THE HOCHSCHILD COHOMOLOGY RING OF REGULAR MAXIMAL PRIMITIVE QUOTIENTS OF ENVELOPING ALGEBRAS OF SEMISIMPLE LIE ALGEBRAS

WOLFGANG SOERGEL

ABSTRACT. Let U be the enveloping algebra of a semisimple complex Lie algebra, χ a regular maximal ideal of its center. We show that the Hochschild cohomology ring of $U/\chi U$ is just the coinvariant algebra of the Weyl group. This turns out to be almost immediate if one uses localization.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $U = U(\mathfrak{g}) \supset Z$ the enveloping algebra with its center. Let $S = S(\mathfrak{h}^*)$ be the regular functions on $\mathfrak{h}$ and $S^+ \subset S$ the maximal ideal of all functions vanishing at $0 \in \mathfrak{h}$.

The Weyl group $\mathcal{W}$ acts on S and we consider the coinvariant algebra $C = S/(S^+)^{\mathcal{W}}S$, the quotient of S by the ideal generated by all $\mathcal{W}$-invariant functions vanishing at zero.

Theorem 1. *Let $\chi \subset Z$ be a regular maximal ideal, i.e. the kernel of a regular central character. Then the Hochschild cohomology ring of $U/\chi U$ is the coinvariant algebra with its obvious grading doubled, so $HH^n(U/\chi U) = 0$ for n odd and*

$$HH^{2\bullet}(U/\chi U) = C^\bullet.$$

Remark . I thank Patrick Polo for telling me this problem along with the expected answer, and for his helpful comments on a first version.

Proof. By the Hochschild cohomology ring of an associative k-algebra A we mean just the ring $HH^\bullet(A) = \mathrm{Ext}^\bullet_{A\otimes A^{\mathrm{opp}}}(A, A)$ of self-extensions of A considered as an $A \otimes_k A^{\mathrm{opp}}$-module.

I now give an outline of the proof. Let X be the variety of all Borel subalgebras of $\mathfrak{g}$. Under localization [BB81] the bimodule $U/\chi U$ becomes a $\mathcal{D}$-module $\mathcal{M}$ for some sheaf of twisted differential operators $\mathcal{D}$ on $X \times X$ such that $X \times X$ is $\mathcal{D}$-affine, and we just need to compute the selfextensions of this $\mathcal{D}$-module $\mathcal{M}$. Now choosing $\mathcal{D}$ correctly we can assume that

1991 *Mathematics Subject Classification.* Primary 17B20; Secondary 22E46.

1. the restriction of $\mathcal{D}$ to the open orbit $Y \subset X \times X$ for the diagonal action of G is just the standard sheaf of differential operators $\mathcal{D}_Y$ on Y and
2. the $\mathcal{D}$-module $\mathcal{M}$ is just the standard module (with unique simple quotient) $i_! \mathcal{O}_Y$, where i is the inclusion of Y into $X \times X$.

Once we know that, we find quickly

$$\mathrm{Ext}^\bullet_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) \quad = \quad \mathrm{Ext}^\bullet_{\mathcal{D}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \quad = \quad H^\bullet(Y, \mathbb{C}),$$

the first step since $i_!$ is fully faithful even on derived categories, the last step by the Riemann-Hilbert-correspondence. Since Y is fibered over X with fibers just affine spaces, the cohomology ring of Y coincides with the cohomology ring of the flag manifold X which is known to be the ring C of coinvariants (with its obvious grading doubled).

Certainly 1 and 2 are known in much greater generality, they are just a special case of the correspondence between the classifications of Beilinson-Bernstein and Langlands. However, I want to avoid these heavy arguments and provide a more simple minded approach. So let us now start the work and find the sheaf of twisted differential operators (tdo for short) $\mathcal{D}$. We fix some notations concerning homogeneous twisted differential operators. I will follow [HMSW87], Appendix A.

For a complex algebraic group G and a closed subgroup $B \subset G$ and a B-invariant linear form λ on Lie$B$ we have a sheaf $\mathcal{D}_\lambda$ of homogeneous twisted differential operators on G/B. Let $\mathfrak{g} \supset \mathfrak{b}$ be the Lie algebras of $G \supset B$. The geometric stalk of $\mathcal{D}_\lambda$ at the point $x = B$ of G/B is given as

$$\mathcal{D}_\lambda / \mathcal{D}_\lambda \mathfrak{m}_x = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes \bigwedge^{\max} (\mathfrak{g}/\mathfrak{b})).$$

The ordinary differential operators are $\mathcal{D}_0$. Suppose now $H \subset G$ is a closed subgroup and let $f : H/H \cap B \hookrightarrow G/B$ be the inclusion. Pulling back the tdo $\mathcal{D}_\lambda$ via f we find $\mathcal{D}_\lambda^f = \mathcal{D}_\mu$ where μ is the restriction of λ to Lie$(H \cap B)$.

Let us return now to our situation. We fix a semisimple connected complex algebraic group G with Lie$G = \mathfrak{g}$, let $T \subset G$ be the maximal torus with Lie$T = \mathfrak{h}$ and fix two Borel subgroups B^+, $B^- \subset G$ containing T which are opposite, i.e. $B^+ \cap B^- = T$. Their Lie algebras will be denoted $\mathfrak{b}^+$ and $\mathfrak{b}^-$ respectively. Any $\lambda \in \mathfrak{h}^*$ determines a homogeneous tdo $\mathcal{D}_\lambda^+$ on $X = G/B^+$ whose geometric stalk at B^+ is the Verma module $M^+(\lambda) = U \otimes_{U(\mathfrak{b}^+)} (\mathbb{C}_\lambda \otimes \bigwedge^{\max}(\mathfrak{g}/\mathfrak{b}^+))$. If we put $\xi^+(\lambda) = \mathrm{Ann}_Z M^+(\lambda)$, then $\Gamma(\mathcal{D}_\lambda^+) = U/\xi^+(\lambda)U$. The same statements hold with $+$ replaced by $-$, and since the $\mathfrak{h}$-finite vectors in the (ordinary) dual of $M^+(\lambda)$ have the same $\mathfrak{h}$-weights as $M^-(-\lambda)$, we see directly that the principal antiautomorphism of $\mathfrak{g}$ induces an isomorphism $U/\xi^+(\lambda)U \to (U/\xi^-(-\lambda)U)^{\mathrm{opp}}$. This way we associate to $\lambda \in \mathfrak{h}^*$ a tdo $\mathcal{D}_{(\lambda)} = \mathcal{D}^+_\lambda \boxtimes \mathcal{D}^-_{-\lambda}$ on $X \times X$ with global sections $\Gamma(\mathcal{D}_{(\lambda)}) = U/\xi^+(\lambda)U \otimes (U/\xi^+(\lambda)U)^{\mathrm{opp}}$. We leave it to the interested reader to verify that $\mathcal{D}_{(\lambda)}$ and this identification of the global sections depend only on the linear form λ on the "abstract Cartan subgroup".

The localization theorem [BB81] says that for a homogeneous tdo $\mathcal{D}$ on X the functor of global sections is an equivalence of categories

$$\Gamma : \mathcal{D}\text{-mod} \to \Gamma(\mathcal{D})\text{-mod}$$

if and only if for one (equivalently any) point $x \in X$ the geometric stalk $\mathcal{D}/\mathcal{D}\mathfrak{m}_x$ at x is a simple $\mathfrak{g}$-module with regular central character. In this case X is called $\mathcal{D}$-affine. It is clear that for any regular maximal ideal $\chi \subset Z$ we can find $\lambda \in \mathfrak{h}^*$ such that $\xi^+(\lambda) = \chi$ and that $X \times X$ is $\mathcal{D}_{(\lambda)}$-affine. This homogeneous tdo $\mathcal{D}_{(\lambda)}$ is the $\mathcal{D}$ we were looking for.

To see this, remark first that by definition of λ the functor

$$\Gamma : \mathcal{D}_{(\lambda)}\text{-mod} \to U/\chi U\text{-mod-}U/\chi U$$

is an equivalence of categories. Furthermore, if $i : Y \hookrightarrow X \times X$ denotes the inclusion of the open G-orbit $Y = G(B^+, B^-) = G/T$, it is plain that $\mathcal{D}_{(\lambda)}$ pulls back to the ordinary differential operators $\mathcal{D}^i_{(\lambda)} = \mathcal{D}_Y$ on Y.

We have to check that $\Gamma(i_!\mathcal{O}_Y) = U/\chi U$. Remark first that since i is an open embedding, the direct and inverse images in the category of sheaves, $\mathcal{O}$-modules and $\mathcal{D}$-modules coincide. We thus have an adjoint pair of functors (i^*, i_*) between sheaves, $\mathcal{O}$-modules and $\mathcal{D}$-modules for any tdo $\mathcal{D}$ on $X \times X$. Certainly i^* is exact, and since i is affine i_* is exact, too. Now $\Gamma(i_*\mathcal{O}_Y)$ coincides with $\Gamma(\mathcal{O}_Y)$ and it is clear that every nonzero $\mathcal{D}_{(\lambda)}$-submodule of $i_*\mathcal{O}_Y$ contains the global section $\mathbf{1}_Y$ on Y. Thus $i_*\mathcal{O}_Y$ has a simple socle. Let us define the adjoint $\mathfrak{g}$-action on a U-bimodule by the formula

$$(\mathrm{ad}A)(m) = Am - mA.$$

It is then clear that the socle of the U-bimodule $\Gamma(i_*\mathcal{O}_Y)$ is simple and generated by an $(\mathrm{ad}\mathfrak{g})$-invariant line.

Now we study $i_!\mathcal{O}_Y$. Recall its definition from [Bei83]. If $\mathcal{D}$ is a tdo on a smooth variety of dimension n and $\mathcal{M}$ a holonomic $\mathcal{D}$-module, one defines its dual $*\mathcal{M} = \mathcal{RH}om^n_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$. This is a holonomic $\mathcal{D}^{\mathrm{opp}}$-module, and $** \mathcal{M} = \mathcal{M}$ naturally. By definition $i_!\mathcal{O}_Y = *i_* * \mathcal{O}_Y$. It is clear that in our case (i an open immersion) the duality commutes with i^*, whence an adjoint pair $(i_!, i^*)$ of functors between holonomic modules. It is known in general (and clear in our case) that $i_!\mathcal{O}_Y$ has as its unique simple quotient the socle of $i_*\mathcal{O}_Y$. Passing to global sections we see that $\Gamma(i_!\mathcal{O}_Y)$ is generated by an $(\mathrm{ad}\mathfrak{g})$-invariant line. From this we already find a surjection of bimodules

$$U/\chi U \twoheadrightarrow \Gamma(i_!\mathcal{O}_Y).$$

Now let us go to the category ${}_\chi\mathcal{H}_\chi$ of all locally $(\mathrm{ad}\mathfrak{g})$-finite $U/\chi U$-bimodules. Under localization this corresponds to the category of all G-equivariant $\mathcal{D}_{(\lambda)}$-modules. All objects of this category are mapped under i^* to a finite direct sum of some copies of $\mathcal{O}_Y$. Hence $i_!\mathcal{O}_Y$ is a projective object in our category, whence $\Gamma(i_!\mathcal{O}_Y)$ is a projective object of ${}_\chi\mathcal{H}_\chi$. But this means that the surjection $U/\chi U \twoheadrightarrow \Gamma(i_!\mathcal{O}_Y)$ is split, and since $U/\chi U$ is indecomposable it has to be an isomorphism. $\square$

References

[BB81] Alexander A. Beilinson and Joseph N. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris, Sér. 1 **292** (1981), 15–18.

[Bei83] Alexander A. Beilinson, *Localization of representations of reductive Lie algebras*, Proceedings of the ICM Warszawa, 1983, pp. 699–709.

[HMSW87] Henryk Hecht, Dragan Miličić, Wilfried Schmid, and Joseph A. Wolf, *Localization and standard modules for real semisimple Lie groups I: The duality theorem*, Inventiones **90** (1987), 297–332.

Universität Freiburg, Mathematisches Institut, Albertstrasse 23b, D-79104 Freiburg, Germany

E-mail address: soergel@sun1.mathematik.uni-freiburg.de