$ denote the full subcategory consisting of all objects of the form $R\pi_*(k_{\mathrm{BS}})$ along with their direct sums and summands and their $[\ ]$-translates, where k_{BS} denotes the constant sheaf k on a Bott-Samelson variety BS and $\pi : \mathrm{BS} \to \mathcal{G}/\mathcal{B}$ is the projection. Put into formulas we take all varieties BS of the form

$$\mathrm{BS} = \mathcal{P}_s \times^{\mathcal{B}} \mathcal{P}_t \times^{\mathcal{B}} \ldots \times^{\mathcal{B}} \mathcal{P}_r \times^{\mathcal{B}} \mathcal{B}/\mathcal{B}$$

for any sequence $s, t, \ldots, r \in S$ of simple reflections with corresponding minimal parabolics $\mathcal{B} \subset \mathcal{P}_s, \mathcal{P}_t, \ldots, \mathcal{P}_r \subset \mathcal{G}$.

Theorem 1.2. *Suppose* $p > h$. *Then we have:*

1. *For every* $y \in W$ *there exists (up to isomorphism) a unique indecomposable object* $\mathcal{L}_y \in \mathcal{K}$ *such that* $\mathcal{L}_y[-l(y)]|_{\mathcal{B}y\mathcal{B}/\mathcal{B}}$ *is the constant sheaf* k *on the Bruhat cell* $\mathcal{B}y\mathcal{B}/\mathcal{B}$ *and* $\mathrm{supp}\mathcal{L}_y \subset \overline{\mathcal{B}y\mathcal{B}/\mathcal{B}}$.
2. *The dimensions of the stalks of the cohomology sheaves of the* $\mathcal{L}_y$ *give us multiplicities, more precisely we have for all* $x, y \in W$ *the formula*

$$[H^0(\mathrm{st} + x\rho) : L(\mathrm{st} + y\rho)] = \sum_i \dim_k(\mathcal{H}^i\mathcal{L}_y)_x.$$

Remarks 1.0.2.

1. The theorem implies that $\mathcal{L}_y$ is selfdual under Verdier duality. For almost all p (I hope, for all $p > h$) our $\mathcal{L}_y$ is just the intersection cohomology complex $\mathcal{IC}(\overline{\mathcal{B}y\mathcal{B}/\mathcal{B}}; k)$ with coefficients in k of the Schubert variety $\overline{\mathcal{B}y\mathcal{B}/\mathcal{B}}$, extended by zero to all of $\mathcal{G}/\mathcal{B}$.
2. We will show that any indecomposable object of $\mathcal{K}$ is isomorphic to one of the $\mathcal{L}_y$, up to a $[\ ]$-translation.

Corollary 1.0.3. *For* $p > h$ *the following are equivalent:*

1. *For all $x, y \in W$ the multiplicities $[H^0(\mathrm{st} + x\rho) : L(\mathrm{st} + y\rho)]$ are as conjectured by Lusztig.*
2. *For every simple reflection $s \in S$ and any $x \in W$ with $xs > x$ the derived direct image $R\pi_{s*}\mathcal{IC}(\overline{\mathcal{B}x\mathcal{B}/\mathcal{B}}; k)$ of the intersection cohomology complex of the Schubert variety under the projection $\pi_s : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_s$ is a semisimple perverse sheaf on $\mathcal{G}/\mathcal{P}_s$.*

The article represents my efforts to translate the arguments of [Soe90] into finite characteristic. This works well most of the time, but for the decomposition theorem there are problems. This is the meaning the above theorems.

2. Arguments from representation theory

In this section 2 we will assume that the characteristic p of k is bigger than the Coxeter number h.

2.1. **Description of some endomorphism rings.** We describe the endomorphism rings of the G-modules one gets by translating the Steinberg module $\mathrm{St} = H^0(\mathrm{st})$ out from some walls.

Let $\lambda \mapsto w\lambda$ denote the action of the Weyl group W on $X(T)$. Let $\lambda \mapsto w \cdot \lambda = w(\lambda + \rho) - \rho$ denote the W-action translated to have fixed point $-\rho$. Let $W_p \subset \mathrm{Aff} X(T)$ denote the subgroup generated by all $(w\cdot)$ for $w \in W$ and the translations $(+\mu)$ with $\mu \in p\mathbb{Z}R$ inside the group of all affine transformations of $X(T)$. We call W_p the affine Weyl group and denote its action on $X(T)$ also $\lambda \mapsto w \cdot \lambda$, since it extends the dot-action of W. By [Jan87] the category G-Mod of all rational G-modules decomposes as

$$G\text{-Mod} = \prod_{\Omega} G\text{-Mod}_{\Omega}$$

where Ω runs through all W_p-orbits in $X(T)$ and $G\text{-Mod}_{\Omega}$ consists of all G-modules having only simple subquotients $L(\lambda)$ with $\lambda \in \Omega$. For any two W_p-orbits $\Omega, \Gamma \subset X(T)$ we define as in [Jan87] the translation functor $T_{\Omega}^{\Gamma} : G\text{-Mod}_{\Omega} \to G\text{-Mod}_{\Gamma}$.

Now we take the integral weights in the (suitably translated) lower closure of the fundamental dominant alcove, more precisely the set

$$I = \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \check{\alpha} \rangle < p \ \ \forall \alpha \in R^+\}$$

as a convenient parameter set for part of the W_p-orbits, and denote for $\lambda, \mu \in I$ by T_{λ}^{μ} the translation functor from the W_p-orbit of $\mathrm{st} + \lambda$ to the W_p-orbit of $\mathrm{st} + \mu$.

Put $\mathfrak{h} = \mathrm{Lie}(T)$, $U = U(\mathfrak{h})$ the symmetric algebra over $\mathfrak{h}$, $U^+ \subset U$ the ideal generated by $\mathfrak{h}$ and $C = C(W, \mathfrak{h}) = U/(U^+)^W U$ the algebra of coinvariants. For $\lambda \in X(T)$ we denote W_λ its stabilizer in the Weyl group. We thus have $W_0 = W$. In [AJS94], 19.8 we show:

Theorem 2.1.1. [AJS94] *For any $\lambda \in I$ there exists an isomorphism $\mathrm{can}_\lambda : \mathrm{End}_G T_0^\lambda \mathrm{St} \to C^{W_\lambda}$ of our endomorphism ring with the ring of W_λ-invariants in the coinvariant algebra C.*

Let more precisely $\lambda, \mu \in I$ be given such that $W_\lambda \supset W_\mu$. Then by [Jan87] there is an isomorphism $\mathrm{iso} : T_\lambda^\mu T_0^\lambda \mathrm{St} \to T_0^\mu \mathrm{St}$, and since the endomorphism rings of our modules are commutative, the map $\mathrm{End}_G T_0^\lambda \mathrm{St} \to \mathrm{End}_G T_0^\mu \mathrm{St}$, $f \mapsto \mathrm{iso} \circ (T_\lambda^\mu f) \circ \mathrm{iso}^{-1}$ does not depend on the choice of iso. We abbreviate our notation and denote this map by

$$T_\lambda^\mu : \mathrm{End}_G T_0^\lambda \mathrm{St} \to \mathrm{End}_G T_0^\mu \mathrm{St}.$$

More precisely than stated in the theorem we actually construct in [AJS94], 19.8 explicit isomorphisms $\mathrm{can}_\lambda : \mathrm{End}_G T_0^\lambda \mathrm{St} \to C^{W_\lambda}$ and our construction shows:

Theorem 2.1.2. [AJS94] *For $\lambda, \mu \in I$ with $W_\lambda \supset W_\mu$ the diagram*

$$\begin{array}{ccc} \mathrm{End}_G T_0^\lambda \mathrm{St} & \xrightarrow{\mathrm{can}_\lambda} & C^{W_\lambda} \\ \downarrow & & \downarrow \\ \mathrm{End}_G T_0^\mu \mathrm{St} & \xrightarrow{\mathrm{can}_\mu} & C^{W_\mu} \end{array}$$

commutes, where the left vertical is our map T_λ^μ and the right vertical the obvious inclusion.

2.2. **Quotient categories.** In this section we recall some definitions and results concerning quotient categories. See [Gab62] for a more detailed discussion.

Definition 2.2.1. A full subcategory $\mathcal{N}$ of an abelian category $\mathcal{A}$ is called a "Serre subcategory" if and only if for any short exact sequence $M' \hookrightarrow M \twoheadrightarrow M''$ in $\mathcal{A}$ we have $M \in \mathcal{N} \Leftrightarrow M', M'' \in \mathcal{N}$. In other words, $\mathcal{N}$ is a full subcategory and is stable under subquotients and extensions.

Let $\mathcal{A}$ be an abelian category and $\mathcal{N} \subset \mathcal{A}$ a Serre subcategory. One defines the quotient category $\mathcal{A}/\mathcal{N}$ as follows: The objects of $\mathcal{A}/\mathcal{N}$ are just the objects of $\mathcal{A}$, the morphisms however are given as

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{N}}(M, N) = \lim_{U,I} \mathrm{Hom}_{\mathcal{A}}(U, I),$$

where the direct limit goes over all subobjects $i : U \hookrightarrow M$ and all quotients $p : N \to I$ such that $\mathrm{coker}(i)$ and $\ker(p)$ respectively belong to $\mathcal{N}$.

One shows that $\mathcal{A}/\mathcal{N}$ is an abelian category, that the obvious functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ is exact, and that $QM \cong 0$ for all $M \in \mathcal{N}$. If on the other hand $\mathcal{B}$ is an abelian category and $F : \mathcal{A} \to \mathcal{B}$ an exact functor such that $FM \cong 0$ for all $M \in \mathcal{N}$, then F factorizes uniquely over an exact functor $F' : \mathcal{A}/\mathcal{N} \to \mathcal{B}$ into $F = F' \circ Q$.

Definition 2.2.2. An artinian category is an abelian category all of whose objects are of finite length.

To give a Serre subcategory of an artinian category we only have to say what should be its simple objects.

Let now $\mathcal{A}$ be an artinian category with a projective generator. In particular $\mathcal{A}$ has only finitely many simple isomorphism classes. Let $\{L_x\}_{x\in W}$ be a system of representatives for the simple isomorphism classes. Any L_x has a projective cover P_x. For any subset $T \subset W$ let $\mathcal{A}_T \subset \mathcal{A}$ be the Serre subcategory of all objects having “support in T”, i.e. with simples $\{L_x\}_{x\in T}$. Let $M \mapsto \overline{M}$ denote the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{A}_T$. For all $x \in W \setminus T$ and $M \in \mathcal{A}$ we have $\mathrm{Hom}_{\mathcal{A}}(P_x, M) = \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_T}(\overline{P_x}, \overline{M})$. In particular for all $x \in W \setminus T$ the $\overline{P_x}$ are projective covers of the $\overline{L_x}$.

For a ring R let $\mathrm{Mod}^{\mathrm{fl}}\text{-}R$ denote the category of all finite length right modules. Is $\mathcal{A}$ an artinian category and $P \in \mathcal{A}$ a projective generator, then it is known [Bas68] that $\mathrm{Hom}_{\mathcal{A}}(P) : \mathcal{A} \to \mathrm{Mod}^{\mathrm{fl}}\text{-}\mathrm{End}_{\mathcal{A}}P$ is an equivalence of categories. For $T \subset W$ let us put $P_T = \bigoplus_{x\in T} P_x$. In the commutative diagram given by the universal property of $\mathcal{A}/\mathcal{A}_T$

$$\begin{array}{ccl} \mathcal{A} & = & \mathcal{A} \\ \downarrow & & \downarrow \mathrm{Hom}_{\mathcal{A}}(P_{W\setminus T},\ \ ) \\ \mathcal{A}/\mathcal{A}_T & \to & \mathrm{Mod}^{\mathrm{fl}}\text{-}\mathrm{End}_{\mathcal{A}}P_{W\setminus T} \end{array}$$

by our above remarks the lower horizontal is an equivalence of the categories.

Later we will use the following terminology:

Definition 2.2.3. An exact functor $F : \mathcal{A} \to \mathcal{B}$ is called a “quotient functor” if and only if it induces an equivalence $\mathcal{A}/\ker F \to \mathcal{B}$.

Remark 2.2.4. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Let $\mathcal{N}_{\mathcal{A}} \subset \mathcal{A}$, $\mathcal{N}_{\mathcal{B}} \subset \mathcal{B}$ be Serre subcategories and suppose $F(\mathcal{N}_{\mathcal{A}}) \subset \mathcal{N}_{\mathcal{B}}$. Then F induces a functor $\overline{F} : \mathcal{A}/\mathcal{N}_{\mathcal{A}} \to \mathcal{B}/\mathcal{N}_{\mathcal{B}}$.

Let us now assume in addition, that F admits a left adjoint G such that G is exact and $G(\mathcal{N}_{\mathcal{B}}) \subset \mathcal{N}_{\mathcal{A}}$. Then also $\overline{G}$ is left adjoint to $\overline{F}$.

2.3. **The regular subquotient category.** We put $G\text{-Mof} = \{M \in G\text{-Mod} \mid \dim_k M < \infty\}$. There is a contravariant involutive functor $d : G\text{-Mof} \to G\text{-Mof}$ such that $dL \cong L$ for any simple object L. More precisely take as dM the contravariant representation M^* twisted by a Chevalley-involution of G. We put $V(\lambda) = dH^0(\lambda)$. We consider the partial order $\uparrow$ on X defined in [Jan87], II.6.4 and put

$$\mathcal{A} = \{M \in G\text{-Mof} \mid [M : L(\lambda)] \neq 0 \Rightarrow \lambda \uparrow \mathrm{st} + \rho\}.$$

Inside we consider the subcategory

$$\mathcal{N} = \{M \in \mathcal{A} \mid [M : L(\lambda)] = 0 \;\; \forall \lambda \in \mathrm{st} + W\rho\}$$

and define the "regular subquotient arround the Steinberg point" $\mathcal{O} = \mathcal{A}/\mathcal{N}$. Let $M \mapsto \overline{M}$ denote the quotient functor $\mathcal{A} \to \mathcal{O}$. We put

$$M_x = \overline{V(\mathrm{st} + x\rho)} \in \mathcal{O}.$$

Now we list some properties of $\mathcal{O}$ and the M_x.

1. Every object of $\mathcal{O}$ has finite length. All spaces of morphisms are finite dimensional.
2. Every M_x has a unique simple quotient L_x. The $\{L_x\}_{x \in W}$ form a system of representatives for the simple isomorphism classes of $\mathcal{O}$.
3. There exists a contravariant involution $d : \mathcal{O} \to \mathcal{O}$ such that $dL_x \cong L_x$ for all $x \in W$.
4. There exists a partial order $\leq$ on W such that $[M_x : L_y] \neq 0 \Leftrightarrow y \leq x$. Furthermore $[M_x : L_x] = 1$ for all $x \in W$. (Certainly this is just the Bruhat order, but we don't care.)
5. M_e is the projective cover of L_e in $\mathcal{O}$. This follows from the fact that $V(\mathrm{st} + \rho)$ is the projective cover of $L(\mathrm{st} + \rho)$ in $\mathcal{A}$.
6. $\overline{T_0^\rho \mathrm{St}} = P_{w_\circ}$ is the projective cover of $L_{w_\circ}$ in $\mathcal{O}$. This $P_{w_\circ}$ admits a filtration, in which any M_x with $x \in W$ appears exactly once as a subquotient. Furthermore can_ρ gives a canonical isomorphism $\mathrm{can} : \mathrm{End}_{\mathcal{O}} P_{w_\circ} \to C$.

2.4. **The semiregular subquotient categories.** For any simple reflection $s \in S$ let $\alpha \in R^+$ be the corresponding simple root and ϖ_α the fundamental weight. We consider $\mu = \mu(s) = \rho - \varpi_\alpha \in I$. This weight lies on the s-wall, $W_\mu = \langle s \rangle$. As before we take

$$\mathcal{A}^s = \{M \in G\text{-Mof} \mid [M : L(\lambda)] \neq 0 \Rightarrow \lambda \uparrow \mathrm{st} + \mu\},$$

consider in there the subcategory

$$\mathcal{N}^s = \{M \in \mathcal{A}^s \mid [M : L(\lambda)] = 0 \;\; \forall \lambda \in \mathrm{st} + W\mu\}$$

and form the quotient $\mathcal{O}^s = \mathcal{A}^s/\mathcal{N}^s$. In there we have the standard objects

$$M_x^s = \overline{V(\mathrm{st} + x\mu)}.$$

Certainly $M_x^s = M_{xs}^s$ for $x \in W$. Any M_x^s has a unique simple quotient L_x^s.

We will need in particular that $\overline{T_0^\mu \mathrm{St}} = P^s_{w_\circ}$ is the projective cover of $L^s_{w_\circ}$ in $\mathcal{O}^s$. Again $P^s_{w_\circ}$ admits a filtration such that every M^s_x with $x \in W/\langle s\rangle$ occurs exactely once as a subquotient. Furthermore can_μ gives a canonical isomorphism

$$\mathrm{can}^s : \mathrm{End}_{\mathcal{O}^s} P^s_{w_\circ} \to C^s.$$

2.5. **Translation.** Let $s \in S$, $\mu = \mu(s)$ be as before. The translation functors onto the wall and out from the wall T_ρ^μ and T_μ^ρ induce functors between our subquotient categories, which we denote $T^s : \mathcal{O} \to \mathcal{O}^s$ and $T_s : \mathcal{O}^s \to \mathcal{O}$. Known properties of T_ρ^μ, T_μ^ρ imply lots of properties of T^s, T_s.

1. The functors T^s, T_s are exact and right- as well as leftadjoint to one another.
2. We have $T^s M_x \cong M_x^s$ and if $y < ys$ in the Bruhat order there is a short exact sequence
$$M_y \hookrightarrow T_s M^s_y \twoheadrightarrow M_{ys}.$$
3. We have $T_s P^s_{w_\circ} \cong P_{w_\circ}$, one gets independently of of such an isomorphism a map $T_s : \mathrm{End}_{\mathcal{O}^s} P^s_{w_\circ} \to \mathrm{End}_{\mathcal{O}} P_{w_\circ}$ and the diagram
$$\begin{array}{ccc} \mathrm{End}_{\mathcal{O}^s} P^s_{w_\circ} & \xrightarrow{\mathrm{can}^s} & C^s \\ T_s \downarrow & & \downarrow \\ \mathrm{End}_{\mathcal{O}} P_{w_\circ} & \xrightarrow{\mathrm{can}} & C \end{array}$$
commutes with the inclusion as right vertical.
4. The translation through the wall
$$\theta_s = T_s T^s : \mathcal{O} \to \mathcal{O}$$
is an exact selfadjoint functor. If $y \in W$ is given such that $y < ys$, we have a short exact sequence
$$M_y \hookrightarrow \theta_s M_y \twoheadrightarrow M_{ys}.$$
5. There are enough projectives in $\mathcal{O}$, and each one of them is the direct summand of an object of the form $\theta_s \ldots \theta_t M_e$. Indeed, M_e is projective by 2.3.5 and the θ_s are exact and selfadjoint, hence all $\theta_s \ldots \theta_t M_e$ are projective. Now if $x = t \ldots s$ is a reduced decomposition, then by the preceeding point we have a surjection $\theta_s \ldots \theta_t M_e \twoheadrightarrow M_x \twoheadrightarrow L_x$. This shows what we need.
6. Let $P_x \twoheadrightarrow L_x$ be the projective cover in $\mathcal{O}$. Every P_x admits a filtration with subquotients of the form M_y. If $(P_x : M_y)$ denotes the multiplicity of M_y as a subquotient, we have the reciprocity formula $(P_x : M_y) = [M_y : L_x]$.

7. For all $s \in S$, $M \in \mathcal{O}$ we have $[\theta_s M : L_{w_\circ}] = 2[M : L_{w_\circ}]$. Indeed we know by 2.3.6 that $(P_{w_\circ} : M_x) = 1 \ \ \forall x \in W$. From 2.5.4 we get comparing the characters $\theta_s P_{w_\circ} \cong P_{w_\circ} \oplus P_{w_\circ}$. Now we have

$$\begin{array}{rcl} [\theta_s M : L_{w_\circ}] & = & \mathrm{dimHom}_{\mathcal{O}}(P_{w_\circ}, \theta_s M) \\ & = & \mathrm{dimHom}_{\mathcal{O}}(\theta_s P_{w_\circ}, M) \\ & = & 2\mathrm{dimHom}_{\mathcal{O}}(P_{w_\circ}, M) \\ & = & 2[M : L_{w_\circ}]. \end{array}$$

8. We have $[M_x : L_{w_\circ}] = (P_{w_\circ} : M_x) = 1$ for all $x \in W$. More precisely we even have $\mathrm{soc} M_x = L_{w_\circ}$ for all $x \in W$. Indeed it is sufficient to show: $\mathrm{Hom}_{\mathcal{O}}(L_y, M_x) \neq 0 \Rightarrow y = w_\circ$. This follows by induction from above over the length of x. For $x = w_\circ$ the statement is clear. Suppose $xs < x$. Then we have a short exact sequence $M_{xs} \hookrightarrow \theta_s M_x \twoheadrightarrow M_x$ and we deduce:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{O}}(L_y, M_{xs}) \neq 0 & \Rightarrow & \mathrm{Hom}_{\mathcal{O}}(L_y, \theta_s M_x) \neq 0 \\ & \Rightarrow & \mathrm{Hom}_{\mathcal{O}}(\theta_s L_y, M_x) \neq 0 \\ & \Rightarrow & [\theta_s L_y : L_{w_\circ}] \neq 0 \\ & \Rightarrow & y = w_\circ. \end{array}$$

2.6. **Homomorphisms between projective objects.** Our canonical isomorphism can : $\mathrm{End}_{\mathcal{O}} P_{w_\circ} \to C$ gives us an exact functor

$$\mathbb{V} = \mathrm{Hom}_{\mathcal{O}}(P_{w_\circ}, \ \ ) : \mathcal{O} \to C\text{-}\mathrm{Mod}^{\mathrm{fl}}.$$

We are going to show

Theorem 2.6.1. *For $M, Q \in \mathcal{O}$ with Q projective the functor $\mathbb{V}$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{O}}(M, Q) \to \mathrm{Hom}_C(\mathbb{V}M, \mathbb{V}Q).$$

Proof. We proceed in steps.

$Q = P_{w_\circ}$**:** In this case we have $Q = \overline{T_0^\rho \mathrm{St}}$ also is the injective hull of $M_{w_\circ} = L_{w_\circ}$. For $M \in C\text{-}\mathrm{Mod}^{\mathrm{fl}}$ also the dual space $M^* = \mathrm{Hom}_k(M, k)$ is a C-Modul. By our restriction on the characteristic C is the cohomology ring of the smooth manifold $C = H^\bullet(\mathcal{G}/\mathcal{B}; k)$, see [Dem73], and Poincaré duality gives us an isomorphism $C \cong C^*$ in $C\text{-}\mathrm{Mod}^{\mathrm{fl}}$. Therefore C is not only the projective cover, but also the injective hull of $k \in C\text{-}\mathrm{Mod}^{\mathrm{fl}}$.

Since $\mathbb{V}Q = C$ in this case, both sides of the hoped-for isomorphism $\mathrm{Hom}_{\mathcal{O}}(M, Q) \to \mathrm{Hom}_C(\mathbb{V}M, \mathbb{V}Q)$ are exact in M. By the five lemma it will be sufficient to show this isomorphism for simple M. This is left to the reader.

$Q = M_e$**:** Let us consider once more our isomorphism from the preceding step and take on both sides the part annihilated by the maximal ideal $C^+ \subset C$. We get an isomorphism $\mathrm{Hom}_{\mathcal{O}}(M, N) \to \mathrm{Hom}_C(\mathbb{V}M, k)$ where $N \subset P_{w_\circ}$ is the biggest subobject killed by C^+ and we use $k = \mathrm{soc}C$ on the right. We only have to show $N = M_e$. Clearly we have a short exact sequence $M_e \hookrightarrow P_{w_\circ} \twoheadrightarrow \mathrm{coker}$ and coker admits a filtration with subquotients M_x.

Since $\mathrm{soc}M_x = L_{w_\circ}$ for all x we deduce for any nonzero subobject $U \subset \mathrm{coker}$, that $[U : L_{w_\circ}] \neq 0$. Now certainly we have $M_e \subset N$. In case $M_e \neq N$, the image of N in coker would be nonzero, thus we would have $[N : L_{w_\circ}] \geq 2$. But $\mathrm{Hom}_{\mathcal{O}}(P_{w_\circ}, N) \subset \mathrm{End}_{\mathcal{O}} P_{w_\circ}$ is an ideal killed by C^+, thus we have $1 \geq \mathrm{dim}\mathrm{Hom}_{\mathcal{O}}(P_{w_\circ}, N) = [N : L_{w_\circ}]$. This contradiction shows that indeed $M_e = N$.

Q **arbitrary:** Any projective $Q \in \mathcal{O}$ is a direct summand of an object of the form $\theta_s \ldots \theta_t M_e$, thus we can assume $Q = \theta_s \ldots \theta_t M_e$. Now we deduce with the last paragraph of 2.2 for any $M \in \mathcal{O}$:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}}(M, \theta_s \ldots \theta_t M_e) &= \mathrm{Hom}_{\mathcal{O}}(\theta_t \ldots \theta_s M, M_e) \\
&= \mathrm{Hom}_C(\mathbb{V}\theta_t \ldots \theta_s M, \mathbb{V}M_e) \\
&= \mathrm{Hom}_C(\overline{\theta_t} \ldots \overline{\theta_s}\mathbb{V}M, \mathbb{V}M_e) \\
&= \mathrm{Hom}_C(\mathbb{V}M, \overline{\theta_s} \ldots \overline{\theta_t}\mathbb{V}M_e) \\
&= \mathrm{Hom}_C(\mathbb{V}M, \mathbb{V}\theta_s \ldots \theta_t M_e)
\end{aligned}$$

□

Next we proof:

Theorem 2.6.2. *i.) For all $x \in W$ we have $\mathbb{V}M_x = k$.*
ii.) For all $M \in \mathcal{O}$, $s \in S$ we have $\mathbb{V}\theta_s M \cong C \otimes_{C^s} \mathbb{V}M$.

Proof. i.) is clear by 2.5.8. For ii.) we consider the adjoint pair (T_s, T^s) of functors between $\mathcal{O}$ and $\mathcal{O}^s$. Let us consider also the quotient functor

$$\mathbb{V}^s = \mathrm{Hom}_{\mathcal{O}^s}(P^s_{w_\circ}, \ \ ) : \mathcal{O}^s \to C^s\text{-}\mathrm{Mod}^{\mathrm{fl}}.$$

Since $T_s P^s_{w_\circ} \cong P_{w_\circ}$ we have $[T^s M : L^s_{w_\circ}] = [M : L_{w_\circ}]$ for $M \in \mathcal{O}$. Furthermore we have $[T_s M : L_{w_\circ}] = 2[M : L^s_{w_\circ}]$ for $M \in \mathcal{O}^s$, for example since this holds for all standard objects $M = M^s_x$.

Therefore the adjoint pair (T_s, T^s) gives us an adjoint pair $(\overline{T_s}, \overline{T^s})$ of functors between the quotient categories C-$\mathrm{Mod}^{\mathrm{fl}}$ and C^s-$\mathrm{Mod}^{\mathrm{fl}}$. By 2.5.3 we deduce that $\overline{T^s} = \mathrm{res} : C\text{-}\mathrm{Mod}^{\mathrm{fl}} \to C^s\text{-}\mathrm{Mod}^{\mathrm{fl}}$ is just the restriction functor. Therefore we necessarily have $\overline{T_s} = C \otimes_{C^s}$ and we deduce $\mathbb{V}\theta_s M \cong \mathbb{V}T_s T^s M \cong \mathbb{V}\overline{T_s}\,\overline{T^s} M \cong C \otimes_{C^s} \mathbb{V}M$. □

2.7. **Decomposition with and without grading.** Let C be a finite dimensional graded k-algebra – we even permit homogeneous components of negative degree. Let $C\text{-mod}^{\mathrm{fl}}$ be the category of all graded finite dimensional C-modules, denote by $\hom_C$ the morphisms in this category, and let $v : C\text{-mod}^{\mathrm{fl}} \to C\text{-Mod}^{\mathrm{fl}}$ be the functor forgetting the grading. For $M = \bigoplus M^n \in C\text{-mod}^{\mathrm{fl}}$ let us define the grading shifted modules $M\langle\nu\rangle \in C\text{-mod}^{\mathrm{fl}}$ by $(M\langle\nu\rangle)^n = M^{n-\nu}$. Thus we have $v(M\langle\nu\rangle) = vM$ and $\mathrm{Hom}_C(vM, vN) = \bigoplus_\nu \hom_C(M, N\langle\nu\rangle)$.

Definition 2.7.1. Take $M \in C\text{-Mod}^{\mathrm{fl}}$. We say "$M$ lifts" if and only if there exists $\tilde{M} \in C\text{-mod}^{\mathrm{fl}}$ such that $v\tilde{M} \cong M$.

Proposition 2.7.2. *Let $M, N \in C\text{-Mod}^{\mathrm{fl}}$ be given such that (1) N, $M \oplus N$ lift and (2) N is indecomposable. Then also M lifts. More precisely, for any lift $\widetilde{M \oplus N}$ of $M \oplus N$ we find a lift $\tilde{M}$ of M, which is a direct summand of $\widetilde{M \oplus N}$.*

Proof. The inclusion $i : N \hookrightarrow M \oplus N$ and the projection $p : M \oplus N \twoheadrightarrow N$ decompose on lifts of these objects into homogeneous components $i = \sum i_\nu$, $p = \sum p_\nu$. Since we assumed N indecomposable we have $\mathrm{End}_C N$ local. Since $\mathrm{id} = p \circ i = \sum p_\nu \circ i_{-\nu}$ is an isomorphism, necessarily for some ν the map $p_\nu \circ i_{-\nu}$ has to be an isomorphism. Let $u \in \mathrm{end}_C \tilde{N}$ be its inverse. Then $i_{-\nu}$ is a splitting of the projection $u \circ p_\nu : \widetilde{M \oplus N} \to \tilde{N}\langle\nu\rangle$ and $\ker(u \circ p_\nu)$ is a lift of M. □

On the other hand we have:

Lemma 2.7.3. *Any two lifts of an indecomposable C-module of finite dimension M are isomorphic up to a shift of the grading*

Proof. Let $\tilde{M}$, $\tilde{M}'$ be two lifts of M. The identity on M decomposes into homogeneous components in $\mathrm{Hom}_C(\tilde{M}, \tilde{M}')$. Since M is indecomposable, one of these homogeneous components has to be an isomorphism. □

2.8. **Definition of the modules D_x.** Let now C be again the coinvariant algebra with its even grading $C = \bigoplus C^{2\nu}$. Let us consider the additive subcategory $\mathcal{C} \subset C\text{-mod}^{\mathrm{fl}}$ consisting of all $C \otimes_{C^s} C \ldots \otimes_{C^t} k\langle\nu\rangle$ with $s, \ldots, t \in S, \nu \in \mathbb{Z}$ and their direct sums and summands.

Theorem 2.8.1. 1. *There exists a family $\{D_x\}_{x \in W}$ of indecomposable objects in $C\text{-mod}^{\mathrm{fl}}$ such that for any reduced decomposition $x = t \ldots s$ $(s, \ldots, t \in S)$ we have*

$$C \otimes_{C^s} C \ldots \otimes_{C^t} k\langle -l(x)\rangle \cong D_x \oplus \mathrm{Rest},$$

where Rest *is a direct sum of $D_y\langle\nu\rangle$ with $y < x$, $\nu \in \mathbb{Z}$.*

2. *If* $\{D'_x\}_{x \in W}$ *is another such family, then we have* $D'_x \cong D_x \ \forall x \in W$.
3. *The* D_x *stay indecomposable, when we forget their grading.*
4. $\{D_x\langle\nu\rangle\}_{x\in W,\nu\in\mathbb{Z}}$ *is a system of representatives for the isomorphism classes of indecomposable objects in* $\mathcal{C}$.

We prove this theorem along with the following theorem which makes its meaning clear:

Theorem 2.8.2. *We have* $\mathbb{V}P_x \cong vD_x$ *for all* $x \in W$.

Proof of both theorems. First we prove the first part of the first theorem along with the second theorem by induction on the length of x. For $x = e$ everything is trivial. For $x = t \dots s$ (reduced decomposition) we have by 2.5, 4, 6 and 2.3, 4

$$\theta_s \dots \theta_t P_e \cong P_x \oplus \mathrm{Rest}$$

where Rest is a sum of P_y with $y < x$. We deduce

$$\begin{array}{rcl} C \otimes_{C^s} C \dots \otimes_{C^t} k & = & \mathbb{V}\theta_s \dots \theta_t P_e \\ & \cong & \mathbb{V}(P_x) \oplus \mathbb{V}(\mathrm{Rest}). \end{array}$$

By induction $\mathbb{V}(\mathrm{Rest})$ is a sum of vD_y with $y < x$. Every one of these vD_y lifts by the inductive assumption. By Proposition 2.7.2 also $\mathbb{V}P_x$ lifts, and even lifts to a direct summand of $C \otimes_{C^s} C \dots \otimes_{C^t} k\langle -l(x)\rangle$. This proves the first part of the first theorem and the second theorem.

We still have to prove the rest of the first theorem. 2 follows easily by Krull-Schmid. 3 follows since $\mathrm{End}_{\mathcal{C}} D_x = \mathrm{End}_{\mathcal{O}} P_x$ is a local ring. Part 4 follows by Krull-Schmidt and Lemma 2.7.3, once we know, that in C-Mod the subcategory consisting of all direct sums of D_x is stable under $C \otimes_{C^s}$ for all $s \in S$. This follows since $C \otimes_{C^s} D_x = \mathbb{V}\theta_s P_x$ and since $\theta_s P_x$ is projective, hence a direct sum of indecomposable projectives. □

2.9. **Properties of the modules D_x.** By [Dem73] our coinvariant algebra $C = H^\bullet(\mathcal{G}/\mathcal{B}; k)$ is just the cohomology ring of the flag manifold $\mathcal{G}/\mathcal{B}$. For the homology the Bruhat cells $\mathcal{B}x\mathcal{B}/\mathcal{B}$ give a canonical basis, and we denote the dual basis in cohomology with $\{p_x\}_{x\in W}$. Any $s \in S$ with corresponding simple root α gives a Demazure operator $\partial_s : S \to S$, $f \mapsto (f - sf)/\check{\alpha}$. This induces a map $\partial_s : C \to C$ on the coinvariants, and it is known for example from [BGG73], 3.14(i), that

$$\partial_s p_x = \begin{cases} p_{xs} & xs < x; \\ 0 & \text{else.} \end{cases}$$

The restriction $H^\bullet(\mathcal{G}/\mathcal{B};k) \to H^\bullet(\overline{\mathcal{B}x\mathcal{B}/\mathcal{B}};k)$ has kernel

$$I_x = \sum_{y \not\leq x} kp_y \subset C.$$

We want to show:

Proposition 2.9.1. *For all $x \in W$ we have $\dim D_x^{-l(x)} = \dim D_x^{l(x)} = 1$ and $\mathrm{Ann}_C D_x = I_x$.*

Proof. First we claim:

$$f \in \mathrm{Ann}_C(C \otimes_{C^s} M) \Leftrightarrow f, \partial_s f \in \mathrm{Ann}_C M.$$

Indeed any element of $C \otimes_{C^s} M$ can be written uniquely in the form $1 \otimes m_1 + \check{\alpha} \otimes m_2$, in addition $2f = (f + sf) + (\check{\alpha}(\partial_s f))$ is the decomposition of $2f \in C$ along $C = C^s \oplus \check{\alpha} C^s$, and thus we have

$$\begin{aligned} 2f(1 \otimes m_1 + \check{\alpha} \otimes m_2) = \ & 1 \otimes ((f + sf)m_1 + (\check{\alpha})^2(\partial_s f)m_2) \\ & + \check{\alpha} \otimes ((\partial_s f)m_1 + (f + sf)m_2). \end{aligned}$$

The claim follows immediately. The same calculation also shows $f \in \mathrm{Ann}_C(1 \otimes m) \Leftrightarrow f, \partial_s f \in \mathrm{Ann}_C(m)$.

Now let $x = t \dots s$ be a reduced decomposition. With induction we deduce $\mathrm{Ann}_C(C \otimes_{C^s} C \dots \otimes_{C^t} k) = I_x$ and even $\mathrm{Ann}_C(1 \otimes 1 \dots \otimes 1) = I_x$. Now the two extremal components of degrees $l(x)$ and $-l(x)$ of $C \otimes_{C^s} C \dots \otimes_{C^t} k\langle -l(x)\rangle$ are one dimensional, and since $(C/I_x)^{2l(x)} \neq 0$ the highest homogeneous component lies in the submodule generated by the lowest one. The proposition follows by induction over the length of x. $\square$

Next we prove:

Lemma 2.9.2. *Let $M \in C$-mod$^{\mathrm{fl}}$, $s \in S$. Then we have $(C \otimes_{C^s} M)^* \cong C \otimes_{C^s} (M^*)\langle -2\rangle$.*

Proof. Again any element of $C \otimes_{C^s} M$ can be written uniquely in the form $1 \otimes m_1 + \check{\alpha} \otimes m_2$. Now we define the pairing

$$(C \otimes_{C^s} M) \times (C \otimes_{C^s} (M^*)) \to k, \quad (v, w) \mapsto \langle v, w\rangle$$

by $\langle (1 \otimes m + \check{\alpha} \otimes n), (1 \otimes f + \check{\alpha} \otimes g)\rangle = f(n) + g(m)$ for all $m, n \in M$, $f, g \in M^*$. It is not degenerated and we check that $\langle cv, w\rangle = \langle v, cw\rangle$ for $c \in C^s$ and $c = \check{\alpha}$, hence for all $c \in C$. $\square$

Proposition 2.9.3. *The modules D_x are selfdual, $D_x \cong D_x^*$.*

Proof. By now we know: If $x = t \dots s$ is a reduced decomposition and $M \subset C \otimes_{C^s} C \dots \otimes_{C^t} k\langle -l(x)\rangle$ is a direct summand with $M^{l(x)} = k$ or $M^{-l(x)} = k$, then we have $M \cong D_x$. But by the preceding lemma $C \otimes_{C^s} C \dots \otimes_{C^t} k\langle -l(x)\rangle$ is selfdual. $\square$

Theorem 2.9.4. *For $x \in W$ are equivalent:*

1. $D_x \cong (C/I_x)\langle -l(x)\rangle$.
2. *The cohomology ring $H^\bullet(\overline{\mathcal{B}x\mathcal{B}/\mathcal{B}}; k) = C/I_x$ with coefficients in k the Schubert variety $\overline{\mathcal{B}x\mathcal{B}/\mathcal{B}}$ satisfies Poincaré duality.*

Proof. $1 \Rightarrow 2$. Certainly C/I_x has exactly one simple quotient and in the light of the preceding proposition 1 implies that C/I_x also has a unique simple submodule, $\mathrm{soc}(C/I_x) = k$. This immediately implies 2.

$2 \Rightarrow 1$. We consider the inclusion

$$(C/I_x)\langle -l(x)\rangle \hookrightarrow C \otimes_{C^s} C \ldots \otimes_{C^t} k\langle -l(x)\rangle$$

with $1 \mapsto 1 \otimes 1 \ldots \otimes 1$. By assumption resp. Lemma 2.9.2 both sides are selfdual, therefore we get a surjection

$$C \otimes_{C^s} C \ldots \otimes_{C^t} k\langle -l(x)\rangle \twoheadrightarrow (C/I_x)\langle -l(x)\rangle.$$

Since I_x is already the annihilator of the left hand side, this surjection has to split. This implies immediately $D_x \cong (C/I_x)\langle -l(x)\rangle$. □

To prove the first theorem from the introduction, we still need some preparations.

2.10. **Grothendieck groups.** To every abelian category $\mathcal{A}$ we associate its Grothendieck group $[\mathcal{A}]$, and every object $A \in \mathcal{A}$ determines an element $[A] \in [\mathcal{A}]$. On the other hand we associate to every additive category $\mathcal{A}$ its split Grothendieck group $\langle \mathcal{A}\rangle$ as follows: $\langle \mathcal{A}\rangle$ is the free abelian group generated by the objects of $\mathcal{A}$ with the relations $A' + A'' = A$ whenever there is a split short exact sequence $A' \hookrightarrow A \twoheadrightarrow A''$. Every object $A \in \mathcal{A}$ gives us an element $\langle A\rangle \in \langle \mathcal{A}\rangle$. Every (co- or contravariant) exact resp. additive functor $F : \mathcal{A} \to \mathcal{B}$ gives us a group homomorphism $F : [\mathcal{A}] \to [\mathcal{B}]$ resp. $F\langle \mathcal{A}\rangle \to \langle \mathcal{B}\rangle$.

Let us now consider our category $\mathcal{O}$. Certainly $\{[L_x]\}_{x\in W}$ is a $\mathbb{Z}$-basis of $[\mathcal{O}]$. For $M \in \mathcal{O}$ the coefficients of $[M]$ in this basis are just the Jordan-Hölder multiplicities $[M] = \sum_x [M : L_x][L_x]$. Now the transition matrix $[M_y : L_x]$ is upper triangular with ones on the diagonal, hence also $\{[M_x]\}_{x\in W}$ is a $\mathbb{Z}$ basis of $[\mathcal{O}]$. We write the coefficients of $[M]$ in this basis $(M : M_x)$, i.e. $[M] = \sum_x (M : M_x)[M_x]$. If M has a Verma flag (i.e. a filtration such that all subquotients are among the M_x), then $(M : M_x)$ is precisely the multiplicity of M_x as a subquotient.

Elements of the group ring $\mathbb{Z}[W]$ will be written in the form $\sum_x n_x x$ with $n_x \in \mathbb{Z}$. We define an isomorphism

$$\begin{array}{rcl} A : [\mathcal{O}] & \to & \mathbb{Z}[W] \\ {[M]} & \mapsto & \sum (M : M_x)x. \end{array}$$

This is an isomorphism of abelian groups, and from 2.5 we can even deduce $A[\theta_s M] = A[M](1+s)$ for all $M \in \mathcal{O}$, $s \in S$. Finally we consider the additive category $p\mathcal{O}$ of all projective objects of $\mathcal{O}$. From 2.5 we deduce that the obvious map $\langle p\mathcal{O}\rangle \to [\mathcal{O}]$, $\langle P\rangle \mapsto [P]$ is an isomorphism. The composition of this isomorphism with A will be denoted $A : \langle p\mathcal{O}\rangle \to \mathbb{Z}[W]$. The functors θ_s map projective objects to projective objects. We deduce that also $A\langle \theta_s P\rangle = A\langle P\rangle(1+s)$ for all $P \in p\mathcal{O}$, $s \in S$.

2.11. **Proof of the first theorem from the introduction.** We need the following commutative diagram:

$$\begin{array}{ccccc}
\langle p\mathcal{O}\rangle & \stackrel{\mathbb{V}}{\to} & \langle v\mathcal{C}\rangle & \stackrel{\dim}{\to} & \mathbb{Z} \\
A\downarrow & & B\downarrow & & || \\
\mathbb{Z}[W] & = & \mathbb{Z}[W] & \stackrel{\mathrm{tr}}{\to} & \mathbb{Z}
\end{array}$$

The map B is defined by the condition that the first square is commutative. The map dim is given by $\dim\langle M\rangle = \dim_k M$ for all $M \in v\mathcal{C}$. The map tr is given by $\mathrm{tr}(\sum n_x x) = \sum n_x$. We leave it to the reader to check that also the second square commutes. Furthermore from 2.6.2 we deduce the formula $B\langle C \otimes_{C^s} M\rangle = B\langle M\rangle(1+s)$ for $s \in \mathcal{S}$, $M \in v\mathcal{C}$. Theorem 1.1 follows using 2.9.4 and 2.5.6 from

Theorem 2.11.1. *Take $y \in W$. The following are equivalent:*

1. *$(P_y : M_x) = 1$ for all $x \in W$ with $x \leq y$ in the Bruhat order.*
2. *$D_y \cong (C/I_y)\langle -l(y)\rangle$.*

Proof. From 1 we deduce $\sum_{x\leq y} x = A\langle P_y\rangle = B\langle vD_y\rangle$, therefore $\sharp\{x \in W \mid x \leq y\} = \dim_k D_y$, and then comparing dimensions the map $(C/I_y)\langle -l(y)\rangle \hookrightarrow D_y$, $1 \mapsto 1 \otimes 1 \ldots 1$ has to be an isomorphism.

If on the other hand we have $(P_y : M_x) > 1$ for some x, we deduce $\dim_k(C/I_y) < \dim_k D_y$ by the same argument. □

For later applications we add a formula for the dimension of Hom-spaces between objects of $\mathcal{C}$. Let us denote by

$$\langle \quad , \quad \rangle : \mathbb{Z}[W] \times \mathbb{Z}[W] \to \mathbb{Z}$$

the bilinear form for which the group elements form an orthonormal basis, $\langle x, y\rangle = \delta_{x,y}$ for $x, y \in W$.

Lemma 2.11.2. *For all $D, D' \in \mathcal{S}$ we have the formula*

$$\dim_k \mathrm{Hom}_C(D, D') = \langle B(D), B(D')\rangle.$$

Proof. Certainly it is sufficient to show for $P, P' \in p\mathcal{O}$ the corresponding formula $\dim_k \mathrm{Hom}_{\mathcal{O}}(P, P') = \langle A(P), A(P')\rangle$. Without restriction of generality we may assume $P = P_x$, $P' = P_y$ and then we have indeed

$$\begin{array}{rcl} \dim_k \mathrm{Hom}_{\mathcal{O}}(P_x, P_y) & = & [P_y : L_x] \\ & = & \sum_z (P_y : M_z)[M_z : L_x] \\ & = & \sum_z (P_y : M_z)(P_x : M_z). \end{array}$$

□

3. Arguments from topology

In this part 3 let k be an arbitrary field.

3.1. **Notations.** As possible sources for the following generalities we cite [Bor84, KS90]. For every topological space X denote by $\mathrm{Sh}X = \mathrm{Sh}_k X$ the category of all sheaves of k-vectorspaces on X. We consider the bounded below derived category $D^+\mathrm{Sh}X$ and the cohomology sheaves $\mathcal{H} : D^+\mathrm{Sh}X \to \mathrm{Sh}X$. Morphisms in $D^+\mathrm{Sh}X$ will be denoted Hom_D. Any continous map $f : X \to Y$ induces functors $f_*, f_! : \mathrm{Sh}X \to \mathrm{Sh}Y$ and $f^{-1} : \mathrm{Sh}Y \to \mathrm{Sh}X$. All these have derived functors. We change our notations and put $f_* = R^+f_*$, $f_! = R^+f_!$ and $f^* = R^+f^{-1}$. Thus we have triangulated functors $f_*, f_! : D^+\mathrm{Sh}X \to D^+\mathrm{Sh}Y$ and $f^* : D^+\mathrm{Sh}Y \to D^+\mathrm{Sh}X$ and an adjunction (f^*, f_*). We also need Verdier duality. This doesn't exist in full generality, therefore we restrict to a comfortable situation.

Let X be a complex algebraic variety and $X(\mathbb{C})$ the corresponding analytic space (with its "metric" topology). We consider the category of "algebraically constructible sheaves" $\mathrm{Sh}_{\mathrm{ac}}X(\mathbb{C}) = \{\mathcal{F} \in \mathrm{Sh}X(\mathbb{C}) \mid$ There is a decomposition $X = \bigcup_\alpha X_\alpha$ into (finitely many) local closed subvarieties such that $j_\alpha^*\mathcal{F} \in \mathrm{Sh}X_\alpha(\mathbb{C})$ is locally constant of finite rank, for all inclusions $j_\alpha : X_\alpha(\mathbb{C}) \hookrightarrow X(\mathbb{C})\}$. Here we also could have written $j_\alpha^{-1}\mathcal{F}$, but since we already renamed j_α^{-1} into j_α^* we stay with it. Finally we associate to any complex algebraic variety X the triangulated k-category

$$\mathcal{D}(X) = \left\{ \mathcal{F} \in D^+\mathrm{Sh}X(\mathbb{C}) \;\middle|\; \begin{array}{l} \mathcal{H}^\nu\mathcal{F} \in \mathrm{Sh}_{\mathrm{ac}}X(\mathbb{C}) \text{ for all } \nu, \\ \mathcal{H}^\nu\mathcal{F} = 0 \text{ for } |\nu| \gg 0 \end{array} \right\}$$

For $\mathcal{F}, \mathcal{E} \in \mathcal{D}(X)$ we can form $\mathcal{RH}om(\mathcal{F}, \mathcal{E}) \in \mathcal{D}(X)$. If $f : X \to Y$ is a morphism of complex varieties, then $f_*, f_!, f^*$ map our $\mathcal{D}(X)$, $\mathcal{D}(Y)$ into one another. Furthermore the functor $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ admits a right adjoint $f^! : \mathcal{D}(Y) \to \mathcal{D}(X)$. I will in addition use the following nonstandard notation: Let $\underline{\mathrm{pt}} \in \mathrm{Sh}_{\mathrm{ac}}(\mathrm{pt}) \subset \mathcal{D}(\mathrm{pt})$ be the constant sheaf k on a point, and $\underline{X} \in \mathcal{D}(X)$ the constant sheaf $\underline{X} = c^*\underline{\mathrm{pt}}$, where c :

$X \to \mathrm{pt}$ is the constant map. Furthermore let $\mathbb{D}\mathcal{F} = \mathcal{RH}om(\mathcal{F}, c^!\underline{\mathrm{pt}})$ be the Verdier dual of $\mathcal{F}$. We have $\mathbb{D}^2 = \mathrm{id}$ and $\mathbb{D}f^! = f^*\mathbb{D}$, $\mathbb{D}f_! = f_*\mathbb{D}$.

3.2. **The category $\mathcal{K}$.** In the introduction we introduced $\mathcal{K}$ as the full subcategory $\mathcal{K} \subset \mathcal{D}(\mathcal{G}/\mathcal{B})$ consisting of all objects $\pi_*\underline{\mathrm{BS}}$ along with their direct sums and summands and the [ ]-translates of those, for all projections

$$\pi : \mathrm{BS} \longrightarrow \mathcal{G}/\mathcal{B}$$

of a Bott-Samelson variety to a flag manifold. For our purposes it is better to use the following more technical description of $\mathcal{K}$. We could have taken the following lemma also as the definition of $\mathcal{K}$, and then wouldn't have to prove it. With the proof we pay for greater elegance in the introduction.

Lemma 3.2.1. *$\mathcal{K}$ is the smallest full subcategory of $\mathcal{D}(\mathcal{G}/\mathcal{B})$ satisfying all of the following:*

1. *$\mathcal{K}$ contains with an object all isomorphic objects.*
2. *$\mathcal{K}$ contains the skyscraper sheaf on the one point Bruhat cell, $j_{e*}\underline{\mathcal{B}/\mathcal{B}} \in \mathcal{K}$ for $j_e : \mathcal{B}/\mathcal{B} \hookrightarrow \mathcal{G}/\mathcal{B}$.*
3. *$\mathcal{F} \in \mathcal{K} \Rightarrow \pi_s^*\pi_{s*}\mathcal{F} \in \mathcal{K}\ \ \forall s \in S$, for $\pi_s : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_s$.*
4. *$\mathcal{F} \in \mathcal{K} \Rightarrow \mathcal{F}[n] \in \mathcal{K}\ \ \forall n \in \mathbb{Z}$.*
5. *$\mathcal{K}$ is stable under direct sums and direct summands, i.e. if $\mathcal{F}, \mathcal{E} \in \mathcal{D}(\mathcal{G}/\mathcal{B})$ are given then we have: $\mathcal{F} \oplus \mathcal{E} \in \mathcal{K} \Leftrightarrow \mathcal{F}, \mathcal{E} \in \mathcal{K}$.*

Proof. The lemma easily follows once we know that

$$\pi_*\underline{\mathrm{BS}} \cong \pi_t^*\pi_{t*} \dots \pi_s^*\pi_{s*}j_*\underline{\mathcal{B}/\mathcal{B}}$$

for $t, \dots, s \in S$ and $\pi : \mathrm{BS} = \mathcal{P}_s \times^{\mathcal{B}} \dots \mathcal{P}_t \times^{\mathcal{B}} \mathcal{B}/\mathcal{B} \to \mathcal{G}/\mathcal{B}$ the multiplication. To show this we rename $t, \dots, s$ into $s_r, \dots, s_1$ and abbreviate $\mathcal{P}_{s_i} = \mathcal{P}_i$. In these notations the maps

$$\begin{aligned} f_i &: Y = \mathcal{G} \times^{\mathcal{B}} \mathcal{P}_1 \times^{\mathcal{B}} \mathcal{P}_2 \dots \times^{\mathcal{B}} \mathcal{P}_r/\mathcal{B} \to \mathcal{G}/\mathcal{B} \\ f_i &: [(g, p_1, \dots, p_r)] \mapsto gp_1 \dots p_i B \ \ (i = 0, \dots, r) \end{aligned}$$

give us an isomorphism

$$Y \xrightarrow{\sim} \left\{ (x_0, \dots, x_r) \in (\mathcal{G}/\mathcal{B})^{r+1} \;\middle|\; \begin{array}{c} x_{i-1} \text{ and } x_i \text{ have} \\ \text{the same image in } \mathcal{G}/\mathcal{P}_i \\ \text{for } i = 1, \dots, r \end{array} \right\}$$

If we write $Y = Y(1, 2, \dots, r)$, then we have in hopefully selfexplaining notations a commutative diagram of cartesian squares that for $r = 3$

has the following form:

$$\begin{array}{ccccccc}
Y(1,2,3) & \to & Y(2,3) & \to & Y(3) & \to & \mathcal{G}/\mathcal{B} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \pi_t \\
Y(1,2) & \to & Y(2) & \to & \mathcal{G}/\mathcal{B} & \underset{\pi_t}{\to} & \mathcal{G}/\mathcal{P}_3 \\
\downarrow & & \downarrow & & \downarrow & & \\
Y(1) & \to & \mathcal{G}/\mathcal{B} & \to & \mathcal{G}/\mathcal{P}_2 & & \\
\downarrow & & \downarrow \pi_s & & & & \\
\mathcal{G}/\mathcal{B} & \underset{\pi_s}{\to} & \mathcal{G}/\mathcal{P}_1 & & & &
\end{array}$$

Now proper base change implies

$$f_{r*}f_0^* = \pi_t^*\pi_{t*}\dots\pi_s^*\pi_{s*}.$$

Finally we remark that the left vertical of our diagram is part of a cartesian square

$$\begin{array}{ccc}
\mathrm{BS} & \hookrightarrow & Y(1,\dots,r) \\
\downarrow & & \downarrow \\
\mathcal{B}/\mathcal{B} & \hookrightarrow & \mathcal{G}/\mathcal{B}.
\end{array}$$

The claim follows. □

Remark 3.2.2. $\mathcal{F} \in \mathcal{K} \Rightarrow \mathbb{D}\mathcal{F} \in \mathcal{K}$. Indeed we have $\mathbb{D}\pi_{s*} = \mathbb{D}\pi_{s!} = \pi_{s*}\mathbb{D}$ since π_s is proper, and $\mathbb{D}\pi_s^* = \pi_s^!\mathbb{D} = \pi_s^*[2]\mathbb{D}$ since π_s is smooth with fibres of real dimension 2.

To study $\mathcal{K}$ we need some auxiliary categories. We put $\mathcal{D}_\mathcal{B}(\mathcal{G}/\mathcal{B}) = \{\mathcal{F} \in \mathcal{D}(\mathcal{G}/\mathcal{B}) \mid \mathcal{H}^\nu j_x^*\mathcal{F}$ is a constant sheaf on $\mathcal{B}x\mathcal{B}/\mathcal{B}$, for all $x \in W, \nu \in \mathbb{Z}\}$ and

$$\mathcal{D}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}} = \{\mathcal{F} \in \mathcal{D}(\mathcal{G}/\mathcal{B}) \mid \mathcal{H}^\nu\mathcal{F} \neq 0 \Rightarrow \nu \text{ is even }\}$$

and consider

$$\mathcal{D}_\mathcal{B}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}} = \mathcal{D}_\mathcal{B}(\mathcal{G}/\mathcal{B}) \cap \mathcal{D}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}}.$$

In addition let $\mathbf{H} = \mathbf{H}(W,S) = \bigoplus_{x\in W}\mathbb{Z}[t,t^{-1}]\mathbf{T}_x$ be the Hecke algebra as in [KL79] or [Soe97], with $t = q^{1/2} = v^{-1}$. We define $h : \mathcal{D}_\mathcal{B}(\mathcal{G}/\mathcal{B}) \to \mathbf{H}$ by

$$h(\mathcal{F}) = \sum_{\nu,x} \mathrm{rk}\mathcal{H}^\nu(j_x^*\mathcal{F})t^\nu\mathbf{T}_x.$$

Clearly we have $h(\mathcal{F}[1]) = t^{-1}h(\mathcal{F})$. Less clear is the following

Lemma 3.2.3. *Take $\mathcal{F} \in \mathcal{D}_\mathcal{B}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}}$ and $s \in S$. Then we also have $\pi_s^*\pi_{s*}\mathcal{F} \in \mathcal{D}_\mathcal{B}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}}$ and $h(\pi_s^*\pi_{s*}\mathcal{F}) = h(\mathcal{F})(\mathbf{T}_s + 1)$.*

Proof. We repeat a proof of [Spr82]. Springer assigns it to MacPherson. Consider the cartesian digaram

$$\begin{array}{ccc} \mathcal{B}x\mathcal{P}_s/\mathcal{B} & \stackrel{j}{\to} & \mathcal{G}/\mathcal{B} \\ \pi\downarrow & & \downarrow\pi_s \\ \mathcal{B}x\mathcal{P}_s/\mathcal{P}_s & \stackrel{j}{\to} & \mathcal{G}/\mathcal{P}_s \end{array}$$

and assume that $xs > x$. Then we have a decomposition of the upper left corner

$$\mathcal{B}xs\mathcal{B}/\mathcal{B} \stackrel{u}{\hookrightarrow} \mathcal{B}x\mathcal{P}_s/\mathcal{B} \stackrel{i}{\hookleftarrow} \mathcal{B}x\mathcal{B}/\mathcal{B}$$

into an open and a closed subset with inclusions u and i. For any $\mathcal{F} \in \mathcal{D}(\mathcal{G}/\mathcal{B})$ proper base change gives $j_x^*(\pi_s^*\pi_{s*}\mathcal{F}) = i^*\pi^*\pi_*j^*\mathcal{F}$ and $j_{xs}^*(\pi_s^*\pi_{s*}\mathcal{F}) = u^*\pi^*\pi_*j^*\mathcal{F}$.

Now the distinguished triangle $\pi^*\pi_*(u_!u^!, \mathrm{id}, i_*i^*)j^*\mathcal{F}$ leads under i^*, u^* to two distinguished triangles which can be written after some standard transformations as

$$(\pi\circ i)^*(\pi\circ u)_!j_{xs}^*\mathcal{F} \to j_x^*(\pi_s^*\pi_{s*}\mathcal{F}) \to (\pi\circ i)^*(\pi\circ i)_*j_x^*\mathcal{F} \stackrel{[1]}{\to}$$

$$(\pi\circ u)^*(\pi\circ u)_!j_{xs}^*\mathcal{F} \to j_{xs}^*(\pi_s^*\pi_{s*}\mathcal{F}) \to (\pi\circ u)^*(\pi\circ i)_*j_x^*\mathcal{F} \stackrel{[1]}{\to} .$$

But $(\pi\circ i)$ is an isomorphism and $(\pi\circ u)$ is a projection of the affine space $\mathcal{B}xs\mathcal{B}/\mathcal{B}$ to the affine space $\mathcal{B}x\mathcal{P}_s/\mathcal{P}_s$ with fibre $\mathbb{C}$. Therefore we have $(\pi\circ u)_!\underline{\mathcal{B}xs\mathcal{B}/\mathcal{B}} \cong \underline{\mathcal{B}x\mathcal{P}_s/\mathcal{P}_s}[-2]$. If we now assume $\mathcal{F} \in \mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}}$, then with our two distinguished triangles we have $\pi_s^*\pi_{s*}\mathcal{F} \in \mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}}$ and

$$\mathrm{rk}\mathcal{H}^{\nu-2}j_{xs}^*\mathcal{F} + \mathrm{rk}\mathcal{H}^{\nu}j_x^*\mathcal{F} = \mathrm{rk}\mathcal{H}^{\nu}j_x^*(\pi_s^*\pi_{s*}\mathcal{F}) = \mathrm{rk}\mathcal{H}^{\nu}j_{xs}^*(\pi_s^*\pi_{s*}\mathcal{F}).$$

These equations imply by an easy calculation in the Hecke algebra $h(\pi_s^*\pi_{s*}\mathcal{F}) = h(\mathcal{F})(\mathbf{T}_s + 1)$. □

Corollary 3.2.4.
1. *Any $\mathcal{F} \in \mathcal{K}$ can be written as $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''[1]$ with $\mathcal{F}', \mathcal{F}'' \in \mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}}$.*
2. *For $\mathcal{F} \in \mathcal{K}$, $s \in S$ we have $h(\pi_s^*\pi_{s*}\mathcal{F}) = h(\mathcal{F})(\mathbf{T}_s + 1)$.*

Proof. Left to the reader. □

Let us now consider as in [KL79] on the Hecke algebra $\mathbf{H}$ the involution $i : \mathbf{H} \to \mathbf{H}$ mit $i(t) = t^{-1}$, $i(\mathbf{T}_x) = (\mathbf{T}_{x^{-1}})^{-1}$.

Lemma 3.2.5. *$\mathcal{F} \in \mathcal{K} \Rightarrow h(\mathbb{D}\mathcal{F}) = ih(\mathcal{F})$.*

Proof. It is sufficient to establish the following three claims: (1) The formula holds for $j_{e*}\underline{\mathcal{B}/\mathcal{B}}$ and (2) if the formula holds for $\mathcal{F}$, then it also holds (a) for $\mathcal{F}[1]$ and (b) for $\pi_s^*\pi_{s*}\mathcal{F}$. Here (1) and (2)(a) are trivial, and we only have to check (2)(b). But indeed we have

$$h(\mathbb{D}\pi_s^*\pi_{s*}\mathcal{F}) = h(\pi_s^!\pi_{s!}\mathbb{D}\mathcal{F}) = h(\pi_s^*\pi_{s*}\mathbb{D}\mathcal{F}[2]) = (ih(\mathcal{F}))(\mathbf{T}_s+1)t^{-2}$$

and

$$ih(\pi_s^*\pi_{s*}\mathcal{F}) = i(h(\mathcal{F})(\mathbf{T}_s+1)) = h(\mathbb{D}\mathcal{F})(\mathbf{T}_s+1)t^{-2}$$

what was to be shown. □

For arbitrary X let us consider the hypercohomology functor

$$c_* = \mathbb{H}^\bullet = \bigoplus_\nu \mathrm{Hom}_{\mathcal{D}}(\underline{X}[\nu], \quad) : \mathcal{D}(X) \to k\text{-mod}^{\mathrm{fl}}.$$

Proposition 3.2.6. *For all $\mathcal{F},\mathcal{E} \in \mathcal{K}$ hypercohomology induces an injection*

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F},\mathcal{E}) \to \mathrm{Hom}_k(\mathbb{H}^\bullet\mathcal{F}, \mathbb{H}^\bullet\mathcal{E}).$$

Proof. Let in full generality X be a topological space. For $\mathcal{F} \in D^+\mathrm{Sh}X$, its hypercohomology $\mathbb{H}^\bullet\mathcal{F}$ can be computed as follows: Let us take an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and then $\mathbb{H}^\bullet\mathcal{F}$ is the cohomology of the complex $\Gamma(\mathcal{I}^\bullet)$ of global sections. Now if $X = X_0 \supset X_1 \supset \ldots \supset X_r = \emptyset$ is a filtration by closed subsets, then $\Gamma(\mathcal{I}^\bullet)$ is filtered by sections with support in the X_i,

$$0 = \Gamma_{X_r}(\mathcal{I}^\bullet) \subset \ldots \subset \Gamma_{X_0}(\mathcal{I}^\bullet) = \Gamma(\mathcal{I}^\bullet).$$

Therefore $\mathbb{H}^\bullet\mathcal{F}$ is the limit of a spectral sequence with E_1-term

$$E_1^{p,q} = \mathbb{H}^{p+q}_{X_p - X_{p+1}}(\mathcal{F})$$

the local hypercohomology of $\mathcal{F}$ along $X_p - X_{p+1}$. If $v : Y \to X$ is a locally closed embedding of algebraic varieties and $\mathcal{F} \in \mathcal{D}(X)$, then we have $\mathbb{H}^\bullet_Y\mathcal{F} = \mathbb{H}^\bullet v^!\mathcal{F}$ in our formalism.

Let us now consider on $X = \mathcal{G}/\mathcal{B}$ the filtration by the

$$X_p = \bigcup_{l(x)+p \le \dim X} \mathcal{B}x\mathcal{B}/\mathcal{B}.$$

Let $j_p : (X_p - X_{p+1}) \hookrightarrow X$ be the inclusion. For $\mathcal{F} \in \mathcal{K}$ our spectral sequence

$$\mathbb{H}^{p+q} j_p^!\mathcal{F} \Rightarrow \mathbb{H}^\bullet\mathcal{F}$$

degenerates already at this E_1-term: The reason is that we may assume without restriction of generality $\mathbb{D}\mathcal{F} \in \mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})^{\mathrm{ev}}$, and then our E_1-terms vanish as a chessboard.

Take now $\mathcal{F},\mathcal{E} \in \mathcal{K}$ and $f \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F},\mathcal{E})$. If $\mathbb{H}^\bullet f$ vanishes, then necessarily $0 = j_p^! f \in \mathrm{Hom}_{\mathcal{D}}(j_p^!\mathcal{F}, j_p^!\mathcal{E})$ for all p, since our spectral sequence degenerates. Let us denote by $i_p : X_p \hookrightarrow X$ the inclusion. We have a decomposition

$$X_p - X_{p+1} \overset{u}{\hookrightarrow} X_p \overset{i}{\hookleftarrow} X_{p+1}$$

into an open and a closed subset and a distinguished triangle

$$(i_! i^!, \mathrm{id}, u_* u^*) i_p^!$$

from which we deduce that $i_{p+1}^! f = 0 = j_{p+1}^! f \Rightarrow i_p^! f = 0$. Thus by induction $j_p^! = 0 \ \forall p$ implies $f = i_0^! f = 0$. □

Next we calculate the dimension of morphism spaces between objects of $\mathcal{K}$. We can define on the Hecke algebra an involutive antiautomorphism $j : \mathbf{H} \to \mathbf{H}$ by the prescription $j(t) = t^{-1}$, $j(\mathbf{T}_x) = (\mathbf{T}_x)^{-1}$. For any two elements $A, B \in \mathbf{H}$ let us define $\langle A, B\rangle(t) \in \mathbb{Z}[t, t^{-1}]$ as the coefficient of $\mathbf{T}_e$ in the product $Aj(B)$.

Theorem 3.2.7. *For $\mathcal{F}, \mathcal{E} \in \mathcal{K}$ we have*

$$\sum_\nu \dim_k \mathrm{Hom}_{\mathcal{D}}^\nu(\mathcal{F}, \mathcal{E}) t^{-\nu} = \langle h(\mathcal{F}), h(\mathcal{E})\rangle(t).$$

Proof. For $\mathcal{E} = j_{e*}\mathcal{B}/\mathcal{B}$ this follows from the definitions and the adjunction (j_e^*, j_{e*}). To get from this the general case we only have to show that our formula is valid for a pair $\mathcal{F}, \pi_s^*\pi_{s*}\mathcal{E}[2]$ if and only if it is valid for the pair $\pi_s^*\pi_{s*}\mathcal{F}, \mathcal{E}$.

But now (π_s^*, π_{s*}) is an adjoint pair and so is $(\pi_{s!}, \pi_s^!) = (\pi_{s*}, \pi_s^*[2])$, therefore also $(\pi_s^*\pi_{s*}, \pi_s^*\pi_{s*}[2])$ is an adjoint pair. Thus the left hand side of the formula in the theorem is the same for both pairs. We therefore only have to show that

$$\langle a, bt^{-2}(T_s + 1)\rangle = \langle a(T_s + 1), b\rangle$$

for all $a, b \in H$. But indeed $j(t^{-2}(T_s + 1)) = T_s + 1$. □

3.3. **Perverse sheaves on $\mathcal{G}/\mathcal{B}$ and $\mathcal{G}/\mathcal{P}_s$.** Let $\mathcal{P}(X) \subset \mathcal{D}(X)$ be the category of perverse sheaves. Let us denote by $H^i : \mathcal{D}(X) \to \mathcal{P}(X)$ the perverse cohomology functors. Let $\mathcal{P}_{\mathcal{B}}(\mathcal{G}/\mathcal{P}_s)$ be the perverse sheaves on $\mathcal{G}/\mathcal{P}_s$ which are smooth along the $\mathcal{B}$-orbits. Let us furthermore denote by $\mathcal{P}_{\mathcal{P}_s^{-1}}(\mathcal{G}/\mathcal{B})$ the perverse sheaves on $\mathcal{G}/\mathcal{B}$ which are smooth along all strata $\mathcal{B}x\mathcal{P}_s/\mathcal{B}$ for $x \in W$. We want to show:

Theorem 3.3.1. *Suppose* $\mathrm{char} k \neq 2$. *Then the functors $\pi_s^![-1]$ and $H^1\pi_{s!}$ define an equivalence of categories $\mathcal{P}_{\mathcal{P}_s^{-1}}(\mathcal{G}/\mathcal{B}) \cong \mathcal{P}_{\mathcal{B}}(\mathcal{G}/\mathcal{P}_s)$.*

Proof. First we show that $H^1\pi_{s!} : \mathcal{P}(\mathcal{G}/\mathcal{B}) \to \mathcal{P}(\mathcal{G}/\mathcal{P}_s)$ is right adjoint to $\pi_s^![-1] : \mathcal{P}(\mathcal{G}/\mathcal{P}_s) \to \mathcal{P}(\mathcal{G}/\mathcal{B})$. Indeed for $\mathcal{F} \in \mathcal{P}(\mathcal{G}/\mathcal{B})$, $\mathcal{E} \in \mathcal{P}(\mathcal{G}/\mathcal{P}_s)$ we have an equation

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \pi_s^![-1]\mathcal{E}) = \mathrm{Hom}_{\mathcal{D}}(\pi_{s!}[1]\mathcal{F}, \mathcal{E}) = \mathrm{Hom}_{\mathcal{D}}(H^1\pi_{s!}[1]\mathcal{F}, \mathcal{E}),$$

the second equality since $H^i\pi_{s!}[1]\mathcal{F} = 0$ for $i > 0$. Now we need

Lemma 3.3.2. *Suppose* char$k \neq 2$. *Then we have a decomposition* $\pi_{s*}\underline{\mathcal{G}/\mathcal{B}} \cong \underline{\mathcal{G}/\mathcal{P}_s} \oplus \underline{\mathcal{G}/\mathcal{P}_s}[-2]$.

Proof. We have completely canonical morphisms

$$\underline{\mathcal{G}/\mathcal{P}_s} \xrightarrow{a} \pi_{s*}\pi_s^*\underline{\mathcal{G}/\mathcal{P}_s} = \pi_{s*}\underline{\mathcal{G}/\mathcal{B}} = \pi_{s!}\pi_s^!\underline{\mathcal{G}/\mathcal{P}_s}[-2] \xrightarrow{b} \underline{\mathcal{G}/\mathcal{P}_s}[-2].$$

Now we choose noncanonically a compact real form $K \subset \mathcal{G}$ with maximal torus $T = K \cap \mathcal{B}$. So we get $\mathcal{G}/\mathcal{B} = K/T$ and $s \in S$ acts on K/T stabilizing the fibres of π_s. Therefore s defines an involution on $\pi_{s*}\underline{\mathcal{G}/\mathcal{B}}$ and even an involution in the category of all complexes of sheaves on $\pi_{s*}\mathcal{I}^\bullet$, where $\mathcal{I}^\bullet$ denotes the canonical flasque resolution of $\underline{\mathcal{G}/\mathcal{B}}$ in the sense of Godement. Now $\pi_{s*}\mathcal{I}^\bullet$ and therefore $\pi_{s*}\underline{\mathcal{G}/\mathcal{B}}$ decompose into eigenspaces under this involution, i.e. $\pi_{s*}\underline{\mathcal{G}/\mathcal{B}} = E^+ \oplus E^-$. Denote by i_+, i_-, p_+, p_- the projections and injections corresponding to this decomposition. It is easy to see that $0 = p_- \circ a = b \circ i_+$ and that $p_+ \circ a$ and $b \circ i_-$ are isomorphisms. The lemma follows. $\square$

Lemma 3.3.3. *Suppose* char$k \neq 2$. *Then there are natural isomorphisms* $\pi_{s!}\pi_s^!\mathcal{F} \cong \mathcal{F} \oplus \mathcal{F}[2]$ *for* $\mathcal{F} \in \mathcal{D}(\mathcal{G}/\mathcal{P}_s)$.

Proof. We find

$$\begin{aligned} \pi_{s!}\pi_s^!\mathcal{F} &= \mathcal{RH}om(\underline{\mathcal{G}/\mathcal{P}_s}, \pi_{s!}\pi_s^!\mathcal{F}) \\ &= \pi_{s*}\mathcal{RH}om(\underline{\mathcal{G}/\mathcal{B}}, \pi_s^!\mathcal{F}) \\ &= \mathcal{RH}om(\pi_{s*}\underline{\mathcal{G}/\mathcal{B}}, \mathcal{F}) \\ &= \mathcal{F} \oplus \mathcal{F}[2] \end{aligned}$$

where the third step is Verdier duality and the last step the preceeding lemma. $\square$

Remark 3.3.4. More precisely we have shown: Any choice of a decomposition

$$\pi_{s*}\underline{\mathcal{G}/\mathcal{B}} \cong \underline{\mathcal{G}/\mathcal{P}_s} \oplus \underline{\mathcal{G}/\mathcal{P}_s}[-2]$$

gives a natural equivalence between the functors $\mathcal{F} \mapsto \pi_{s!}\pi_s^!\mathcal{F}$ and $\mathcal{F} \mapsto \mathcal{F} \oplus \mathcal{F}[2]$.

We deduce that for any $\mathcal{F} \in \mathcal{P}(\mathcal{G}/\mathcal{P}_s)$ the adjointness map

$$(H^1\pi_{s!})(\pi_s^![-1])\mathcal{F} \to \mathcal{F}$$

is an isomorphism.

Furthermore we need:

Lemma 3.3.5. *The functor* $H^1\pi_{s!} : \mathcal{P}_{\mathcal{P}_s^{-1}}(\mathcal{G}/\mathcal{B}) \to \mathcal{P}_{\mathcal{B}}(\mathcal{G}/\mathcal{P}_s)$ *is exact.*

Proof. It is sufficient to show that for any object $\mathcal{F} \in \mathcal{P}_{\mathcal{P}_s^{-1}}(\mathcal{G}/\mathcal{B})$ we have $H^{2i}(\pi_{s!}\mathcal{F}) = 0 \ \forall i$. It is even sufficient to show this for all simple objects of $\mathcal{P}_{\mathcal{P}_s^{-1}}(\mathcal{G}/\mathcal{B})$, i.e. for all $\mathcal{IC}(\overline{\mathcal{B}x\mathcal{B}/\mathcal{B}}) = \pi_s^![-1]\mathcal{IC}(\overline{\mathcal{B}x\mathcal{P}_s/\mathcal{P}_s})$ with $x \in W$, $xs < x$. For these objects however this follows from the preceeding lemma. $\square$

Finally we show that for all $\mathcal{F} \in \mathcal{P}_{\mathcal{P}_s^{-1}}(\mathcal{G}/\mathcal{B})$ the adjointness map $\mathcal{F} \to (\pi_s^![-1])(H^1\pi_{s!})\mathcal{F}$ is an isomorphism. This is immediate for all $\mathcal{F}$ of the form $\pi_s^![-1]\mathcal{E}$ with $\mathcal{E} \in \mathcal{P}_{\mathcal{B}}(\mathcal{G}/\mathcal{P}_s)$, thus for all simple $\mathcal{F}$. It follows for all $\mathcal{F}$ by the five lemma. $\square$

Corollary 3.3.6. *Take* $\mathcal{E} \in \mathcal{P}_{\mathcal{B}}(\mathcal{G}/\mathcal{P}_s)$. *Then we also have* $\pi_s^![-1]\mathcal{E} \in \mathcal{P}(\mathcal{G}/\mathcal{B})$, *and* $\mathcal{E}$ *is semisimple if and only if* $\pi_s^![-1]\mathcal{E}$ *is semisimple.*

4. Topology and representation theory

Let from now on the charakteristic of k be again bigger than the Coxeter number. (In the following subsection 4.1 this condition is not yet needed.)

4.1. **Preliminary remarks.** If X is a topological space, we can interprete its cohomology ring as the endomorphism algebra of the constant sheaf in the derived category

$$H^\bullet X = \mathrm{End}_D^\bullet \underline{X} = \bigoplus_\nu \mathrm{Hom}_D(\underline{X}, \underline{X}[\nu]).$$

In the same way we can interprete the hypercohomology of a complex of sheaves $\mathcal{F} \in D^+\mathrm{Sh}X$ as

$$\mathbb{H}^\bullet \mathcal{F} = \mathrm{Hom}_D^\bullet(\underline{X}, \mathcal{F}) = \bigoplus_\nu \mathrm{Hom}_D(\underline{X}, \mathcal{F}[\nu]).$$

The composition of morphisms in the derived category thus makes $\mathbb{H}^\bullet\mathcal{F}$ in a natural way into a right module over $H^\bullet X$. If $f : X \to Y$ is a continous map and $\mathcal{F} \in D^+\mathrm{Sh}X$, then we clearly have

$$\mathbb{H}^\bullet f_*\mathcal{F} = \mathrm{Hom}_D^\bullet(\underline{Y}, f_*\mathcal{F}) = \mathrm{Hom}_D(\underline{X}, \mathcal{F}) = \mathbb{H}^\bullet\mathcal{F}$$

via $\underline{X} = f^*\underline{Y}$, and one sees that the $H^\bullet Y$-action on $\mathbb{H}^\bullet f_*\mathcal{F}$ comes by restriction via $H^\bullet Y \to H^\bullet X$ from the $H^\bullet X$-action on $\mathbb{H}^\bullet\mathcal{F}$.

If on the other hand $\mathcal{F} \in D^+\mathrm{Sh}Y$ is given, we have an obvious morphism

$$\mathbb{H}^\bullet\mathcal{F} \otimes_{H^\bullet Y} H^\bullet X \to \mathbb{H}^\bullet f^*\mathcal{F}$$

or written differently

$$\begin{array}{rcl}\mathrm{Hom}_D^\bullet(\underline{Y},\mathcal{F})\otimes_{\mathrm{End}_D^\bullet\underline{Y}}\mathrm{End}_D^\bullet\underline{X} & \rightarrow & \mathrm{Hom}_D^\bullet(\underline{X},f^*\mathcal{F}) \\ \gamma\otimes a & \mapsto & f^*(\gamma)\circ a.\end{array}$$

We show that this map is an isomorphism in a special case.

Proposition 4.1.1. *Suppose* char$k \neq 2$. *Let $\pi_s:\mathcal{G}/\mathcal{B}\to\mathcal{G}/\mathcal{P}_s$ be the projection map. Then for all $\mathcal{F}\in\mathcal{D}(\mathcal{G}/\mathcal{P}_s)$ the map*

$$\mathbb{H}^\bullet\mathcal{F}\otimes_{H^\bullet(\mathcal{G}/\mathcal{P}_s)}H^\bullet(\mathcal{G}/\mathcal{B})\to\mathbb{H}^\bullet(\pi_s^*\mathcal{F})$$

explained above is an isomorphism of $\mathbb{Z}$-graded right $\mathbb{H}^\bullet(\mathcal{G}/\mathcal{B})$-modules.

Proof. For this proof we abreviate $\pi_s:\mathcal{G}/\mathcal{B}\to\mathcal{G}/\mathcal{P}_s$ as $\pi: X\to Y$. Since π is smooth with complex one dimensional fibres we have $\pi^*\cong\pi^![-2]$. Thus it is sufficient to show that the map

$$\begin{array}{rcl}\mathrm{Hom}_D^\bullet(\underline{Y},\mathcal{F})\otimes_{\mathrm{End}_D^\bullet\underline{Y}}\mathrm{End}_D^\bullet(\pi^!\underline{Y}) & \rightarrow & \mathrm{Hom}_D^\bullet(\pi^!\underline{Y},\pi^!\mathcal{F}), \\ \gamma\otimes a & \mapsto & \pi^!(\gamma)\circ a\end{array}$$

is an isomorphism. Using the adjunction $(\pi_!,\pi^!)$ it is even sufficient to show that

$$\begin{array}{rcl}\mathrm{Hom}_D^\bullet(\underline{Y},\mathcal{F})\otimes_{\mathrm{End}_D^\bullet\underline{Y}}\mathrm{Hom}_D^\bullet(\pi_!,\pi^!\underline{Y},\underline{Y}) & \rightarrow & \mathrm{Hom}_D^\bullet(\pi_!\pi^!\underline{Y},\mathcal{F}) \\ \gamma\otimes b & \mapsto & \gamma\circ b\end{array}$$

is an isomorphism. But now we can use the result $\pi_!\pi^!\underline{Y}\cong\underline{Y}\oplus\underline{Y}[2]$ from 3.3.2. □

4.2. **Equivalence of $\mathcal{K}$ and $\mathcal{C}$.** Let again $C=H^\bullet(\mathcal{G}/\mathcal{B};k)$ be the coinvariant algebra alias the cohomology ring of the flag manifold. Certainly hypercohomology is a functor $\mathbb{H}^\bullet:\mathcal{D}(\mathcal{G}/\mathcal{B})\to C\text{-mod}^{\mathrm{fl}}$. We want to show:

Theorem 4.2.1. *The hypercohomology functor induces an equivalence of categories $\mathbb{H}^\bullet:\mathcal{K}\to\mathcal{C}$.*

Proof. We procede in steps.

1. By 4.1.1 for all $s\in S$, $\mathcal{F}\in\mathcal{D}(\mathcal{G}/\mathcal{B})$ we have an isomorphism of graded C-modules $\mathbb{H}^\bullet\pi_s^*\pi_{s*}\mathcal{F}\cong C\otimes_{C^s}\mathbb{H}^\bullet\mathcal{F}$. Now certainly we have $\mathbb{H}^\bullet(j_{e*}\underline{\mathcal{B}/\mathcal{B}})=k$ and from the definitions we deduce easily that $\mathbb{H}^\bullet$ induces a functor from $\mathcal{K}$ to $\mathcal{C}$.
2. Next we show that $\mathbb{H}^\bullet:\mathcal{K}\to\mathcal{C}$ is fully faithful. Since we know already by proposition 3.2.6 that our functor is faithful we only have to show for all $\mathcal{F},\mathcal{E}\in\mathcal{K}$ that

$$\dim_k\mathrm{Hom}_{\mathcal{D}}^\bullet(\mathcal{F},\mathcal{E})=\dim_k\mathrm{Hom}_C(\mathbb{H}^\bullet\mathcal{F},\mathbb{H}^\bullet\mathcal{E}).$$

To show this we claim that there is a commutative diagram

$$\begin{array}{ccc} \langle\mathcal{K}\rangle & \xrightarrow{h} & \mathbf{H} \\ \mathbb{H}^\bullet \downarrow & & \downarrow t=1 \\ \langle v\mathcal{C}\rangle & \xrightarrow{B} & \mathbb{Z}[W]. \end{array}$$

Indeed the class of $\pi_t^*\pi_{t*}\dots\pi_s^*\pi_{s*}j_{e*}\underline{\mathcal{B}/\mathcal{B}}$ gets maped both ways to the same element $(t+1)\dots(s+1)$, for all $t,\dots,s\in S$. Equality of dimensions now is a consequence of 2.11.2, 3.2.7.

3. Finally we have to show that every object of $\mathcal{C}$ is isomorphic to the hypercohomology of an object of $\mathcal{K}$. It will be sufficient to find for all $x\in W$ an $\mathcal{L}_x\in\mathcal{K}$ such that $\mathbb{H}^\bullet\mathcal{L}_x\cong D_x$. We procede by induction on the length of x. Certainly we can assume $\mathcal{L}_e = j_{e*}\underline{\mathcal{B}/\mathcal{B}}$. To construct $\mathcal{L}_x$ we choose a reduced decomposition $x = t\dots s$ $(s,\dots,t\in S)$ and consider $\pi_s^*\pi_{s*}\dots\pi_t^*\pi_{t*}\mathcal{L}_e[l(x)]$. We know that
$$\begin{aligned}\mathbb{H}^\bullet(\pi_s^*\pi_{s*}\dots\pi_t^*\pi_{t*}\mathcal{L}_e[l(x)]) &= C\otimes_{C^s}C\dots\otimes_{C^t}k\langle -l(x)\rangle \\ &= D_x\oplus \text{Rest},\end{aligned}$$
where Rest is a sum of $D_y\langle\nu\rangle$ with $y<x$, $\nu\in\mathbb{Z}$. By the induction hypothesis we have Rest $=\mathbb{H}^\bullet\mathcal{L}_{\text{Rest}}$ for suitable $\mathcal{L}_{\text{Rest}}\in\mathcal{K}$. Since we know already that $\mathbb{H}^\bullet:\mathcal{K}\to\mathcal{C}$ is fully faithful, we get a decomposition of the identity on $\mathcal{L}_{\text{Rest}}$ into $\mathrm{id}=p\circ i$,
$$\mathcal{L}_{\text{Rest}}\xrightarrow{i}\pi_s^*\pi_{s*}\dots\pi_t^*\pi_{t*}\mathcal{L}_e[l(x)]\xrightarrow{p}\mathcal{L}_{\text{Rest}}.$$
If we complete i to a distinguished triangle
$$\mathcal{L}_{\text{Rest}}\xrightarrow{i}\pi_s^*\pi_{s*}\dots\pi_t^*\pi_{t*}\mathcal{L}_e[l(x)]\to\mathcal{L}_x\xrightarrow{d}$$
the boundery map $d=(p\circ i)\circ d = p\circ(i\circ d)$ disappears. Therefore the middle is isomorphic to $\mathcal{L}_{\text{Rest}}\oplus\mathcal{L}_x$ and $\mathbb{H}^\bullet\mathcal{L}_x\cong D_x$.

□

Corollary 4.2.2. $\{\mathcal{L}_x[\nu]\}_{x\in W,\nu\in\mathbb{Z}}$ *is a system of representatives for the isomorphism classes of indecomposable objects of* $\mathcal{K}$.

We also can characterize the $\mathcal{L}_x$ as follows:

Proposition 4.2.3. *The following conditions on* $\mathcal{L}\in\mathcal{K}$ *are equuvivalent:*

1. $\mathcal{L}$ *is indecomposable,* $\operatorname{supp}\mathcal{L}\subset\overline{\mathcal{B}x\mathcal{B}/\mathcal{B}}$ *and* $j_x^*\mathcal{L}\cong\underline{\mathcal{B}x\mathcal{B}/\mathcal{B}}[l(x)]$.
2. $\mathcal{L}\cong\mathcal{L}_x$.

Proof. For any reduced decomposition $x=t\dots s$ the object $\mathcal{F} = \pi_s^*\pi_{s*}\dots\pi_t^*\pi_{t*}\mathcal{L}_e[l(x)]$ already satisfies $j_x^*\mathcal{F}\cong\underline{\mathcal{B}x\mathcal{B}/\mathcal{B}}[l(x)]$ and it satisfies as well $\operatorname{supp}\mathcal{F}\subset\overline{\mathcal{B}x\mathcal{B}/\mathcal{B}}$. Now the proposition is clear. □

Corollary 4.2.4. $\mathbb{D}\mathcal{L}_x \cong \mathcal{L}_x \ \forall x \in W$.

Proof. Left to the reader. □

We now show part 2 of the theorem 1.2 from the introduction.

Theorem 4.2.5. *For all $x, y \in W$ we have*

$$[M_x : L_y] = \sum_i \dim(\mathcal{H}^i \mathcal{L}_y)_x.$$

Proof. We recall the reciprocity formula $[M_x : L_y] = (P_y : M_x)$. Let us consider the commutative diagram

$$\begin{array}{ccccc} \langle \mathcal{K} \rangle & \stackrel{\mathbb{H}^\bullet}{\to} & \langle v\mathcal{C} \rangle & \xrightarrow{\mathbb{V}} & \langle p\mathcal{O} \rangle \\ \downarrow h & & \downarrow B & & \downarrow A \\ \mathbf{H} & \xrightarrow{t=1} & \mathbb{Z}[W] & = & \mathbb{Z}[W] \end{array}$$

from the proof of 4.2.1 and section 2.11. We know already $\mathbb{H}^\bullet \mathcal{L}_x \cong D_x \cong \mathbb{V} P_x$. We deduce that $h(\mathcal{L}_x)$ specializes to $A(P_x)$ at $t = 1$. □

Finally we show Corollary 1.0.3 from the introduction.

Proof. 2 $\Rightarrow$ 1. From 2 we deduce by induction that all $\mathcal{L}_x$ are simple perverse sheaves. By the characterization of intersection cohomology in [BBD82] we have then $h(\mathcal{L}_x) \in H_x + \sum_y v\mathbb{Z}[v] H_y$ in the notations of [Soe97]. But we know already by 4.2.4, 3.2.5 that $h(\mathcal{L}_x) = h(\mathbb{D}\mathcal{L}_x) = ih(\mathcal{L}_x)$ and this implies $h(\mathcal{L}_x) = C'_x$ with C'_x as in [KL79]. And the conjectures of Lusztig say in our case (after translation by the reciprocity formula) precisely that $A(P_x) = C'_x(1)$.

1 $\Rightarrow$ 2. Left to the reader. □

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