KAZHDAN-LUSZTIG POLYNOMIALS AND A COMBINATORIC FOR TILTING MODULES

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ABSTRACT. This article gives a self-contained treatment of the theory of Kazhdan-Lusztig polynomials with special emphasis on affine reflection groups. There are only few new results but many new proofs. We close with a conjectural character formula for tilting modules, which formed the starting point of these investigations.

1. Introduction

While trying to write down conjectural character formulas for tilting modules, I dived into the literature on Kazhdan-Lusztig polynomials, notably the works of Kazhdan-Lusztig [KL79], Lusztig [Lus80a], Andersen [And86], Kato [Kat85], Kaneda [Kan87] and Deodhar [Deo87, Deo91]. It seemed reasonable to me, to make a synopsis of all these sources, to make them more easily accessible. That is done in the first sections of this manuscript. The only new result there is Theorem 5.1. However, many proofs and also the presentation as a whole (which fully develops the point of view adopted in [Mil] and [Lus91]) are new. In particular the so-called R-polynomials don't appear at all in my presentation of the theory. In the last section I finally reach my goal and give conjectural character formulas for tilting modules. After that follows a graphically presented sample computation and an index of notation. For a presentation of the basics of this article including the results of the following section one might consult [Hum90]. For the third section compare also [Deo94].

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2. The ordinary Kazhdan-Lusztig polynomials

Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, $l : \mathcal{W} \to \mathbb{N}$ the corresponding length function and $\leq$ the Bruhat order on $\mathcal{W}$. In particular $x < y$ means

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$x \leq y$, $x \neq y$. Let $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials with integer coefficients in one variable v. On the free $\mathcal{L}$-module with basis indexed by $\mathcal{W}$,

$$\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathcal{S}) = \bigoplus_{x \in \mathcal{W}} \mathcal{L}T_x$$

there is exactly one structure of associative $\mathcal{L}$-algebra such that $T_x T_y = T_{xy}$ if $l(x) + l(y) = l(xy)$ and $T_s^2 = v^{-2}T_e + (v^{-2} - 1)T_s$ for all $s \in \mathcal{S}$, see [Bou81], IV, §2, Exercice 23. This associative algebra $\mathcal{H}$ is called the Hecke algebra corresponding to $(\mathcal{W}, \mathcal{S})$.

It can also be given as the associative algebra over $\mathcal{L}$ with generators $\{H_s\}_{s \in \mathcal{S}}$ (for $H_s = vT_s$), the quadratic relations $H_s^2 = 1 + (v^{-1} - v)H_s$ and the so-called braid relations $H_s H_t \ldots H_s = H_t H_s \ldots H_t$ resp. $H_s H_t H_s \ldots H_t = H_t H_s H_t \ldots H_s$ if $st \ldots s = ts \ldots t$ resp. $sts \ldots t = tst \ldots s$ for $s, t \in \mathcal{S}$. All H_s are invertible, more precisely, one checks that $H_s^{-1} = H_s + (v - v^{-1})$.

From now on we work with $H_x = v^{l(x)}T_x$. Certainly we have $H_x H_y = H_{xy}$ if $l(x) + l(y) = l(xy)$. Hence with the H_s all H_x are units in $\mathcal{H}$. There is exactly one ring homomorphism $d : \mathcal{H} \to \mathcal{H}$, $H \mapsto \overline{H}$ such that $\overline{v} = v^{-1}$ and $\overline{H_x} = (H_{x^{-1}})^{-1}$. Certainly d is an involution. We call $H \in \mathcal{H}$ self-dual if $\overline{H} = H$.

Theorem 2.1 ([KL79])**.** *For all $x \in \mathcal{W}$ there exists a unique self-dual element $\underline{H}_x \in \mathcal{H}$ such that $\underline{H}_x \in H_x + \sum_y v\mathbb{Z}[v]H_y$.*

Remark 2.2. In [KL79] this $\underline{H}_x$ is called C'_x. Furthermore Kazhdan and Lusztig use the variable $q = v^{-2}$ and the $\mathcal{L}$-basis consisting of the T_x.

Proof. As we know already we have $\overline{H_s} = H_s^{-1} = H_s + (v - v^{-1})$ for all $s \in \mathcal{S}$. In particular $C_s = H_s + v$ is self-dual, $\overline{C_s} = C_s$. (The expert reader should be cautioned, that our C_s is called C'_s in [KL79], and in this source C_s means another element of the Hecke algebra. Once the Theorem is established, we could as well write $C_s = \underline{H}_s$.)

The multiplication from the right of C_s on $\mathcal{H}$ is given by the formulas

$$H_x C_s = \begin{cases} H_{xs} + vH_x & \text{if } xs > x; \\ H_{xs} + v^{-1}H_x & \text{if } xs < x. \end{cases}$$

We now start proving the existence. To this end we show by induction on the Bruhat order the stronger

Claim 2.3. For all $x \in \mathcal{W}$ there exists a self-dual $\underline{H}_x \in \mathcal{H}$ such that $\underline{H}_x \in H_x + \sum_{y<x} v\mathbb{Z}[v]H_y$.

Certainly we can start our induction with $\underline{H}_e = H_e = 1$. Now let $x \in \mathcal{W}$ be given and suppose we know the existence of $\underline{H}_y$ for all $y < x$. If $x \neq e$ we find $s \in \mathcal{S}$ such that $xs < x$ and by our induction hypothesis we have

$$\underline{H}_{xs} C_s = H_x + \sum_{y<x} h_y H_y$$

for suitable $h_y \in \mathbb{Z}[v]$. We form

$$\underline{H}_x = \underline{H}_{xs} C_s - \sum_{y<x} h_y(0) \underline{H}_y,$$

and our induction works. Hence there exists $\underline{H}_x$ as in the Claim. The unicity of the $\underline{H}_x$ follows immediately from

Claim 2.4. For $H \in \sum_y v\mathbb{Z}[v] H_y$ self-duality $H = \overline{H}$ implies $H = 0$.

Certainly $H_z \in \underline{H}_z + \sum_{y<z} \mathcal{L}\underline{H}_y$ for $\underline{H}_x$ as in the preceding Claim, whence $\overline{H_z} \in H_z + \sum_{y<z} \mathcal{L} H_y$ for all $z \in \mathcal{W}$. If we write $H = \sum h_y H_y$ and choose z maximal such that $h_z \neq 0$,then $H = \overline{H}$ implies $h_z = \overline{h}_z$ contradicting $h_z \in v\mathbb{Z}[v]$. This proves the Claim and the Theorem. □

Definition 2.5. For $x, y \in \mathcal{W}$ we define $h_{y,x} \in \mathcal{L}$ by the equality

$$\underline{H}_x = \sum_y h_{y,x} H_y.$$

Remark 2.6. The $h_{y,x}$ are given in terms of the polynomials $P_{y,x}$ from [KL79] as $h_{y,x} = v^{l(x)-l(y)} P_{y,x}$. These equations are to be understood in $\mathcal{L} = \mathbb{Z}[v, v^{-1}] \supset \mathbb{Z}[q]$, with $q = v^{-2}$ as before. By induction one may check directly, that $v^{l(y)-l(x)} h_{y,x}$ is even a polynomial in q with constant term 1.

The original definition of the Kazhdan-Lusztig polynomials in [KL79] was along the lines of another characterization we give in the sequel. Let us look once more at Theorem 2.1. Up to the signs there is no reason to prefer v over v^{-1} there.

Theorem 2.7 ([KL79])**.** *For all $x \in \mathcal{W}$ there exists a unique self-dual $\underline{\tilde{H}}_x \in \mathcal{H}$ such that $\underline{\tilde{H}}_x \in H_x + \sum_y v^{-1}\mathbb{Z}[v^{-1}] H_y$.*

Proof. Let us look at the two involutive anti-automorphisms a and i of $\mathcal{H}$ given as

$$\begin{array}{ll} a(v) = v, & a(H_x) = (-1)^{l(x)} H_x^{-1} \quad \text{resp.} \\ i(v) = v, & i(H_x) = H_{x^{-1}}. \end{array}$$

They commute and both of them commute with our involution $d : H \mapsto \overline{H}$. Their composite $dia : \mathcal{H} \to \mathcal{H}$ satisfies $dia(H_y) = (-1)^{l(y)} H_y$ and $dia(v) = v^{-1}$.

So we are allowed and forced to take $\underline{\tilde{H}}_x = (-1)^{l(x)} dia(\underline{H}_x)$ and get in addition to the existence of $\underline{\tilde{H}}_x$ the formula

$$\underline{\tilde{H}}_x = \textstyle\sum_y (-1)^{l(x)+l(y)} \overline{h}_{y,x} H_y. \qquad \square$$

Remark 2.8. Instead of dia we could as well use the automorphism $b : \mathcal{H} \to \mathcal{H}$ given by $b(H_x) = H_x$, $b(v) = -v^{-1}$. It commutes with d, and we are allowed and forced to take $\underline{\tilde{H}}_x = b(\underline{H}_x)$. By the way b commutes with i and a, and these four pairwise commuting involutions d, b, i and a define a faithful action of $(\mathbb{Z}/2\mathbb{Z})^4$ on $\mathcal{H}$.

Let us finish this section with an explicit formula for finite $\mathcal{W}$.

Proposition 2.9 ([KL79]). *Let $\mathcal{W}$ be finite, $w \in \mathcal{W}$ the longest element, and $r = l(w)$ its length. Then we have $\underline{H}_w = \sum_{y \in \mathcal{W}} v^{r-l(y)} H_y$.*

Proof. Let R denote the right hand side. Our formulas for the action of C_s show that

$$\{H \in \mathcal{H} \mid HC_s = (v + v^{-1})H \quad \forall s \in \mathcal{S}\} = \mathcal{L}R.$$

Hence we have $\overline{R} \in \mathcal{L}R$ and then immediately $\overline{R} = R$, thus $R = \underline{H}_w$. $\square$

The expert reader will miss here the inversion formulas [KL79] for finite Coxeter groups. We treat them in the next section in greater generality (see 3.10).

3. The parabolic case

Let $\mathcal{S}_f \subset \mathcal{S}$ be a subset, $\mathcal{W}_f = \langle \mathcal{S}_f \rangle \subset \mathcal{W}$ the subgroup it generates, $\mathcal{W}^f \subset \mathcal{W}$ the set of minimal length representatives for the right cosets $\mathcal{W}_f \backslash \mathcal{W}$. So multiplication gives a bijection $\mathcal{W}_f \times \mathcal{W}^f \xrightarrow{\sim} \mathcal{W}$. Let $\mathcal{H}_f = \mathcal{H}(\mathcal{W}_f, \mathcal{S}_f) \subset \mathcal{H}$ be the Hecke algebra of $(\mathcal{W}_f, \mathcal{S}_f)$. One sees that the quadratic relation in the Hecke algebra can also be written $(H_s + v)(H_s - v^{-1}) = 0$. If we fix $u \in \{-v, v^{-1}\}$, the prescription $H_s \mapsto u \ \forall s \in \mathcal{S}_f$ defines a surjection of $\mathcal{L}$-algebras

$$\varphi_u : \mathcal{H}_f \twoheadrightarrow \mathcal{L}.$$

In this way $\mathcal{L}$ becomes an $\mathcal{H}_f$-bimodule, which we denote $\mathcal{L}(u)$. We induce to obtain two right $\mathcal{H}$-modules

$$\begin{array}{rcccl} \mathcal{M} & = & \mathcal{M}^f & = & \mathcal{L}(v^{-1}) \otimes_{\mathcal{H}_f} \mathcal{H}, \\ \mathcal{N} & = & \mathcal{N}^f & = & \mathcal{L}(-v) \otimes_{\mathcal{H}_f} \mathcal{H}. \end{array}$$

In both these modules the $M_x = 1 \otimes H_x$ resp. $N_x = 1 \otimes H_x$ with $x \in \mathcal{W}^f$ form an $\mathcal{L}$-basis. The action of C_s for $s \in \mathcal{S}$ is given in these bases as:

$$M_x C_s = \begin{cases} M_{xs} + vM_x & \text{if } xs \in \mathcal{W}^f,\ xs > x; \\ M_{xs} + v^{-1}M_x & \text{if } xs \in \mathcal{W}^f,\ xs < x; \\ (v + v^{-1})M_x & \text{if } xs \notin \mathcal{W}^f, \end{cases}$$

$$N_x C_s = \begin{cases} N_{xs} + vN_x & \text{if } xs \in \mathcal{W}^f,\ xs > x; \\ N_{xs} + v^{-1}N_x & \text{if } xs \in \mathcal{W}^f,\ xs < x; \\ 0 & \text{if } xs \notin \mathcal{W}^f. \end{cases}$$

To see this one has to use the fact that $x \in \mathcal{W}^f$, $xs \notin \mathcal{W}^f$ implies $xs = rx$ for some $r \in \mathcal{S}_f$. (In particular $xs < x$ implies $xs \in \mathcal{W}^f$.) Indeed for arbitrary $x \in \mathcal{W}$ and $r, s \in \mathcal{S}$ the relations $rx > x$ and $rxs < xs$ together imply $rxs = x$.

For all $s \in \mathcal{S}_f$ one easily checks

$$\varphi_u(C_s) = \begin{cases} (v + v^{-1}) & \text{if } u = v^{-1}; \\ 0 & \text{if } u = -v. \end{cases}$$

Since the C_s for $s \in \mathcal{S}_f$ generate $\mathcal{H}_f$ as an $\mathcal{L}$-algebra, we have $\varphi_u(\overline{H}) = \overline{\varphi_u(H)}\ \forall H \in \mathcal{H}_f$. Hence the prescription $a \otimes H \mapsto \overline{a} \otimes \overline{H}$ defines a homomorphism of additive groups

$$\mathcal{M} \to \mathcal{M},\ M \mapsto \overline{M}$$

such that $\overline{M_e} = M_e$ and $\overline{MH} = \overline{M}\ \overline{H}$ for all $M \in \mathcal{M}$, $H \in \mathcal{H}$. Analogous results hold for $\mathcal{N}$.

We call an additive map F between two right $\mathcal{L}$- resp. $\mathcal{H}$-modules "$\mathcal{L}$-skew-linear" resp. "$\mathcal{H}$-skew-linear" iff $F(MH) = F(M)\overline{H}$ for all M and all $H \in \mathcal{L}$ resp. $H \in \mathcal{H}$. If a module is given a fixed skew-linear involution, we call the elements stable under this involution "self-dual" For example $N \in \mathcal{N}$ is self-dual iff $\overline{N} = N$.

Theorem 3.1 ([Deo87]). 1. *For all $x \in \mathcal{W}^f$ there exists a unique self-dual $\underline{M}_x \in \mathcal{M}$ such that $\underline{M}_x \in M_x + \sum_y v\mathbb{Z}[v]M_y$.*
2. *For all $x \in \mathcal{W}^f$ there exists a unique self-dual $\underline{N}_x \in \mathcal{N}$ such that $\underline{N}_x \in N_x + \sum_y v\mathbb{Z}[v]N_y$.*

Proof. We show (1), the proof of (2) is identical. To show the existence of $\underline{M}_x$ we proceed by induction the length of x and show more precisely, that we can find $\underline{M}_x$ of the form

$$\underline{M}_x = M_x + \sum_{y<x} m_{y,x} M_y.$$

Certainly we can start our induction with $\underline{M}_e = M_e$. Now suppose $\underline{M}_y$ is already constructed for all $y \in \mathcal{W}^f$, $y < x$, and let $s \in \mathcal{S}$ be given such that $xs < x$, $xs \in \mathcal{W}^f$. Then we have

$$\underline{M}_{xs} C_s = M_x + \sum_{z<x} m_z M_z$$

for suitable $m_z \in \mathbb{Z}[v]$. By induction possible $\underline{M}_z$ for $z < x$ are known already. We form

$$\underline{M}_x = \underline{M}_{xs} C_s - \sum_z m_z(0) \underline{M}_z$$

and find this way a possible $\underline{M}_x$. The existence of these $\underline{M}_x$ implies as in the proof of Theorem 2.1 first $\overline{\underline{M}_x} \in M_x + \sum_{y<x} \mathcal{L} M_y$ and then the unicity of the $\underline{M}_x$. □

Remark 3.2. 1. Let us define $m_{y,x} \in \mathbb{Z}[v]$ by

$$\underline{M}_x = \sum_y m_{y,x} M_y.$$

In particular we have $m_{x,x} = 1$, and $m_{y,x} \neq 0 \Rightarrow y \leq x$. Again we consider the variable $q = v^{-2} \in \mathcal{L}$. By induction we deduce easily that even $v^{l(y)-l(x)} m_{y,x} \in \mathbb{Z}[q]$. The same holds for the $n_{y,x}$ defined by

$$\underline{N}_x = \sum_y n_{y,x} N_y.$$

The $v^{l(y)-l(x)} m_{y,x}$ resp. $v^{l(y)-l(x)} n_{y,x}$ are Deodhar's [Deo87] parabolic polynomials $P^J_{\tau,\sigma}$ for Deodhar's cases $u = -1$ resp. $u = q$, if $\tau = y^{-1}\mathcal{W}_f$, $\sigma = x^{-1}\mathcal{W}_f$, and $W_J = \mathcal{W}_f$. The comparison with Deodhar's definition however will suceed only with the help of Theorem 3.5.

2. Possible interpretations of these parabolic polynomials in a representation theoretic context are summarized in Theorem 3.11.4 of [BGS96]. Up to a transformation $v = t$ and with $W_Q = \mathcal{W}_f$ the polynomials $(P^Q(t))_{x,y}$ are precisely the $m_{x,y}$ here, and the $(P_Q(t))_{x,y}$ coincide up to a change of parameters with our $n_{x,y}$, compare 3.10.
3. The proof gives an inductive description of the $\underline{M}_x$. By induction on the length of x we deduce, that for all $y \leq x$ the leading term of $m_{y,x}$ is $v^{l(x)-l(y)}$. This statement has no analogue for the $\underline{N}_x$, since $N_y C_s = 0$ for certain y and s.
4. To simplify the task of calculating the $n_{y,x}$ one may use the well-known formula $\underline{N}_x C_s = (v + v^{-1})\underline{N}_x$ for all $x \in \mathcal{W}^f$, $s \in \mathcal{S}$ such that $xs < x$. This is proved by induction on x, where one has

to use, that $C_s^2 = (v + v^{-1})C_s$ and $n_z(0) \neq 0 \Rightarrow zs < z$ in the preceeding proof. In particular we have $n_{ys,x} = vn_{y,x}$ if $y, x \in \mathcal{W}^f$, $s \in \mathcal{S}$ are such that $ys < y$, $xs < x$.

In the same way one proves that $\underline{M}_x C_s = (v + v^{-1})\underline{M}_x$ for all $x \in \mathcal{W}^f$, $s \in \mathcal{S}$ such that $xs < x$ or $xs \notin \mathcal{W}^f$ and deduces $m_{ys,x} = vm_{y,x}$ for all $y, x \in \mathcal{W}^f$, $s \in \mathcal{S}$ such that $ys < y$, $xs < x$.

5. For $\mathcal{S}_f = \emptyset$ we certainly have $\mathcal{M} = \mathcal{N} = \mathcal{H}$, $\underline{M}_x = \underline{N}_x = \underline{H}_x$, $m_{y,x} = n_{y,x} = h_{y,x}$.

For an abelian group E with involution d let $E^+ \subset E$ be the subgroup of self-dual elements $E^+ = \{e \in E \mid de = e\}$.

Proposition 3.3. 1. *$\mathcal{H}^+ \subset \mathcal{H}$ is the subalgebra generated over $\mathcal{L}^+ = \mathbb{Z}[(v + v^{-1})]$ by the C_s with $s \in \mathcal{S}$.*
2. *$\mathcal{M}^+ = M_e\mathcal{H}^+$ and the $\underline{M}_x$ form an $\mathcal{L}^+$-basis of $\mathcal{M}^+$.*
3. *$\mathcal{N}^+ = N_e\mathcal{H}^+$ and the $\underline{N}_x$ form an $\mathcal{L}^+$-basis of $\mathcal{N}^+$.*

Proof. For this proof only let $\mathcal{H}^+ \subset \mathcal{H}$ be the subalgebra generated over $\mathcal{L}^+ = \mathbb{Z}[(v + v^{-1})]$ by the C_s, $s \in \mathcal{S}$. If we show (2) or (3) for this $\mathcal{H}^+$, then (1) follows.

We show (2), the proof of (3) being identical. First remark that by the inductive construction of the $\underline{M}_x$ all $\underline{M}_x$ lie in $M_e\mathcal{H}^+$. On the other hand the $\underline{M}_x$ form an $\mathcal{L}$-basis of $\mathcal{M}$, and $M = \sum m_x \underline{M}_x$ is self-dual iff all m_x are. □

The $m_{y,x}$, $n_{y,x}$ are related to the ordinary Kazhdan-Lusztig polynomials as follows.

Proposition 3.4 ([Deo87])**.** *Let $x, y \in \mathcal{W}^f$.*
1. *If $\mathcal{W}_f$ is finite and $w_f \in \mathcal{W}_f$ is its longest element, we have $m_{y,x} = h_{w_f y, w_f x}$.*
2. *For $\mathcal{S}_f$ arbitrary we have $n_{y,x} = \sum_{z \in \mathcal{W}_f} (-v)^{l(z)} h_{zy,x}$.*

Proof. (1) Consider the $\mathcal{L}$-linear embedding

$$\begin{array}{lcl} \mathcal{L}(v^{-1}) & \to & \mathcal{H}_f \\ 1 & \mapsto & \underline{H}_{w_f}. \end{array}$$

It commutes with the dualities, and by the proof of Proposition 2.9 it is even compatible with the right $\mathcal{H}_f$ action. Therefore we get by induction an embedding

$$\zeta : \mathcal{M} \hookrightarrow \mathcal{H}$$

of right $\mathcal{H}$-modules, which is compatible with the dualities as well. We put $r = l(w_f)$. By Proposition 2.9 we have

$$\zeta(M_x) = \sum_{z \in \mathcal{W}_f} v^{r - l(z)} H_{zx}$$

and thus we get $\zeta(\underline{M}_x) = \underline{H}_{w_f x}$. This even implies $m_{y,x} = v^{r-l(z)} h_{zy, w_f x}$ for all $y, x \in \mathcal{W}^f$, $z \in \mathcal{W}_f$.

(2) Consider the obvious surjection

$$\xi : \mathcal{H} \twoheadrightarrow \mathcal{N} = \mathcal{L}(-v) \otimes_{\mathcal{H}_f} \mathcal{H}$$

with $\xi(H) = 1 \otimes H$. It commutes with the dualities, and one may check, that $\xi(H_{zx}) = (-v)^{l(z)} N_x$ for all $z \in \mathcal{W}_f$, $x \in \mathcal{W}^f$. Thus we get

$$\xi(\underline{H}_x) = \begin{cases} \underline{N}_x & \text{if } x \in \mathcal{W}^f; \\ 0 & \text{otherwise,} \end{cases}$$

and the Proposition follows. $\square$

In the definition of $\underline{N}_x$, $\underline{M}_x$ we may ask, whether v couldn't be replaced by v^{-1}. The answer is given by the following

Theorem 3.5 ([Deo91]). 1. *For all $x \in \mathcal{W}^f$ there is a unique self-dual $\tilde{\underline{N}}_x \in \mathcal{N}$ such that $\tilde{\underline{N}}_x \in N_x + \sum_y v^{-1}\mathbb{Z}[v^{-1}] N_y$. This $\tilde{\underline{N}}_x$ can be given by the formula $\tilde{\underline{N}}_x = \sum_y (-1)^{l(x)+l(y)} \overline{m}_{y,x} N_y$.*

2. *For all $x \in \mathcal{W}^f$ there is a unique self-dual $\tilde{\underline{M}}_x \in \mathcal{M}$ such that $\tilde{\underline{M}}_x \in M_x + \sum_y v^{-1}\mathbb{Z}[v^{-1}] M_y$. This $\tilde{\underline{M}}_x$ can be given by the formula $\tilde{\underline{M}}_x = \sum_y (-1)^{l(x)+l(y)} \overline{n}_{y,x} M_y$.*

Proof. We start with the relation $\varphi_{-v} = \varphi_{v^{-1}} \circ ia$, in other words the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_f & \longrightarrow & \mathcal{L}(-v) \\ ia \downarrow & & \| \\ \mathcal{H}_f & \longrightarrow & \mathcal{L}(v^{-1}). \end{array}$$

We can thus define an $\mathcal{L}$-skew-linear bijection $\phi : \mathcal{N} \to \mathcal{M}$ by the formula $\phi(c \otimes H) = \overline{c} \otimes dia(H)$, and clearly $\phi(\overline{N}) = \overline{\phi(N)} \quad \forall N \in \mathcal{N}$. Certainly we have $\phi(N_x) = (-1)^{l(x)} M_x$. Thus we are allowed and forced to put $\tilde{\underline{M}}_x = (-1)^{l(x)} \phi(\underline{N}_x)$ and $\tilde{\underline{N}}_x = (-1)^{l(x)} \phi^{-1}(\underline{M}_x)$. $\square$

Next we discuss inversion formulas. For this we consider the $\mathcal{L}$-modules

$$\begin{array}{rcl} \mathcal{M}^* & = & \mathrm{Hom}_{\mathcal{L}}(\mathcal{M}, \mathcal{L}) \\ \mathcal{N}^* & = & \mathrm{Hom}_{\mathcal{L}}(\mathcal{N}, \mathcal{L}) \end{array}$$

and define on them an $\mathcal{L}$-skew-linear involution $F \mapsto \overline{F}$ by the formula $\overline{F}(M) = \overline{F(\overline{M})}$. Furthermore we define $M_x^* \in \mathcal{M}^*$ by $M_x^*(M_y) = \delta_{x,y}$ and put $M^x = (-1)^{l(x)} M_x^*$. Analogously we define $N_x^* \in \mathcal{N}^*$ by $N_x^*(N_y) = \delta_{x,y}$ and put $N^x = (-1)^{l(x)} N_x^*$. Why I prefer to work with the M^x resp. N^x will become clear later. Right now it rather complicates all formulas.

We write the elements of $\mathcal{M}^*$ as formal linear combinations $F = \sum^\infty m^z M^z$ with $m^z = (-1)^{l(z)} F(M_z) \in \mathcal{L}$. The ∞ sign above the sum should remind us that formal infinite sums are allowed. The elements of $\mathcal{N}^*$ are written in the same way. Now we have $\overline{M^x} \in M^x + \sum_{z>x}^\infty \mathcal{L} M^z$ and similarly for $\overline{N^x}$, since the matrices of the dualities on $\mathcal{M}$ and $\mathcal{M}^*$ (resp. $\mathcal{N}$ and $\mathcal{N}^*$) are transposed up to signs.

Theorem 3.6. 1. *For all $x \in \mathcal{W}^f$ there exists a unique self-dual $\underline{M}^x \in \mathcal{M}^*$ such that $\underline{M}^x \in M^x + \sum^\infty v\mathbb{Z}[v] M^z$.*
2. *For all $x \in \mathcal{W}^f$ there exists a unique self-dual $\underline{N}^x \in \mathcal{N}^*$ such that $\underline{N}^x \in N^x + \sum^\infty v\mathbb{Z}[v] N^z$.*

Proof. We show (1), the proof of (2) is identical. For the unicity we have to show that $F = 0$ is the only self-dual element of $\sum^\infty v\mathbb{Z}[v] M^z$. But let $F = \sum^\infty m^z M^z$. If $F \neq 0$, we find y minimal such that $m^y \neq 0$. Then $\overline{m^y} = m^y$ and this contradicts $m^y \in v\mathbb{Z}[v]$.

To prove existence, we just define $\underline{M}^x \in \mathcal{M}^*$ by the formula

$$\underline{M}^x(\underline{M}_y) = (-1)^{l(x)} \delta_{x,y}$$

and only have to check our properties. Certainly this $\underline{M}^x$ is self-dual. If we put $\underline{M}^x = \sum_z^\infty m^{z,x} M^z$, then clearly

$$\sum_z (-1)^{l(z)+l(x)} m^{z,x} m_{z,y} = \delta_{x,y}.$$

However the matrix $m_{z,y}$ is lower triangular with ones on the diagonal and entries from $v\mathbb{Z}[v]$ outside the diagonal, whence the same holds for its inverse and we get $m^{z,x} \in v\mathbb{Z}[v]$ if $z \neq x$ and $m^{x,x} = 1$ (and even $m^{z,x} \neq 0 \Rightarrow z \geq x$). $\square$

In the same way we introduce the $n^{z,x} \in \mathbb{Z}[v]$ by $\underline{N}^x = \sum^\infty n^{z,x} N^z$ and get the inversion formulas

$$\sum_z (-1)^{l(z)+l(x)} n^{z,x} n_{z,y} = \delta_{x,y}.$$

In case $\mathcal{S}_f = \emptyset$ we write $\mathcal{H}^*$, H_x^*, H^x, $\underline{H}^x$, $h^{z,x}$ instead of $\mathcal{M}^*$, M_x^*, M^x, $\underline{M}^x$, $m^{z,x}$. Thus the $h^{z,x}$ with $\underline{H}^x = \sum^\infty h^{z,x} H^z$ are the renormalized inverse Kazhdan-Lusztig polynomials,

$$\sum_z (-1)^{l(z)+l(x)} h^{z,x} h_{z,y} = \delta_{x,y}.$$

As in Proposition 3.4 the parabolic inverse polynomials $m^{x,y}$, $n^{x,y}$ can be expressed in terms of the ordinary inverse polynomials $h^{x,y}$. More precisely, we have

Proposition 3.7. 1. *If $\mathcal{W}_f$ is finite, $w_f \in \mathcal{W}_f$ its longest element and $r = l(w_f)$ its length, then for all $x, y \in \mathcal{W}^f$ we have*

$$m^{y,x} = \sum_{z \in \mathcal{W}_f} (-v)^{r-l(z)} h^{zy, w_f x}.$$

2. *For arbitrary $\mathcal{S}_f$ we have $n^{y,x} = h^{y,x}$ for all $x, y \in \mathcal{W}^f$.*

Proof. (1) We transpose the map ζ considered in the proof of 3.4 (1) and get

$$\zeta^* : \mathcal{H}^* \to \mathcal{M}^*.$$

The formula for $\zeta(M_x)$ implies $\zeta^*(H^{zx}) = (-v)^{r-l(z)}(-1)^r M^x$ for all $x \in \mathcal{W}^f$, $z \in \mathcal{W}_f$. Since also $\zeta(\underline{M}_x) = \underline{H}_{w_f x}$ we get

$$\zeta^*(\underline{H}^{tx}) \;\;=\;\; \begin{cases} (-1)^r \underline{M}^x & \text{if } t = w_f; \\ 0 & \text{otherwise,} \end{cases}$$

again for all $x \in \mathcal{W}^f$, $t \in \mathcal{W}_f$. If we apply ζ^* to the equation $\underline{H}^{w_f x} = \sum^{\infty} h^{zy, w_f x} H^{zy}$ where the sum runs over $z \in \mathcal{W}_f$, $y \in \mathcal{W}^f$, we get

$$\underline{M}^x = \sum_y^{\infty} \sum_{z \in \mathcal{W}_f} (-v)^{r-l(z)} h^{zy, w_f x} M^y$$

and this proves our claim. By the way we could also apply ζ^* to $\underline{H}^{tx}$ with $t \neq w_f$ to get $\sum_{z \in \mathcal{W}_f} (-v)^{-l(z)} h^{zy,tx} = 0$ for all $x, y \in \mathcal{W}^f$.

(2) We transpose the map ξ from the proof of 3.4 (2) and get

$$\xi^* : \mathcal{N}^* \to \mathcal{H}^*.$$

The formula for $\xi(H_{zx})$ implies

$$\xi^*(N^x) = \sum_{z \in \mathcal{W}_f} v^{l(z)} H^{zx}$$

and from the formula for $\xi(\underline{H}_{zx})$ we get $\xi^*(\underline{N}^x) = \underline{H}^x$. When we apply ξ^* to the equation $\underline{N}^x = \sum_y^{\infty} n^{y,x} N^y$ we get

$$\underline{H}^x = \sum_y^{\infty} \sum_{z \in \mathcal{W}_f} v^{l(z)} n^{y,x} H^{zy}$$

and deduce even $v^{l(z)} n^{y,x} = h^{zy,x}$ for all $x, y \in \mathcal{W}^f$ and $z \in \mathcal{W}_f$. □

To formulate the next theorem, I have to introduce a convention. Let $\varphi : A \to A'$ be a ring homomorphism, $\mathcal{M}$ an A-module and $\mathcal{M}'$ an A'-module. A homomorphism of additive groups $\psi : \mathcal{M} \to \mathcal{M}'$ is called "φ-linear" iff $\psi(rm) = \varphi(r)\psi(m) \quad \forall r \in A,\ m \in \mathcal{M}$. Thus for our involution $d : \mathcal{H} \to \mathcal{H}$ with $d(H) = \overline{H}$ a d-linear map is the same thing as an $\mathcal{H}$-skew-linear one.

Certainly $\mathcal{M}^*$ is a left $\mathcal{H}$-module via $(HF)(M) = F(MH)\ \forall H \in \mathcal{H}, F \in \mathcal{M}^*, M \in \mathcal{M}$. Recall the involutions d, a, and $da = d \circ a$ on $\mathcal{H}$ from the proof of Theorem 2.7.

Theorem 3.8. *There exists a da-linear map $\psi : \mathcal{N} \to \mathcal{M}^*$ such that $\psi(N_x) = M^x \quad \forall x$.*

Proof. Let's first check the forumlas

$$C_s M_x^* \quad = \quad \begin{cases} M_{xs}^* + vM_x^* & \text{if } xs \in \mathcal{W}^f,\ xs > x; \\ M_{xs}^* + v^{-1}M_x^* & \text{if } xs \in \mathcal{W}^f,\ xs < x; \\ (v + v^{-1})M_x^* & \text{if } xs \notin \mathcal{W}^f. \end{cases}$$

Indeed, the matrix of the right action of C_s on $\mathcal{M}$ expressed in the basis of the M_x decomposes in 1×1-blocks and 2×2-blocks of the form $\left(\begin{smallmatrix} v & 1 \\ 1 & v^{-1} \end{smallmatrix}\right)$. Thus this matrix is its own transposed matrix, and this gives the above formulas.

We deduce the existence of an i-linear map $\mathcal{M} \to \mathcal{M}^*$ such that $M_x \mapsto M_x^*$. On the other hand from the proof of Theorem 3.5 we know there is a dia-linear map $\mathcal{N} \to \mathcal{M}$ such that $N_x \mapsto (-1)^{l(x)} M_x$. Composing these two maps the Theorem follows. □

Certainly all our arguments and results stay valid when we exchange the roles of $\mathcal{N}$, N and $\mathcal{M}$, M. For completeness I finish this section with Douglass' inversion formulas for finite $\mathcal{W}$.

Proposition 3.9 ([Dou90])**.** *Let $\mathcal{W}$ be finite and let $w \in \mathcal{W}$ resp. $w_f \in \mathcal{W}_f$ be the longest elements. Then we have*

$$\sum_z (-1)^{l(x)+l(z)} m_{z,x} n_{w_f zw, w_f yw} = \delta_{x,y}.$$

Remark 3.10. In particular this gives the inversion formulas of Kazhdan-Lusztig [KL79]

$$\sum_z (-1)^{l(x)+l(z)} h_{z,x} h_{zw,yw} = \delta_{x,y}.$$

Proof. If we put $\mathcal{S}_g = w\mathcal{S}_f w$, the map $x \mapsto w_f wx$ gives an order-reversing bijection $\mathcal{W}^g \xrightarrow{\sim} \mathcal{W}^f$. Therefore we get an $\mathcal{H}$-skew-linear map

$$\mathcal{N}^g \to \mathcal{N}, \quad N_x^g \mapsto N_{w_f wx},$$

where we put $\mathcal{N} = \mathcal{N}^f$. Composing this with our $\psi : \mathcal{N} \to \mathcal{M}^*$, we get an a-linear map

$$\mathcal{N}^g \to \mathcal{M}^*, \quad N_x^g \mapsto M^{w_f wx}.$$

This map even commutes with the dualities on our modules, since N_e^g is self-dual and so is $M^{w_f w}$, because $w_f w$ is the maximal element in $\mathcal{W}^f$. But then our map necessarily transforms $\underline{N}_x^g$ into $\underline{M}^{w_f w x}$, and we deduce

$$m^{w_f w y, w_f w x} = n_{y,x}^g = n_{wyw,wxw}.$$

After a transformation of the variables this gives

$$m^{y,x} = n_{w_f y w, w_f x w}.$$

□

4. Affine reflection groups and the periodic Hecke module

For an explanation of the terminology used in the sequel one may look at [Bou81]. Let $V \supset R \supset R^+ \supset \Delta$ be a vector space over the reals, a root system, a system of positive roots and the corresponding set of simple roots. Let $W \subset GL(V)$ be the Weyl group and $\mathcal{W} = W \ltimes \mathbb{Z}R$ the affine Weyl group. For $\mu \in V$ let $\mathcal{W}_\mu$ resp. W_μ be its stabilizer in $\mathcal{W}$ resp. W. Thus we have $W = \mathcal{W}_0$. The group $\mathcal{W}$ is generated by its (affine) reflections, and we let $\mathcal{F}$ be the set of all its reflection hyperplanes. For $F \in \mathcal{F}$ let $s_F \in \mathcal{W}$ be the reflection leaving F invariant.

The connected components of the complement of all reflection hyperplanes $V - \cup_{F \in \mathcal{F}} F$ are called "alcoves". We denote by $\mathcal{A}$ the set of all alcoves. The obvious action of $\mathcal{W}$ on $\mathcal{A}$ is free and transitive. Let

$$\mathcal{C} = \{\tau \in V \mid \langle \tau, \alpha^\vee \rangle > 0 \quad \forall \alpha \in R^+\}$$

be the dominant Weyl chamber. Let $A^+ \in \mathcal{A}$ be the unique alcove contained in $\mathcal{C}$ and having the null vector in its closure.

Let $\mathcal{S} \subset \mathcal{W}$ be the the set of all reflections, which pointwise fix some wall of A^+. Then $(\mathcal{W}, \mathcal{S})$ is a Coxeter system. We also consider the bijection $\mathcal{W} \xrightarrow{\sim} \mathcal{A}, w \mapsto wA^+$. The obvious right action of $\mathcal{W}$ on itself corresponds under such a bijection to a right action of $\mathcal{W}$ on $\mathcal{A}$, denoted $A \mapsto Aw$. For $A \in \mathcal{A}, s \in \mathcal{S}$ one may visualize As as follows: Consider the wall of A^+ fixed by s. Exactly one wall of A is conjugate to this wall of A^+ under the action of $\mathcal{W}$ on V. Then As meets A exactly along this wall of A.

A reflecting hyperplane $F \in \mathcal{F}$ divides V into two halfspaces

$$V - F = F^+ \cup F^-,$$

where we let F^+ the unique halfspace, which meets every translate of the dominant Weyl chamber, $F^+ \cap (\tau + \mathcal{C}) \neq \emptyset \quad \forall \tau \in V$. For

$A \in \mathcal{A}, s \in \mathcal{S}$ we write $As \succ A$ (resp. $As \prec A$) iff $As \subset F^+$ (resp. $As \subset F^-$) for the reflecting hyperplane $F \in \mathcal{F}$ separating As and A.

Now we may define the "periodic" Hecke module $\mathcal{P}$. As an $\mathcal{L}$-module $\mathcal{P}$ is just free with basis $\mathcal{A}$,

$$\mathcal{P} = \bigoplus_{A \in \mathcal{A}} \mathcal{L}A.$$

Lemma 4.1 ([Lus80a]). *On $\mathcal{P}$ there can be defined a right $\mathcal{H}$-action such that for all $s \in \mathcal{S}$ we have:*

$$AC_s = \begin{cases} As + vA & \text{if } As \succ A; \\ As + v^{-1}A & \text{if } As \prec A. \end{cases}$$

Remark 4.2. To identify the $\mathcal{M}$ from [Lus80a] with our $\mathcal{P}$, one needs a length function $\delta : \mathcal{A} \to \mathbb{Z}$ as in [Lus80a]. Our A would be called $q^{-\delta(A)/2}A$ in Lusztig's notation. In addition we let $\mathcal{H}$ act from the right.

Proof. First let us consider for $s \in \mathcal{S}$ the $\mathcal{L}$-linear map $\rho_s : \mathcal{P} \to \mathcal{P}$ given by

$$\rho_s(A) = \begin{cases} As + vA & \text{if } As \succ A; \\ As + v^{-1}A & \text{if } As \prec A. \end{cases}$$

For $\mu \in \mathbb{Z}R$ we also consider $\langle \mu \rangle : \mathcal{P} \to \mathcal{P}$, $A \mapsto \mu + A$. Certainly we have $\langle \mu \rangle \circ \rho_s = \rho_s \circ \langle \mu \rangle$ for all $\mu \in \mathbb{Z}R$, $s \in \mathcal{S}$.

In any case we get a right action of $\mathcal{H}$ on $\mathcal{P}$ by transport of structure via the $\mathcal{L}$-linear bijection $\mathcal{H} \to \mathcal{P}$ given by $H_x \mapsto xA^+ \quad \forall x \in \mathcal{W}$. Let us denote this right action by $P * H$ for $P \in \mathcal{P}, H \in \mathcal{H}$. The map $\mathcal{P} \to \mathcal{P}$, $P \mapsto P * H$ will be denoted by $\rho^*(H)$.

Let $\mathcal{A}^+ \subset \mathcal{A}$ be the set of all alcoves contained in the dominant Weyl chamber, $\mathcal{A}^+ = \{A \in \mathcal{A} \mid A \subset \mathcal{C}\}$. For $x \in \mathcal{W}$, $s \in \mathcal{S}$ such that xA^+, $xsA^+ \in \mathcal{A}^+$ the relation $x > xs$ is equivalent to $xA^+ \succ xA^+s$. Thus for all $A \in \mathcal{A}$, $s \in \mathcal{S}$ such that $A, As \in \mathcal{A}^+$ we have

$$\rho_s(A) = A * C_s.$$

Choose $\mu \in \mathcal{C} \cap \mathbb{Z}R$. For any alcove A the translated alcove $n\mu + A$ lies inside $\mathcal{C}$, for $n \gg 0$. We deduce

$$\begin{aligned} \rho_s(A) &= \langle -n\mu \rangle \circ \rho_s \circ \langle n\mu \rangle(A) \\ &= \langle -n\mu \rangle \circ \rho^*(C_s) \circ \langle n\mu \rangle(A), \end{aligned}$$

if $n \gg 0$. Thus for all $H \in \mathcal{H}$, $P \in \mathcal{P}$ the expression $\langle -n\mu \rangle \circ \rho^*(H) \circ \langle n\mu \rangle(P)$ is independent of n for $n \gg 0$. We call this expression PH and have thus defined the looked-for right action of $\mathcal{H}$ on $\mathcal{P}$. $\square$

Let $X \subset V$ be the lattice of integral weights. For $\lambda \in X$ we define $E_\lambda \in \mathcal{P}$ by

$$E_\lambda = \sum_{z \in W} v^{l(z)}(\lambda + zA^+).$$

Let $\mathcal{P}^\circ \subset \mathcal{P}$ be the $\mathcal{H}$-submodule generated by all the E_λ .

Theorem 4.3 ([Lus80a])**.** 1. *On $\mathcal{P}^\circ$ there exists a unique $\mathcal{H}$-skew-linear involution $\mathcal{P}^\circ \to \mathcal{P}^\circ$, $P \mapsto \overline{P}$ such that $\overline{E_\lambda} = E_\lambda$ for all $\lambda \in X$.*

2. *For all $A \in \mathcal{A}$ there exists a unique $\underline{P}_A \in \mathcal{P}^\circ$ which is self-dual with respect to this involution and such that $\underline{P}_A \in A + \sum_B v\mathbb{Z}[v]B$. The $\underline{P}_A$ form an $\mathcal{L}$-basis of $\mathcal{P}^\circ$.*

Remark 4.4. For $A, B \in \mathcal{A}$ we define $p_{B,A} \in \mathbb{Z}[v]$ by $\underline{P}_A = \sum p_{B,A}B$. Let $d(B, A) \in \mathbb{Z}$ be the weighted sum of reflecting hyperplanes $H \in \mathcal{F}$ separating B from A, where we count H with weight 1 resp. (-1) if $B \subset H^-$ resp. $B \subset H^+$. The polynomials $Q_{B,A}$ of Lusztig [Lus80a] are related to these $p_{B,A}$ by $p_{B,A} = v^{-d(B,A)}Q_{B,A}$. I call the $p_{B,A}$ the "periodic polynomials".

The proof of the Theorem needs some preparations and will be complete towards the end of this section. We start by repeating Lusztig's construction of an action of $\mathcal{W}$ on $\mathcal{P}^\circ$.

Proposition 4.5 ([Lus80a])**.** *For all $w \in \mathcal{W}$ there exists a homomorphism of $\mathcal{H}$-modules $\langle w \rangle : \mathcal{P}^\circ \to \mathcal{P}^\circ$ such that $\langle w \rangle E_\lambda = E_{w\lambda}$ for all $\lambda \in X$.*

Remark 4.6. Certainly $\langle w \rangle$ is uniquely defined by this condition and we get thus an action of $\mathcal{W}$ on $\mathcal{P}^\circ$. In addition for $w = \mu \in \mathbb{Z}R$ this $\langle \mu \rangle$ is obviously the restriction to $\mathcal{P}^\circ$ of our translation $\langle \mu \rangle$ from above.

Proof. For $\alpha \in R^+$ let $\mathcal{F}_\alpha \subset \mathcal{F}$ be the set of reflecting hyperplanes orthogonal to α. Thus $\mathcal{F} = \cup_{\alpha \in R^+} \mathcal{F}_\alpha$ is a partition of $\mathcal{F}$. The connected components of $V - \cup_{F \in \mathcal{F}_\alpha} F$ are called "α-strips". Every α-strip U has the form $U = F^+ \cap G^-$ for unique $F, G \in \mathcal{F}_\alpha$. We put $F = \partial^- U$ and $G = \partial^+ U$. For $A \in \mathcal{A}$, $\alpha \in R^+$ we define $\alpha{\uparrow}A = s_G A$, $\alpha{\downarrow}A = s_F A$, if A lies in the α-strip U and $F = \partial^- U$, $G = \partial^+ U$ as above. For a simple root $\alpha \in \Delta$ let us consider the $\mathcal{L}$-submodule $\mathcal{P}_\alpha \subset \mathcal{P}$ generated by all $A + v(\alpha{\downarrow}A)$ with $A \in \mathcal{A}$. Certainly these expressions form even an $\mathcal{L}$-basis of $\mathcal{P}_\alpha$. So we can define for all $F \in \mathcal{F}_\alpha$ an $\mathcal{L}$-linear map

$$\langle s_F \rangle : \mathcal{P}_\alpha \to \mathcal{P}_\alpha$$

by the prescription $\langle s_F \rangle(A + v(\alpha{\downarrow}A)) = v(s_F A) + \alpha{\uparrow}(s_F A)$.

Lemma 4.7 ([Lus80a]). *$\mathcal{P}_\alpha$ is an $\mathcal{H}$-submodule of $\mathcal{P}$ and $\langle s_F\rangle : \mathcal{P}_\alpha \to \mathcal{P}_\alpha$ is $\mathcal{H}$-linear.*

Proof. We have to show for all $A \in \mathcal{A}, s \in \mathcal{S}$ that
(i) $(A + v(\alpha{\downarrow}A))C_s \in \mathcal{P}_\alpha$.
(ii) $\langle s_F\rangle\{(A + v(\alpha{\downarrow}A))C_s\} = \{\langle s_F\rangle(A + v(\alpha{\downarrow}A))\}C_s$.

Let U be the α-strip of A. Let $G \in \mathcal{F}$ be the reflecting hyperplane separating As and A. We have to consider three cases.

1. G is not a wall of U. From there $G \in \mathcal{F}_\beta$ with $\beta \in R^+ - \{\alpha\}$. In particular $s_F(G^+) = (s_F G)^+$. Then (i) and (ii) follow easily.
2. $G = \partial^+ U$, left to the reader.
3. $G = \partial^- U$, left to the reader.

The Lemma is proved. □

Lemma 4.8 ([Lus80a]). *We have $E_\lambda \in \mathcal{P}_\alpha$ for all simple roots $\alpha \in \Delta$ and $\langle s_F\rangle E_\lambda = E_{s_F\lambda}$ for all $F \in \mathcal{F}_\alpha$.*

Proof. Left to the reader. □

In particular we have $\mathcal{P}^\circ \subset \mathcal{P}_\alpha$ and $\langle s_F\rangle\mathcal{P}^\circ \subset \mathcal{P}^\circ$. Now for $w \in \mathcal{W}$ we get $\langle w\rangle : \mathcal{P}^\circ \to \mathcal{P}^\circ$ as follows: We write $w = s_F \cdots s_G$ with $F, \dots, G \in \cup_{\alpha\in\Delta}\mathcal{F}_\alpha$ and put $\langle w\rangle = \langle s_F\rangle \circ \cdots \circ \langle s_G\rangle$. □

It will be important to know, that the $\mathcal{H}$-linear action of $\mathcal{W}$ on $\mathcal{P}^\circ$ can be extended to an $\mathcal{L}$-linear action of the "extended affine Weyl group" $\tilde{\mathcal{W}} = W \ltimes X$.

For any $\mu \in X$ let us consider the $\mathcal{L}$-linear map $\langle\mu\rangle : \mathcal{P} \to \mathcal{P}$ given by $A \mapsto \mu + A \quad \forall A \in \mathcal{A}$. It doesn't commute with the right action of $\mathcal{H}$ in general. If C_s denotes for the moment the map $\mathcal{P} \to \mathcal{P}$, $P \mapsto PC_s$, we rather have

$$\langle\mu\rangle \circ C_s = C_{[\mu]s} \circ \langle\mu\rangle$$

for a suitable permutation $[\mu] : \mathcal{S} \to \mathcal{S}$ of the simple reflections, and we have $[\mu + \nu] = [\mu] \circ [\nu]$ for all $\mu, \nu \in X$ and $[\mu] = \mathrm{id}$ for $\mu \in \mathbb{Z}R$, hence in particular $[w\mu] = [\mu] = [-\mu]^{-1}$ for all $w \in \mathcal{W}$, $\mu \in X$.

Lemma 4.9. *There exists an $\mathcal{L}$-linear action $\varphi : \tilde{\mathcal{W}} \to \operatorname{Aut}\mathcal{P}^\circ$ of $\tilde{\mathcal{W}}$ on $\mathcal{P}^\circ$ such that $\varphi(w) = \langle w\rangle$ for all $w \in \mathcal{W}$ and $\varphi(\mu) = \langle\mu\rangle$ for all $\mu \in X$.*

Remark 4.10. Once the Lemma is established, we will abbreviate $\varphi(w)$ by $\langle w\rangle$ for all $w \in \tilde{\mathcal{W}}$.

Proof. It is sufficient to show that the map $\langle w\rangle\langle\mu\rangle\langle w^{-1}\rangle\langle -w\mu\rangle$ is the identity on $\mathcal{P}^\circ$, for all $\mu \in X$, $w \in \mathcal{W}$. However this map commutes with the right action of C_s and maps E_λ to itself. □

Now we can prove part (1) of Theorem 4.3.

Proposition 4.11 ([Lus80a])**.** *There exists a unique skew-linear map* $\mathcal{P}^\circ \to \mathcal{P}^\circ$, $P \mapsto \overline{P}$ *such that* $\overline{E_\lambda} = E_\lambda \quad \forall \lambda \in X$.

Proof. Certainly by skew-linear we mean $\mathcal{H}$-skew-linear here, but writing this produced an overfull box. Unicity is clear, we only have to construct such a map. Let $w_0 \in W$ be the longest element and $r = l(w_0)$ its length. Let $c : \mathcal{P} \to \mathcal{P}$ denote the $\mathcal{L}$-skew-linear map given by $c(A) = w_0 A$. For all $s \in \mathcal{S}$ we have $A \succ As \Leftrightarrow w_0 A \prec w_0 As$. Hence we have $c(AC_s) = c(A)C_s$ for all $s \in \mathcal{S}$ and c is even $\mathcal{H}$-skew-linear. Certainly we have $c(E_\lambda) = v^{-r} E_{w_0\lambda}$. In particular we get $c(\mathcal{P}^\circ) \subset \mathcal{P}^\circ$. We put $\overline{P} = v^r c\langle w_0\rangle P$ and are done. □

This duality even commutes with the $\tilde{\mathcal{W}}$-action.

Proposition 4.12. *We have* $\overline{\langle w\rangle P} = \langle w\rangle \overline{P}$ *for all* $w \in \tilde{\mathcal{W}}$, $P \in \mathcal{P}^\circ$.

Proof. Let $d : \mathcal{P}^\circ \to \mathcal{P}^\circ$ denote our duality $P \mapsto \overline{P}$. We have to show that $\langle w\rangle d = d\langle w\rangle$ for all $w \in \tilde{\mathcal{W}}$. It will be sufficient to show that $\langle \mu\rangle d\langle -\mu\rangle d$ resp. $\langle w\rangle d\langle w^{-1}\rangle d$ are the identity, for all $\mu \in X$ resp. $w \in \mathcal{W}$. However these maps commute with the right action of the C_s and map E_λ to itself. □

We now establish the existence of the $\underline{P}_A$. Let us consider the partial order $\preceq$ on $\mathcal{A}$ generated by the relations

$$A \preceq s_F A \qquad \text{if } A \in \mathcal{A},\ F \in \mathcal{F},\ A \subset F^-.$$

So $A \preceq B$ means, that there exists some sequence of alcoves, say $A = A_0, A_1, \dots, A_n = B$ and some sequence of reflecting hyperplanes $F_i \in \mathcal{F}$ such that $A_i \subset F_i^-$ and $A_{i+1} = s_{F_i} A_i$ for $i = 0, \dots, n-1$. To check that $\preceq$ is indeed a partial order, we may proceed as follows: Let us denote for an alcove $A \in \mathcal{A}$ by $b(A) \in V$ its barycenter. Then $A \preceq B$ implies $b(A) \in b(B) + \mathbb{R}_{\leq 0} R^+$. Thus $B \preceq A \preceq B$ implies $b(A) = b(B)$ and hence $A = B$.

Obviously our new notation is compatible with our old notation $A \prec As$ for $s \in \mathcal{S}$. Obviously our partial order on $\mathcal{A}$ is invariant under translation by $\mu \in X$. In addition it has the following property:

Lemma 4.13 ([Lus80a])**.** *Let* $A, B \in \mathcal{A}$ *and* $s \in \mathcal{S}$. *Then* $B \preceq A \prec As$ *implies* $Bs \preceq As$.

Proof. The Bruhat order on $\mathcal{W}$ defines via our bijection $\mathcal{W} \to \mathcal{A}$, $w \mapsto wA^+$ another partial order $\leq$ on $\mathcal{A}$. Deep inside $\mathcal{C}$ now $\preceq$ and $\leq$ coincide. More precisely we have:

Claim 4.14. Let $\mu \in X \cap \mathcal{C}$. For $A, B \in \mathcal{A}$ are equivalent:

1. $A \preceq B$.
2. $n\mu + A \leq n\mu + B$ for $n \gg 0$, i. e. for all n above a suitable lower bound depending on A, B and μ.

This claim follows from the definition of Bruhat order. Indeed $A \leq B$ means, that there exists a sequence of alcoves $A = A_0, A_1, \dots A_n = B$ and a sequence of reflecting hyperplanes $F_i \in \mathcal{F}$ such that $A_{i+1} = s_{F_i} A_i$ and that A_i isn't separated from A^+ by F_i. So for $A, B \in \mathcal{A}^+$ and $F \in \mathcal{F}$ a hyperplane such that $B = s_F A$, we have

$$\begin{aligned} A \preceq B \quad &\Leftrightarrow \quad A \subset F^- \\ &\Leftrightarrow \quad A \text{ and } A^+ \text{ are not separated by } F \\ &\Leftrightarrow \quad A \leq B. \end{aligned}$$

The equivalence of (1) and (2) in general can easily be deduced from this special case. Using the claim we deduce the lemma from the analogous property of the Bruhat order. □

Let $\Pi \subset V$ be the fundamental box

$$\Pi = \{\tau \in V \mid 0 < \langle \tau, \alpha^\vee \rangle < 1 \quad \forall \alpha \in \Delta\}.$$

For $\lambda \in X$ we abbreviate $\lambda + \Pi = \Pi_\lambda$. For any alcove $A \in \mathcal{A}$ there exists a unique $\lambda = \lambda(A) \in X$ such that $A \subset \Pi_\lambda$.

Lemma 4.15. *Let λ be a dominant weight, i.e. $\lambda \in X \cap \overline{\mathcal{C}}$. Then we have $B \preceq \lambda + B$ for every alcove B.*

Proof. We choose $\tau \in B$ and consider the line segment joining τ and $\lambda + \tau$. It meets in that order, say, the alcoves $B = A_0, A_1, A_2, \dots, A_n = \lambda + B$. Choosing τ properly we can assume that subsequent alcoves in our sequence are separated just by one wall, $A_{i+1} = A_i s_i$ for $s_i \in \mathcal{S}$. Now $\lambda \in \overline{\mathcal{C}}$ implies $A_i \prec A_{i+1}$, hence $B \preceq \lambda + B$. □

Now we are ready to prove the existence part of Theorem 4.3 (2). More precisely we show

Proposition 4.16 ([Lus80a])**.** *For $A \in \mathcal{A}$ there exists a self-dual $\underline{P}_A \in \mathcal{P}^\circ$ such that $\underline{P}_A \in A + \sum_{B \prec A} v\mathbb{Z}[v]B$ and $\langle w \rangle \underline{P}_A = \underline{P}_A$ for all $w \in \mathcal{W}_{\lambda(A)}$.*

Proof. Certainly it will be sufficient to construct $\underline{P}_A$ for $A \subset \Pi$. We proceed by induction on the ordered set of all alcoves in Π and start the induction with

$$\underline{P}_{A^+} = E_0 = \sum_{z \in W} v^{l(z)} (zA^+).$$

Now let $A \subset \Pi$ be an alcove and suppose we already know possible $\underline{P}_B$ for $B \prec A$, $B \subset \Pi$. If $A \neq A^+$ we find $s \in \mathcal{S}$ such that $As \prec A$ and $As \subset \Pi$. Clearly $\underline{P}_{As}C_s$ is self-dual and by Lemma 4.13 and the definition of AC_s we get

$$\underline{P}_{As}C_s = \sum_{B \preceq A} p_B B$$

with $p_A = 1$ and $p_B \in \mathbb{Z}[v]$ for all B. However certain p_B with $B \neq A$ could also have a nonzero constant term. To eliminate these terms, we need a longer argument.

First let us define as in [Lus80a] a new action of $\tilde{\mathcal{W}}$ on $\mathcal{A}$, denoted $B \mapsto w * B$, by the formula

$$w * (\lambda + A) = (w\lambda) + A$$

for all $\lambda \in X, A \subset \Pi$.

Lemma 4.17. *Let $P \in \mathcal{P}^\circ$ be of the form $P = \sum p_B B$ with $p_B \in \mathbb{Z}[v]$ for all B. Take $w \in \tilde{\mathcal{W}}$. Then $\langle w \rangle P = \sum q_B B$ with $q_B \in \mathbb{Z}[v]$ and $p_B(0) = q_{w*B}(0)$ for all B.*

Proof. Without restriction we assume $w = s_F$ with $F \in \mathcal{F}_\alpha$ for $\alpha \in \Delta$. Then our definitions imply that the claim even holds for all $P \in \mathcal{P}_\alpha$. □

For our special situation this means that $p_B(0) = p_{z*B}(0)$ for all $z \in W$. But for any alcove B we find $z \in W$ such that $B' = z * B \in \mathcal{A}^+$. Now $p_B \neq 0$ and $B \neq A$ imply $p_{B'} \neq 0$ and $B' \neq A$, hence $B' \prec A$. On the the other hand $B' \in \mathcal{A}^+$ implies $B' = \lambda + B''$ for suitable $\lambda \in X \cap \overline{\mathcal{C}}$ and $B'' \subset \Pi$. From Lemma 4.15 we get $B'' \prec A$, so we know a possible $\underline{P}_{B''}$ by induction, and translating this $\underline{P}_{B''}$ by λ we get a possible $\underline{P}_{B'}$. We now consider

$$\underline{P}_A = \underline{P}_{As}C_s - \sum_{B \in \mathcal{A}^+, B \neq A} p_B(0) \sum_z \langle z \rangle \underline{P}_B,$$

where in the second sum z runs over (a set of representatives for) the cosets $W/W_{\lambda(B)}$. This completes the induction step and the existence of the $\underline{P}_A$ is established. □

Next we have to care for unicity of the $\underline{P}_A$. By Proposition 4.16 we even know, that there exists a family of self-dual elements $\{\underline{P}_A\}_{A \in \mathcal{A}}$ in $\mathcal{P}^\circ$ such that

1. $\langle w \rangle \underline{P}_A = \underline{P}_{w*A} \quad \forall A \in \mathcal{A}, w \in \tilde{\mathcal{W}}$.
2. $\underline{P}_A \in A + \sum_{B \prec A} v\mathbb{Z}[v]B$.

Indeed we get such a family by choosing possible $\underline{P}_A$ for $A \subset \Pi$ as in the Proposition and defining the remaining $\underline{P}_A$ for $A \not\subset \Pi$ as translations of these.

Proposition 4.18 ([Lus80a]). 1. *Such a family $\{\underline{P}_A\}_{A \in \mathcal{A}}$ already is an $\mathcal{L}$-basis of $\mathcal{P}^\circ$.*
2. *For $P \in \mathcal{P}^\circ \cap \sum v\mathbb{Z}[v]B$ self-duality $P = \overline{P}$ implies $P = 0$.*

Remark 4.19. Certainly (2) implies the unicity of the $\underline{P}_A$ claimed in Theorem 4.3. The preceeding considerations or Lemma 4.17 then show $\langle w \rangle \underline{P}_A = \underline{P}_{w*A}$.

Proof. Let us start with (1). Clearly the $\underline{P}_A$ are linearly independent. We have to show they generate $\mathcal{P}^\circ$ over $\mathcal{L}$. Thus we have to show, that for all $\lambda \in X$, $H \in \mathcal{H}$ the element $E_\lambda H$ lies in the $\mathcal{L}$-submodule generated by $\underline{P}_A$. Without restriction we can assume $\lambda = 0$. Clearly it will be sufficient to show that every W-invariant $Q \in \mathcal{P}^\circ$ (i.e. $\langle z \rangle Q = Q \quad \forall z \in W$) lies in the $\mathcal{L}$-submodule generated by the $\underline{P}_A$. So suppose

$$Q = \sum_{B \in \mathcal{A}} q_B B.$$

For $Q \neq 0$ there exists $B \in \mathcal{A}^+$ such that $q_B \neq 0$. (To see this one may take the smallest $n \in \mathbb{Z}$ such that $v^n Q \in \sum_B \mathbb{Z}[v]B$ and apply Lemma 4.17 to $v^n Q$.) Now we proceed by induction on $\#\{A \in \mathcal{A}^+ \mid \exists B \in \mathcal{A}^+ \text{ such that } q_B \neq 0 \text{ and } B \succeq A\}$. Let $C \in \mathcal{A}^+$ be maximal with $q_C \neq 0$. We consider

$$Q' = Q - \sum_z \langle z \rangle q_C \underline{P}_C,$$

where z runs over (a system of representations for) the cosets $W/W_{\lambda(C)}$. Then Q' is W-invariant, and by induction Q' lies in the $\mathcal{L}$-submodule generated by $\underline{P}_A$. This proves (1).

Next we show (2). Certainly any self-dual P has the form $P = \sum_{A \in \mathcal{A}} c_A \underline{P}_A$ with $\overline{c}_A = c_A$. On the other hand $P = \sum p_A A$ with $p_A \in v\mathbb{Z}[v]$ by assumption. If $P \neq 0$, there is A maximal such that $p_A \neq 0$. Then $p_A = c_A$, hence $\overline{p}_A = p_A$ and then $p_A \in v\mathbb{Z}[v]$ implies $p_A = 0$. This contradiction proves the Proposition and completes the proof of Theorem 4.3. □

To simplify the calculation of $p_{B,A}$ one may use

Proposition 4.20 ([Lus80a]). *For $s \in \mathcal{S}$ and $A \in \mathcal{A}$ such that $As \prec A$ we have $\underline{P}_A C_s = (v + v^{-1})\underline{P}_A$. In other words, $p_{Bs,A} = v p_{B,A}$ for all $B \in \mathcal{A}$ such that $Bs \prec B$.*

Proof. We put $P = \underline{P}_A C_s - (v+v^{-1})\underline{P}_A$. From $C_s^2 = (v+v^{-1})C_s$ we deduce $PC_s = 0$. By construction P is of the form $P = \sum p_B B$ with $p_B \in \mathbb{Z}[v]$, and since P is self-dual we get $P = \sum p_B(0)\underline{P}_B$. In case $P \neq 0$ we would find D maximal such that $p_D \neq 0$, and for this D we would even get $p_D = p_D(0)$. Now we write

$$PC_s = \sum q_B B$$

and deduce $q_{Ds} = p_D \neq 0$ if $Ds \succ D$, and $q_D = p_{Ds} + v^{-1}p_D \neq 0$ if $Ds \prec D$, contradicting $PC_s = 0$. Thus $P = 0$ and the Proposition is proved. □

For later use we have to discuss an additional symmetry of $\underline{P}_A$. As in the proof of Proposition 4.11 let $w_0 \in W$ be the longest element and r its length. We define a bijection $\mathcal{A} \to \mathcal{A}$, $A \mapsto \check{A}$ as follows: Write $A = \lambda + B$ with $\lambda \in X$, $B \subset \Pi$ and put $\check{A} = \lambda + w_0 B$. The inverse bijection is denoted $A \mapsto \hat{A}$.

Lemma 4.21 ([Lus80a])**.** $\underline{P}_A \in v^r\left(\check{A} + \sum_{B \succ \check{A}} v^{-1}\mathbb{Z}[v^{-1}]B\right)$.

Proof. We may assume $A \subset \Pi$. Then $\underline{P}_A = \overline{\underline{P}_A} = v^r c\langle w_0\rangle \underline{P}_A = v^r c \underline{P}_A$ and the Lemma follows from the definition of c. □

We will also need a bound on the support of $\underline{P}_A$.

Proposition 4.22 ([Lus80a])**.** *Let A, $B \in \mathcal{A}$ be such that $p_{B,A} \neq 0$. Then $xB \preceq A \quad \forall x \in \mathcal{W}_{\lambda(A)}$.*

Proof. This is proved by induction using the inductive construction of the $\underline{P}_A$ in the proof from Proposition 4.16. □

5. Relations between different sorts of polynomials for affine reflection groups

We continue with the notations of the preceding section and put $\mathcal{S}_0 = \{s \in \mathcal{S} \mid s \text{ stabilizes zero}\}$. Then we can use our notations $\mathcal{M} = \mathcal{M}^0$, $\mathcal{N} = \mathcal{N}^0$, $\mathcal{M}^*$, $\mathcal{N}^*$ etc. from Section 3. In particular $\mathcal{W}_0 = W$ is the (finite) Weyl group. The bijection $\mathcal{W} \to \mathcal{A}$, $w \mapsto wA^+$ restricts to a bijection $\mathcal{W}^0 \to \mathcal{A}^+$. We use this bijection to rename our distinguished elements of $\mathcal{N}, \mathcal{M}^*$, etc. and put $N_x = N_A$, $\underline{N}_x = \underline{N}_A$, $n_{x,y} = n_{A,B}$, $M^x = M^A$, etc. if $x, y \in \mathcal{W}^0$ and $A, B \in \mathcal{A}^+$ are given as $xA^+ = A$, $yA^+ = B$.

Let $\mathcal{A}^{++}$ denote the set of all alcoves contained in $\rho + \mathcal{C}$ (where ρ is the half-sum of positive roots). Then $A \mapsto \hat{A}$ is a bijection $\mathcal{A}^+ \xrightarrow{\sim} \mathcal{A}^{++}$. Recall the $\mathcal{L}$-skew-linear map $\psi : \mathcal{N} \to \mathcal{M}^*$ from Theorem 3.8 given by

$\psi(N_A) = M^A$. The only essential result of this article, which I could't find in the literature, is the following

Theorem 5.1. *For all $A \in \mathcal{A}^+$ we have $\underline{M}^A = v^r \psi \underline{N}_{\hat{A}}$, in other words $m^{B,A} = v^r \overline{n}_{B,\hat{A}}$.*

This formula was suggested by the theory of tilting modules, as will be explained in more detail at the end of this article. The proof of the Theorem needs some preparation. We consider in $\mathcal{P}$ the $\mathcal{H}$-submodule

$$\mathcal{P}^{\mathrm{sgn}} = \big\{ P \in \mathcal{P}^\circ \mid \langle z \rangle P = (-1)^{l(z)} P \quad \forall z \in W \big\}.$$

Proposition 5.2. *The $\mathcal{L}$-linear "restriction" $\mathrm{res} : \mathcal{P} \to \mathcal{N}$ given by $\mathrm{res}\, A = N_A$ if $A \in \mathcal{A}^+$ and $\mathrm{res}\, A = 0$ else induces a homomorphism of right $\mathcal{H}$-modules $\mathrm{res} : \mathcal{P}^{\mathrm{sgn}} \to \mathcal{N}$.*

Proof. We have to show that res commutes with all C_s $(s \in \mathcal{S})$. The formula for the action of $\mathcal{W}$ on $\mathcal{P}^\circ$ implies that for $P = \sum p_A A$ in $\mathcal{P}^{\mathrm{sgn}}$ we have $p_A = -v p_{As}$ for $A \in \mathcal{A}^+$, $s \in \mathcal{S}$ such that $As \notin \mathcal{A}^+$. The Proposition follows. □

We define next the $\mathcal{H}$-linear map

$$\begin{array}{rccl} \mathrm{alt} : & \mathcal{P}^\circ & \to & \mathcal{P}^{\mathrm{sgn}} \\ & P & \mapsto & \displaystyle\sum_{x \in W} (-1)^{l(x)} \langle x \rangle P. \end{array}$$

Now our theorem follows from the following more detailed:

Theorem 5.3. 1. *For all $A \in \mathcal{A}^{++}$ we have $\underline{N}_A = \mathrm{res}\,\mathrm{alt}\, \underline{P}_A$.*
2. *For all $A \in \mathcal{A}^+$ we have $\underline{M}^A = v^r \psi\, \mathrm{res}\,\mathrm{alt}\, \underline{P}_{\hat{A}}$.*

Remark 5.4. The second statement is a reformulation of the main result in [Kan87] and therefore contains some results of [And86]. Indeed, since $\langle x \rangle \underline{P}_A = \underline{P}_{x*A}$ we have

$$\mathrm{alt}\, \underline{P}_A = \sum_{x \in W} (-1)^{l(x)} \underline{P}_{x*A}.$$

Proof. We start with (2) and for simplicity abbreviate the $\mathcal{L}$-skew-linear map $(v^r \psi\, \mathrm{res}\,\mathrm{alt})$ to $\varphi : \mathcal{P}^\circ \to \mathcal{M}^*$. With respect to $\mathcal{H}$ both φ, ψ are *da*-linear. Now $\underline{M}^A$ can be characterized by a degree condition and self-duality. By Lemma 4.21 our $\varphi(\underline{P}_{\hat{A}})$ satisfies the degree condition. We only need to show that all $\varphi(\underline{P}_{\hat{A}}) \in \mathcal{M}^*$ are self-dual.

By our definitions $F \in \mathcal{M}^*$ is self-dual iff $(\overline{H} F)(M_{A^+}) = \overline{(HF)(M_{A^+})}$ for all $H \in \mathcal{H}$. We will check this for all $F = \varphi(\underline{P}_A)$ with $A \in \mathcal{A}$.

Certainly we can write $\underline{P}_A H = \sum c_B \underline{P}_B$ and deduce $\underline{P}_A \overline{H} = \sum \overline{c}_B \underline{P}_B$. Then we get

$$\begin{aligned} (da(H)\varphi(\underline{P}_A))(M_{A^+}) &= \sum \overline{c}_B \varphi(\underline{P}_B)(M_{A^+}), \\ (a(H)\varphi(\underline{P}_A))(M_{A^+}) &= \sum c_B \varphi(\underline{P}_B)(M_{A^+}), \end{aligned}$$

and hence only have to show, that all $\varphi(\underline{P}_B)(M_{A^+})$ are invariant for the substitution $v \mapsto v^{-1}$. I claim that we have even

$$\varphi(\underline{P}_B)(M_{A^+}) = \begin{cases} (-1)^{l(z)} & \text{if } B = z * \widehat{A^+} \text{ with } z \in W; \\ 0 & \text{otherwise.} \end{cases}$$

To see this, we consider again $\underline{P}_B = \sum p_{C,B} C$ and prove

Lemma 5.5. *Let $B \in \mathcal{A}$ be such that $p_{A^+,B} \neq 0$. Then either $B = \widehat{A^+}$ or $\lambda(B)$ lies on a reflecting hyperplane for W.*

Proof. To avoid introducing more notation we rather show the equivalent statement that for $B \subset \Pi$ and $\lambda \in X$ both $p_{\lambda+A^+,B} \neq 0$ and $\lambda \neq -\rho$ together imply $W_\lambda \neq 1$. (Here $\rho \in X$ denotes as usual the half-sum of positive roots.) Indeed we can take $x \in W$ such that $x(\lambda + A^+) \subset \mathcal{C}$ and get by Proposition 4.22 and the definition of Π the relations

$$x(\lambda + A^+) \preceq B \preceq \rho + w_0 A^+.$$

But for an alcove $C \subset \mathcal{C}$ such that $C \preceq \rho + w_0 A^+$ it is clear that either all its corners from X lie on walls of the dominant chamber or $C = \rho + w_0 A^+$. For us this means that either $W_{x\lambda} \neq 1$ hence $W_\lambda \neq 1$ or $\lambda = -\rho$, $x = w_0$. □

Now the formula for $\varphi(\underline{P}_B)(M_{A^+})$ follows from the observation that $\mathrm{alt}(\underline{P}_B) = 0$ and a forteriori $\varphi(\underline{P}_B) = 0$ if $\lambda(B)$ lies on a a reflecting hyperplane of W. This proves (2).

Next we attack (1). Let $\mathcal{H}^+ \subset \mathcal{H}$ be the subring generated by $v + v^{-1}$ and all C_s with $s \in \mathcal{S}$. By Proposition 3.3 we have $\mathcal{H}^+ = \{H \in \mathcal{H} \mid \overline{H} = H\}$. However we will only use the obvious inclusion $\subset$. The proof of (1) rests on the following

Lemma 5.6. *All $\underline{M}^A$ lie in the $\mathcal{H}^+$-submodule of $\mathcal{M}^*$ generated by the element $\underline{M}^{\rho+w_0A^+}$.*

Proof. First note that by (2) we have $m^{\hat{A},A} = v^r$ and $\deg_v m^{B,A} < r$ if $B \neq \hat{A}$. Now consider $A \in \mathcal{A}^+$, $s \in \mathcal{S}$ such that $As \prec A$ (but not necessarily $As \in \mathcal{A}^+$) and write

$$a(C_s)\underline{M}^A = \sum q_B M^B.$$

By Theorem 3.8 we know that

$$a(C_s)M^B = \begin{cases} M^{Bs} + v^{-1}M^B, & \text{if } Bs \in \mathcal{A}^+, \quad Bs \succ B; \\ M^{Bs} + vM^B, & \text{if } Bs \in \mathcal{A}^+, \quad Bs \prec B; \\ 0 & \text{if } Bs \notin \mathcal{A}^+. \end{cases}$$

Thus we have $q_B \in \mathbb{Z}[v]$ for all $B \in \mathcal{A}^+$ and we deduce

$$a(C_s)\underline{M}^A = \sum q_B(0)\underline{M}^B.$$

Indeed any element of $\mathcal{M}^*$ can be written uniquely as a formal linear combination of the $\underline{M}^B$, if the element is self-dual all its coefficients are, and for an element from $\sum^\infty \mathbb{Z}[v]M^B$ all its coefficients lie in $\mathbb{Z}[v]$. Thus if $\sum^\infty q_B M^B$ is self-dual with all q_B in $\mathbb{Z}[v]$, then we have

$$\sum^\infty q_B M^B = \sum^\infty q_B(0)\underline{M}^B.$$

For $q \in \mathcal{L}$ let $\hat{q}(\nu)$ denote the coefficient of v^ν, thus $q = \sum_\nu \hat{q}(\nu)v^\nu$ and $\hat{q}(0) = q(0)$. We can further conclude $\hat{q}_{\check{B}}(r) = q_B(0)$ for all $B \in \mathcal{A}^+$ and $\hat{q}_B(r) = 0$ for $B \notin \mathcal{A}^{++}$. Thus we get even

$$a(C_s)\underline{M}^A = \sum \hat{q}_B(r)\underline{M}^{\check{B}}.$$

Changing variables, we have proved: If $D \in \mathcal{A}^{++}$ and $s \in \mathcal{S}$ are given such that $Ds \succ D$, when we write

$$a(C_s)\underline{M}^{\check{D}} = \sum_{B\in\mathcal{A}^+} q_B M^B,$$

then we have

$$a(C_s)\underline{M}^{\check{D}} = \sum_{B\in\mathcal{A}^{++}} \hat{q}_B(r)\underline{M}^{\check{B}}.$$

On the other hand we know by (2) that $m^{B,\check{D}} \neq 0 \Rightarrow B \preceq D$ and $m^{D,\check{D}} = v^r$ for all $D \in \mathcal{A}^{++}$. Thus we get more precisely

$$a(C_s)\underline{M}^{\check{D}} = \underline{M}^{(Ds)^\vee} + \sum_{\substack{B\in\mathcal{A}^{++} \\ B\prec Ds}} \hat{q}_B(r)\underline{M}^{\check{B}}.$$

Using this formula it is easy to show by induction on $\mathcal{A}^{++}$, that all $\underline{M}^{\check{A}}$ with $A \in \mathcal{A}^{++}$ lie in the $\mathcal{H}^+$-submodule of $\mathcal{M}^*$ generated by $\underline{M}^{(\rho+A^+)^\vee}$. □

To show (1) we still need

Lemma 5.7. $\underline{N}_{\rho+A^+} = \sum_{z\in W} v^{l(z)} N_{\rho+zA^+}$.

Proof. Consider more generally for all $\lambda \in (\rho + \mathbb{Z}R) \cap \mathcal{C}$ the expression $F_\lambda = \sum_{z \in W} v^{l(z)} N_{\lambda + zA^+}$. We have to show $\underline{N}_{\rho+A^+} = F_\rho$. Certainly it will be sufficient to show $F_\rho = \overline{F_\rho}$. To show this consider the set

$$\mathcal{S}_\rho = \{s \in \mathcal{S} \mid \rho \in \overline{A} \Rightarrow \rho \in \overline{As} \quad \forall A \in \mathcal{A}\},$$

where exceptionally $\overline{A}$ resp. $\overline{As}$ means the closure of A resp. As. We claim that $F_\rho \in \mathcal{N}$ is the unique element $F = \sum f_A N_A$ of $\mathcal{N}$ such that

1. $FC_s = (v + v^{-1})F \quad \forall s \in \mathcal{S}_\rho$
2. $f_A \neq 0 \Rightarrow A \leqslant \rho + A^+$
3. $f_A = 1$ for $A = \rho + A^+$

Here $\leqslant$ means the order we get on $\mathcal{A}^+$ by transporting the Bruhat order from $\mathcal{W}^0$.

First we see that condition (1) is satisfied precisely by all $\mathcal{L}$-linear combinations of the F_λ with $\lambda \in (\rho + \mathbb{Z}R) \cap \mathcal{C}$. From there we see easily that F_ρ is the unique element of $\mathcal{N}$ satisfying (1)–(3). However these conditions are self-dual, hence $\overline{F_\rho}$ also satisfies (1)–(3) and we deduce $\overline{F_\rho} = F_\rho$. □

Now by Lemma 5.6 all $(\operatorname{res} \operatorname{alt} \underline{P}_A)$ are contained in the $\mathcal{H}^+$-submodule of $\mathcal{N}$ generated by $(\operatorname{res} \operatorname{alt} \underline{P}_{\rho+A^+})$, and $(\operatorname{res} \operatorname{alt} \underline{P}_{\rho+A^+})$ is self-dual in $\mathcal{N}$ by Lemma 5.7. This means all $(\operatorname{res} \operatorname{alt} \underline{P}_A)$ are self-dual in $\mathcal{N}$, and since they satisfy the degree conditions characterizing the $\underline{N}_A$, we deduce $\underline{N}_A = \operatorname{res} \operatorname{alt} \underline{P}_A \quad \forall A \in \mathcal{A}^{++}$. □

6. The generic polynomials

Once we are far enough inside the dominant chamber, the $m_{B,A}$ depend only on the relative position of the alcoves A and B. More precisely we show

Theorem 6.1 ([Lus80a])**.** 1. *For all $A, B \in \mathcal{A}$ there exists $q_{B,A} \in \mathbb{Z}[v]$ such that $q_{B,A} = m_{\lambda+B,\lambda+A}$, if λ is sufficiently far inside the dominant chamber, i.e. if $\lambda \in X \cap (n\rho + \mathcal{C})$ for suitable $n = n(A, B) \in \mathbb{Z}$.*

2. *For the periodic polynomials $p_{B,C}$ we have the inversion formulas*

$$\sum_B (-1)^{d(A,B)} q_{w_0B, w_0A}\, p_{B,C} = \delta_{A,C},$$

where $(-1)^{d(A,B)}$ means the parity of the number of affine reflection hyperplanes separating A and B.

Remark 6.2. The "generic Kazhdan-Lusztig polynomials" $\hat{P}_{A,B}$ of Kato [Kat85] are related to our $q_{A,B}$ by the formula $\hat{P}_{A,B} = v^{-d(A,B)} q_{A,B}$.

To prove the Theorem we consider the "completed below" Hecke module

$$\hat{\mathcal{P}} = \{f : \mathcal{A} \to \mathcal{L} \mid \text{there is } C \in \mathcal{A} \text{ such that } f(A) \neq 0 \Rightarrow A \preceq C\}.$$

For two alcoves C_1, $C_2 \in \mathcal{A}$ there always exists $C \in \mathcal{A}$ such that $C_1 \preceq C$, $C_2 \preceq C$. Thus $\hat{\mathcal{P}}$ is an $\mathcal{L}$-submodule of the space of all maps from $\mathcal{A}$ to $\mathcal{L}$. We write elements $f \in \hat{\mathcal{P}}$ as formal linear combinations $f = \sum^{\infty} f_A A$ with $f_A = f(A)$, where the upper index ∞ again should remind us that also certain infinite formal linear combinations are permitted. We extend the right action of $\mathcal{H}$ on $\mathcal{P}$ to $\hat{\mathcal{P}}$ in the obvious way. For $\lambda \in \mathbb{Z}R$ we extend also $\langle\lambda\rangle : \mathcal{P} \to \mathcal{P}$ to a map $\langle\lambda\rangle : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ in the obvious way. For $\alpha \in R^+$ we define the operator

$$\vartheta_\alpha : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$$

as the formal sum

$$\vartheta_\alpha = (1 + v^2\langle-\alpha\rangle + v^4\langle-2\alpha\rangle + v^6\langle-3\alpha\rangle + \cdots).$$

Certainly ϑ_α commutes with the right $\mathcal{H}$-action. Also the ϑ_α commute among themselves. We put

$$\eta = \prod_{\alpha \in R^+} \vartheta_\alpha : \hat{\mathcal{P}} \to \hat{\mathcal{P}}.$$

This η is closely related to Kostant's partition function. It gives another relation between the periodic and the generic polynomials, namely the following

Theorem 6.3 ([Kat85])**.** $\eta \underline{P}_A = \sum_B q_{B,A} B$.

Proof. Will be given later. ☐

Finally we could also ask whether one could define alternative periodic polynomials by changing v to v^{-1} in the definition of $\underline{P}_A$. It turns out that the $\tilde{\underline{P}}_A$ so defined exist only in $\hat{\mathcal{P}}$. More precisely we extend our skew-linear duality $P \mapsto \overline{P}$ to $\hat{\mathcal{P}}$ as follows: We can write P uniquely as a formal sum $P = \sum_A^{\infty} p_A \underline{P}_A$ with $p_A \in \mathcal{L}$, where we start in the highest alcoves where P has a nonzero coefficient, and then work our way down. Then we define $\overline{P} = \sum_A^{\infty} \overline{p}_A \underline{P}_A$. It is easy to see that $\overline{A} \in A + \sum_{B \prec A}^{\infty} \mathcal{L}B$ for $A \in \mathcal{A}$.

Theorem 6.4.
1. $\underline{P}_A$ *is the unique self-dual element of* $\hat{P}$ *contained in* $A + \sum_B^{\infty} v\mathbb{Z}[v]B$.
2. $\tilde{\underline{P}}_A = \sum(-1)^{d(A,B)}\overline{q}_{B,A}B$ *is the unique self-dual element of* $\hat{\mathcal{P}}$ *contained in* $A + \sum_B^{\infty} v^{-1}\mathbb{Z}[v^{-1}]B$.

Remark 6.5. Here (1) comes from [Lus80a] and (2) from [Kat85].

Now we prove the three preceding Theorems. Let $\mathrm{Alt} = \langle -\rho\rangle \circ \mathrm{alt} \circ \langle \rho \rangle : \mathcal{P}^\circ \to \mathcal{P}^\circ$ be antisymmetrization around $-\rho$, thus

$$\mathrm{Alt}(P) = \sum_{x \in \mathcal{W}_{-\rho}} (-1)^{l(x)} \langle x \rangle P.$$

We also define the $\mathcal{L}$-linear restriction

$$\begin{array}{rlcl} \mathrm{Res}: & \hat{\mathcal{P}} & \to & \mathcal{M} \\ & \sum_A^\infty f_A A & \mapsto & \sum_{A \in \mathcal{A}^+} f_A M_A. \end{array}$$

This restriction doesn't commute with the $\mathcal{H}$-actions on our spaces. However we have

Proposition 6.6. *The composition* $\mathrm{Res} \circ \eta \circ \mathrm{Alt} : \mathcal{P}^\circ \to \mathcal{M}$ *is a homomorphism of right* $\mathcal{H}$*-modules.*

Proof. It will be sufficient to show that this map commutes with all C_s. With our definitions this is easily deduced from the following

Claim 6.7. Let $A, B \in \mathcal{A}$ be neighbouring alcoves such that $A \in \mathcal{A}^+$, $B \notin \mathcal{A}^+$. Then for $f = \sum^\infty f_C C \in \eta \circ \mathrm{Alt}(\mathcal{P}^\circ)$ we have $f_B = v f_A$.

Let us check this claim. By our assumptions A and B meet along a wall of the dominant chamber. Let $\beta \in \Delta$ be the corresponding simple root. By a β-string in $\mathcal{A}$ we mean a minimal nonempty subset containing with A also $\beta{\uparrow}A$ and $\beta{\downarrow}A$. Now let us consider in $\hat{\mathcal{P}}$ the $\mathcal{H}$-submodule

$$\hat{\mathcal{P}}_\beta = \{\sum_{A \in \mathcal{A}}^\infty f_A A \in \hat{\mathcal{P}} \mid \text{for every } \beta\text{-string } \mathcal{K} \text{ we have } \sum_{A \in \mathcal{K}}^\infty f_A A \in \mathcal{P}_\beta\}.$$

Certainly $\vartheta_\alpha \hat{\mathcal{P}}_\beta \subset \hat{\mathcal{P}}_\beta$ for $\alpha \in R^+$, $\alpha \neq \beta$. Let $\tilde{s_\beta} \in \mathcal{W}_{-\rho}$ be the reflection along the β-wall passing through $(-\rho)$. Then $\langle \tilde{s_\beta} \rangle : \mathcal{P}_\beta \to \mathcal{P}_\beta$ can be extended to $\langle \tilde{s_\beta} \rangle : \hat{\mathcal{P}}_\beta \to \hat{\mathcal{P}}_\beta$ in an obvious way, and we have $\langle \tilde{s_\beta} \rangle \circ \vartheta_\alpha = \vartheta_{s_\beta(\alpha)} \circ \langle \tilde{s_\beta} \rangle$ for all $\alpha \in R^+$, $\alpha \neq \beta$. Now we choose a system of representatives $\mathrm{Rep} \subset \mathcal{W}_{-\rho}$ for the cosets $\{e, \tilde{s_\beta}\} \backslash \mathcal{W}_{-\rho}$ and get

$$\begin{aligned} \eta \circ \mathrm{Alt} &= \prod_{\alpha \in R^+} \vartheta_\alpha \circ (1 - \langle \tilde{s_\beta} \rangle) \prod_{x \in \mathrm{Rep}} (-1)^{l(x)} \langle x \rangle \\ &= \vartheta_\beta \circ (1 - \langle \tilde{s_\beta} \rangle) \prod_{\substack{\alpha \in R^+ \\ \alpha \neq \beta}} \vartheta_\alpha \prod_{x \in \mathrm{Rep}} (-1)^{l(x)} \langle x \rangle. \end{aligned}$$

Thus our claim will follow immediately from the much more elementary

Claim 6.8. Let A, $B \in \mathcal{A}$ be neighbouring alcoves separated only by the β-wall of the dominant chamber. If A lies above this wall, then for all $f = \sum^\infty f_C C \in \vartheta_\beta (1 - \langle \tilde{s_\beta} \rangle) \hat{\mathcal{P}}_\beta$ we have $f_B = v f_A$.

This claim can be checked separately for every β-string, thus we have only to check the case

$$f = \vartheta_\beta(1 - \langle \tilde{s_\beta} \rangle)(C + v(\beta{\downarrow}C))$$

with $C \in \mathcal{A}$. But this case is clear from the definitions. □

Corollary 6.9. $\underline{M}_A = \mathrm{Res} \circ \eta \circ \mathrm{Alt}\, \underline{P}_A$ *for all* $A \in \mathcal{A}^+$.

Remark 6.10. This is Theorem 4.2 of Kato [Kat85]. Remark that $\mathrm{Alt}\, \underline{P}_A = \sum_{x \in \mathcal{W}_{-\rho}} (-1)^{l(x)} \langle x\lambda(A) \rangle \underline{P}_A$ and thus

$$\eta\, \mathrm{Alt}\, \underline{P}_A = \sum_{x \in \mathcal{W}_{-\rho}} (-1)^{l(x)} \langle x\lambda(A) \rangle \eta \underline{P}_A.$$

Thus the Corollary implies in particular part (1) of Theorem 6.1 (where the generic polynomials $q_{B,A}$ are defined) and Theorem 6.3.

Proof. For $A = A^+$ both sides equal $M_{A^+} = \underline{M}_{A^+}$ and our formula is true. But by Theorem 5.3 and Lemma 5.6 we know already that $\mathrm{Alt}\, \underline{P}_A \in (\mathrm{Alt}\, \underline{P}_{A^+})\mathcal{H}^+$ for all $A \in \mathcal{A}$. Indeed, translating by ρ it will be sufficient to show that $\mathrm{alt}\, \underline{P}_{\hat{A}} \in (\mathrm{alt}\, \underline{P}_{\rho + A^+})\mathcal{H}^+$ for all $A \in \mathcal{A}^+$. By Theorem 5.3 (2) and its proof the map $v^r \psi\, \mathrm{res}$ defines a da-linear injection $\mathrm{alt}\, \mathcal{P}^\circ \hookrightarrow \mathcal{M}^*$ with $\mathrm{alt}\, \underline{P}_{\hat{A}} \mapsto \underline{M}^A$ for all $A \in \mathcal{A}^+$, thus it will be sufficient to show that $\underline{M}^A \in (\underline{M}^{\rho + w_\circ A})\mathcal{H}^+$ for all $A \in \mathcal{A}^+$. But this is precisely Lemma 5.6.

From $\mathrm{Alt}\, \underline{P}_A \in (\mathrm{Alt}\, \underline{P}_{A^+})\mathcal{H}^+$ we deduce immediately that

$$\mathrm{Res} \circ \eta \circ \mathrm{Alt}\, \underline{P}_A \in \underline{M}_{A^+}\mathcal{H}^+$$

is self-dual for all $A \in \mathcal{A}$. On the other hand certainly

$$\mathrm{Res} \circ \eta \circ \mathrm{Alt}\, \underline{P}_A \in M_A + \sum_B v\mathbb{Z}[v] M_B$$

for all $A \in \mathcal{A}^+$ and the Corollary is established. □

Next we show Theorem 6.4.

Proof. Here everything is left to the reader, except the proof that the formula claimed for $\tilde{\underline{P}}_A$ gives indeed a self-dual element of $\hat{\mathcal{P}}$. Let $\xi : \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ be the $\mathcal{L}$-skew-linear map such that

$$\xi(\sum_A^\infty p_A A) = \sum_A^\infty (-1)^{d(A^+, A)} \overline{p}_A A.$$

We have to show that $\xi \eta \underline{P}_A \in \hat{\mathcal{P}}$ is self-dual. We prove this by contradiction. Let us write

$$\xi \eta \underline{P}_A - \overline{\xi \eta \underline{P}_A} = \sum^\infty f_C C$$

and choose $B \in \mathcal{A}$ such that $f_B \neq 0$. Moving the pair (A, B) sufficiently far inside the dominant chamber we may assume that $\eta \operatorname{Alt} \underline{P}_A$ and $\eta \underline{P}_A$ coincide on all alcoves C such that $C \succeq B$. By Corollary 6.9 on these alcoves also $\operatorname{Res} \eta \underline{P}_A$ coincides with $\underline{M}_A$ and $\operatorname{res} \xi \eta \underline{P}_A$ with $\phi^{-1} \underline{M}_A$, where ϕ as in Section 3 denotes the $\mathcal{L}$-skew-linear map $\phi : \mathcal{N} \to \mathcal{M}$ such that $\phi(N_x) = (-1)^{l(x)} M_x$, and we extended our old $\operatorname{res} : \mathcal{P} \to \mathcal{N}$ to $\hat{\mathcal{P}}$ in the obvious way.

Now by Theorem 5.3 we know that $\underline{N}_D = \operatorname{res} \operatorname{alt} \underline{P}_D$ for $D \in \mathcal{A}^{++}$, in particular we have $\underline{N}_D = \operatorname{res} \underline{P}_D$ for all $D \subset \mathcal{C}$ which are sufficiently far from all walls of the dominant chamber. Moving (A, B) if necessary still further inside the dominant chamber we can assume in addition that all alcoves C such that $A \succeq C \succeq B$ are already so far from the walls, that $\underline{N}_C = \operatorname{res} \underline{P}_C$. Now we can write

$$\begin{aligned} \xi \eta \underline{P}_A &= \sum_C^{\infty} p_C \underline{P}_C, \\ \phi^{-1} \underline{M}_A &= \sum n_C \underline{N}_C, \end{aligned}$$

and deduce $p_C = n_C$ for $A \succeq C \succeq B$. But since $\phi^{-1} \underline{M}_A = \pm \underline{\tilde{N}}_A$ is self-dual (by the end of the proof of Theorem 3.5), all n_C have to be self-dual, hence all p_C for $A \succeq C \succeq B$ have to be self-dual as well, and this finally leads to the contradiction $f_B = 0$. $\square$

We are left with proving part (2) of Theorem 6.1.

Proof. Let us start with the almost tautological formula

$$\sum_B (-1)^{d(A,B)} m_{B,A} m^{B,C} = \delta_{A,C}.$$

If the pair (C, A) is sufficiently far inside the dominant chamber, we have $m_{B,A} = q_{B,A}$ for all $B \succeq C$, thus for all B such that $m^{B,C} \neq 0$. On the other hand we also get $m^{B,C} = v^r \overline{p}_{B,\hat{C}} = p_{w_0 B, w_0 C}$. Here the first equality follows from Theorem 5.3, and the last equality follows from the fact that $\underline{P}_{\hat{C}}$ is self-dual. Indeed from there we get $\underline{P}_{\hat{C}} = v^r c \langle w_0 \rangle \underline{P}_{\hat{C}} = v^r c \underline{P}_{w_0 C}$ by the construction of the duality on $\mathcal{P}^\circ$, where we use the formula $w_0 * \hat{C} = w_0 C$. $\square$

All three Theorems of this section are established.

7. Relation with tilting modules

Let h be the Coxeter number of our root system R, and let $l \geqslant h$ be odd. For a primitive l-th root of unity ζ we form following Lusztig the quantum group with divided powers U_ζ. Let U_ζ-mof be the category of all finite dimensional U_ζ-modules, and let $\mathcal{B} \subset U_\zeta$-mof be the principal block, i. e. the smallest direct summand containing the trivial representation. Certainly $\mathcal{B}$ is a k-category for $k = \mathbb{Q}(\zeta)$.

The simple objects of $\mathcal{B}$ are parametrized in a natural way by the set $\mathcal{A}^+$ of alcoves in the dominant chamber. For $A \in \mathcal{A}^+$ let $L_A \in \mathcal{B}$ be the corresponding simple object. L_A is the socle resp. the unique simple quotient of the standard modules ∇_A resp. Δ_A. For example $L_{A^+} = \nabla_{A^+} = \Delta_{A^+} = k$ is the trivial representation. In $\mathcal{B}$ there are enough projectives. The projective cover of L_A is denoted P_A. By a ∇-flag (resp. Δ-flag) of an object of $\mathcal{B}$ we mean a filtration such that all subquotients are of the form ∇_A resp. Δ_A for suitable $A \in \mathcal{A}^+$.

Definition 7.1. An object $T \in \mathcal{B}$ is called a tilting module if and only if T admits a ∇-flag and a Δ-flag.

We recall without proof some facts from the theory of tilting modules. As standard reference for the completely analogous case of algebraic groups in finite characteristic compare [Don93]. First of all a direct summand of a tilting module is also tilting. Furthermore for an $A \in \mathcal{A}^+$ there exists a unique indecomposable tilting module T_A, which admits a Δ-flag starting with $\Delta_A \subset T_A$. Here unicity follows easily from the following property of the standard objects:

$$\mathrm{Ext}^n_{\mathcal{B}}(\Delta_A, \nabla_B) = \begin{cases} k & \text{if } A = B \text{ and } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

One may define a duality on $\mathcal{B}$, i.e. an involutive exact contravariant k-functor $d : \mathcal{B} \to \mathcal{B}$ fixing the simple objects and exchanging the standard objects, $d\Delta_A \cong \nabla_A$. It is known that tilting modules are self-dual for such a duality, $dT_A \cong T_A$. In particular T_A is the unique indecomposable tilting module which admits a ∇-flag ending with a surjection $T_A \twoheadrightarrow \nabla_A$.

The present work was motivated by the problem to determine the multiplicity $(T_A : \nabla_B)$ of ∇_B as a subquotient in a ∇-flag of T_A, for all $A, B \in \mathcal{A}^+$. For this problem I propose the following

Conjecture 7.1. $(T_A : \nabla_B) = n_{B,A}(1)$.

Remark 7.2.
1. Recently I found a proof for this conjecture. However it is quite far from the reason for the conjecture explained below. An interpretation of the coefficients of the $n_{B,A}$ was proposed by [And96].
2. The conjecture implies also character formulas for indecomposable tilting modules "on the walls". More precisely we will show

that for an indecomposable tilting module T on walls and Ψ the translation from the walls ΨT is indecomposable as well.

To see this, let Φ be the translation onto the walls, i.e. the adjoint of Ψ. Let $A(1), \ldots, A(r)$ be the alcoves containing the highest weight of T in their closure. Then we have $\Phi\Psi T \cong T \oplus \ldots \oplus T$ (r copies), since both sides are tilting and have the same character. This already means that only the $T_{A(i)}$ are possible direct summands of ΨT.

Now let $A(1)$ be maximal among the $A(i)$. Since we know the highest weight of ΨT, we also know that $T_{A(1)}$ has to be a summand of ΨT. Using remark 3.2 (4), the conjecture implies that $(T_{A(1)} : \nabla_{A(i)}) = 1$ for $i = 1, \ldots, r$. But on the other hand we know that $(\Psi T : \nabla_{A(i)}) = 1$ for $i = 1, \ldots, r$. Thus in ΨT there is no room for other summands $T_{A(i)}$, and we deduce $\Psi T \cong T_{A(1)}$.

Most indecomposable tilting modules are indeed projective, more precisely we have $P_A \cong T_{\hat{A}}$ by [And92], 5.8. For projective objects the looked-for multiplicities are given already by the reciprocity formulas $(P_A : \nabla_B) = [\nabla_B : L_A]$. Now the Lusztig conjecture says that in the Grothendieck group of $\mathcal{B}$ we have

$$L_A = \sum_B (-1)^{d(B,A)} m_{B,A}(1) \nabla_B.$$

(Use Proposition 3.4 to see that this coincides with the conjecture formulated by Lusztig in [Lus80b].) We can invert this equality to get

$$\sum_A m^{C,A}(1) L_A = \nabla_C.$$

Thus the Lusztig conjecture implies $(P_A : \nabla_B) = m^{B,A}(1)$, and as a first test of our conjecture we should check for $m^{B,A}(1) = n_{B,\hat{A}}(1)$. This is an easy consequence of Theorem 5.1.

The true motivation for our conjecture however comes from the philosophy of "$\mathbb{Z}$-graded representation theory," as explained in the sequel. One expects, that the category $\mathcal{B}$ admits a $\mathbb{Z}$-graded version $\tilde{\mathcal{B}}$ in the meaning of [BGS96], 4.3. The objects ∇_A, Δ_A, L_A should admit $\mathbb{Z}$-graded versions $\tilde{\nabla}_A, \tilde{\Delta}_A, \tilde{L}_A \in \tilde{\mathcal{B}}$, there should exist on $\tilde{\mathcal{B}}$ a "shift of $\mathbb{Z}$-grading" $M \mapsto M\langle i \rangle$ (for $i \in \mathbb{Z}$), the duality should lift to a duality $\tilde{d} : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ such that $\tilde{d}(M\langle i \rangle) = (\tilde{d}M)\langle -i \rangle$, $\tilde{d}(\tilde{\nabla}_A) = \tilde{\Delta}_A$, and we would have

$$\operatorname{Ext}^n_{\tilde{\mathcal{B}}}(\tilde{\Delta}_A, \tilde{\nabla}_B\langle i \rangle) = \begin{cases} k & \text{if } A = B \text{ and } i = n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Recall every $s \in \mathcal{S}$ gives an exact functor $\Theta_s : \mathcal{B} \to \mathcal{B}$, the so called "translation through the wall". It commutes with the duality $\Theta_s d = d\Theta_s$ and is easily determined on standard modules: For $A \in \mathcal{A}^+$ we have short exact sequences

$$\begin{array}{lclll} \nabla_A & \hookrightarrow & \Theta_s \nabla_A \twoheadrightarrow & \nabla_{As} & \text{if } As \succ A,\ As \in \mathcal{A}^+; \\ \nabla_{As} & \hookrightarrow & \Theta_s \nabla_A \twoheadrightarrow & \nabla_A & \text{if } As \prec A,\ As \in \mathcal{A}^+, \end{array}$$

and $\Theta_s \nabla_A = 0$ if $As \notin \mathcal{A}^+$.

One may hope that Θ_s admits a graded version $\tilde{\Theta}_s : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ as well, which commutes with $\tilde{d}$ and is such, that again for $A \in \mathcal{A}^+$ there are short exact sequences

$$\begin{array}{lclll} \tilde{\nabla}_A\langle 1\rangle & \hookrightarrow & \tilde{\Theta}_s \tilde{\nabla}_A \twoheadrightarrow & \tilde{\nabla}_{As} & \text{if } As \succ A,\ As \in \mathcal{A}^+; \\ \tilde{\nabla}_{As} & \hookrightarrow & \tilde{\Theta}_s \tilde{\nabla}_A \twoheadrightarrow & \tilde{\nabla}_A\langle -1\rangle & \text{if } As \prec A,\ As \in \mathcal{A}^+, \end{array}$$

resp. that $\tilde{\Theta}_s \tilde{\nabla}_A = 0$ if $As \notin \mathcal{A}^+$. These expectations are supported by the fact that up to existence of $\tilde{d}$ they can be proved in the analogous situation concerning G_1T-modules, see [AJS94].

Now we try to inductively build up graded tilting modules $\tilde{T}_A$. Thus a $\tilde{\nabla}$-flag of $\tilde{T}_A$ should end with a surjection $\tilde{T}_A \twoheadrightarrow \tilde{\nabla}_A$. We start with $\tilde{T}_{A^+} = \tilde{\nabla}_{A^+} = \tilde{L}_{A^+}$. If $\tilde{T}_A$ is constructed already, we choose $s \in \mathcal{S}$ such that $As \succ A$ and form $\tilde{\Theta}_s \tilde{T}_A$. Certainly this is tilting and has even a $\tilde{\nabla}$-flag finishing with $\tilde{\nabla}_{As}$. However it should not be indecomposable in general, but should rather decompose as

$$\tilde{\Theta}_s \tilde{T}_A = \tilde{T}_{As} \oplus \bigoplus_B \tilde{T}_B,$$

where the sum runs over a suitable multiset of alcoves $B \prec As$. Now one might expect, that all homomorphisms in $\tilde{\mathcal{B}}$ (i.e. all $\mathcal{B}$-homomorphisms "of degree zero") from $\tilde{T}_B$ to $\tilde{\Theta}_s \tilde{T}_A$ split, and this assumption leads precisely to the conjecture above. Indeed let us consider the Grothendieck group $[\tilde{\mathcal{B}}]$ of $\tilde{\mathcal{B}}$ and define the homomorphism $h : [\tilde{\mathcal{B}}] \longrightarrow \mathcal{N}$ by $h(\tilde{\nabla}_A\langle i\rangle) = v^i N_A$. By our formulas we have $h(\tilde{\Theta}_s \tilde{M}) = h(\tilde{M}) C_s$ for all $s \in \mathcal{S}$. Now assume we already know by induction that $h(\tilde{T}_A) = \underline{N}_A$. Certainly we have

$$\begin{array}{rcl} \dim \operatorname{Hom}_{\tilde{\mathcal{B}}}(\tilde{T}_B, \tilde{\Theta}_s \tilde{T}_A) & = & \sum_{i,C} (\tilde{T}_B : \tilde{\Delta}_C\langle i\rangle)(\tilde{\Theta}_s \tilde{T}_A : \tilde{\nabla}_C\langle i\rangle) \\ & = & (\tilde{\Theta}_s \tilde{T}_A : \tilde{\nabla}_B), \end{array}$$

since by induction $\tilde{T}_B = \tilde{d}\tilde{T}_B$, thus $(\tilde{T}_B : \tilde{\Delta}_C\langle i\rangle) = (\tilde{T}_B : \tilde{\nabla}_C\langle -i\rangle) = 0$ if $i > 0$ or $i = 0$, $B \neq C$. But we have $h(\tilde{\Theta}_s \tilde{T}_A) = \underline{N}_A C_s = \sum_B m_B N_B$

and by our definitions $(\tilde{\Theta}_s\tilde{T}_A : \tilde{\nabla}_B) = m_B(0)$, thus

$$\begin{aligned} h(\tilde{T}_{As}) &= h(\tilde{\Theta}_s\tilde{T}_A) - \sum_{B \prec As} \dim \mathrm{Hom}_{\tilde{\mathcal{B}}}(\tilde{T}_B, \tilde{\Theta}_s\tilde{T}_A) h(\tilde{T}_B) \\ &= \underline{N}_A C_s - \sum_{B \prec As} m_B(0)\underline{N}_B \\ &= \underline{N}_{As}, \end{aligned}$$

as I wished to explain.

I want to add that the formula $h(\tilde{T}_A) = \underline{N}_A$ also implies T_A is indecomposable. Indeed one may check as above that under our assumptions $E = \mathrm{End}_{\mathcal{B}} T_A$ admits a $\mathbb{Z}$-grading

$$E = \bigoplus_i \mathrm{Hom}_{\tilde{\mathcal{B}}}(\tilde{T}_A\langle i\rangle, \tilde{T}_A)$$

which starts in degree zero with $E_0 = k$ and has no components of negative degree. However a finite dimensional k-algebra which admits such a grading is necessarily local. Hence under our assumptions T_A is indecomposable.

8. The example B_2

In the sequel I want to show for B_2 the algorithm computing the $\underline{N}_A$. An element $\sum n_A N_A \in \mathcal{N}$ will be represented by a picture, where the Laurent polynomial n_A is written inside the alcove A. We put $\mathcal{S} = \{k, l, a\}$ with a for affine, l long and k for short, as in the picture

BILD FEHLT

We start our computation with a picture of $\underline{N}_{A^+}$. A picture where only one alcove A contains a 1 gives the corresponding $\underline{N}_A$. Right multiplication by C_s will be written $\xrightarrow{C_s}$. In the element $\underline{N}_A C_s$ thus obtained there could be additional ones, which have to be eliminated by subtraction of suitable $\underline{N}_B$ with $B \prec As$. This is symbolized by a dotted arrow $- - - \rightarrow$.

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The last picture represents $\mathrm{res}\,\mathrm{alt}\, E_\rho$ and thus illustrates 5.3 (1).

9. Notations

$(\mathcal{W}, \mathcal{S})$	a Coxeter group, in Section 3 and after the affine Weyl group
$\tilde{\mathcal{W}}$	the extended affine Weyl group $W \ltimes X$
$(\mathcal{W}_f, \mathcal{S}_f)$	a parabolic subgroup of $\mathcal{W}$
$\mathcal{W}^f$	the shortest representatives of the right cosets $\mathcal{W}_f\backslash\mathcal{W}$

$\mathcal{W}_\lambda$	the isotropy group of λ in $\mathcal{W}$
$W = \mathcal{W}_0$	the finite Weyl group
W_λ	the isotropy group of λ in W
ρ	the half sum of positive roots
$\mathcal{C}$	the dominant Weyl chamber
$\mathcal{A}$	the set of all alcoves
$\mathcal{A}^+$	the set of all alcoves in the dominant chamber $\mathcal{C}$
$\mathcal{A}^{++}$	the set of all alcoves in $\rho + \mathcal{C}$
A^+	the fundamental dominant alcove
$\mathcal{F}$	the set of all affine reflection hyperplanes
F^+, F^-	the open positive and negative halfspace in the complement of $F \in \mathcal{F}$
$\preceq$	Lusztig's partial order on the alcoves
$\beta \uparrow A$, $\beta \downarrow A$	defined in the proof of 4.5
X	the lattice of integral weights
$l(x)$	the length of the element x of a Coxeter group
w_0	the longest element of W
r	the length of w_0
Π	the fundamental box
Π_λ	the translated box $\lambda + \Pi$
$\lambda(A)$	the lower left corner of the box containing A
$\breve{A}$	is obtained by moving A with w_0 around $\lambda(A)$
$\hat{A}$	$A \mapsto \hat{A}$ is the inverse of $A \mapsto \breve{A}$
$\mathcal{H}$	the Hecke algebra
d	its standard involution $d : H \mapsto \overline{H}$
i, a	two involutive antiautomorphisms of $\mathcal{H}$, defined in the proof of Theorem 2.7
φ-linear	defined right above Theorem 3.8
$\mathcal{L}$	the ring of Laurent-polynomials $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$
C_s	self-dual generators $C_s = v(T_s + 1)$ of $\mathcal{H}$

Some Hecke modules with duality, standard basis,
self-dual basis and transition matrix

$(\mathcal{H}, H_x, \underline{H}_x, h_{x,y})$	the Hecke algebra itself, $H_x = v^{l(x)} T_x$
$(\mathcal{M}, M_x, \underline{M}_x, m_{x,y})$	Deodhar's parabolic analogs,
$(\mathcal{N}, N_x, \underline{N}_x, n_{x,y})$	for $x, y \in \mathcal{W}^f$
$(\mathcal{M}^*, M^x, \underline{M}^x, m^{x,y})$	The dual Hecke-modules,
$(\mathcal{N}^*, N^x, \underline{N}^x, n^{x,y})$	for $x, y \in \mathcal{W}^f$

From Section 4 we identify $\mathcal{W}^0 \cong \mathcal{A}^+$ and write M_A, $\underline{M}_A$, $m_{A,B} \dots$ for M_x, $\underline{M}_x$, $m_{x,y} \dots$

$(\mathcal{P}, A, \underline{P}_A, p_{A,B})$	Lusztig's periodic Hecke module
$\mathcal{P}^\circ$	submodule of $\mathcal{P}$ admitting a duality
$q_{A,B}$	the (renormalized) generic polynomials
$w * A$	a new action of $\tilde{\mathcal{W}}$ on $\mathcal{A}$,
	$w * (\lambda + B) = (w\lambda) + B$ for $\lambda \in X$, $B \subset \Pi$
$\langle w \rangle$	action of $w \in \tilde{\mathcal{W}}$ on $\mathcal{P}^\circ$, see 4.10

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