Corrections for W. Soergel: $\mathfrak{n}$-Cohomology of limits of discrete series [Soe97]

In this note I want to point out a flaw in the argumentation of said article and explain why it has no effect on the main result. Namely, preceding Theorem 3.2 an equivalence of categories $S = S_\psi : \mathcal{O}_\psi \xrightarrow{\sim} \mathcal{O}'_\psi$ is cited from [Soe89]. However, there is no such equivalence mapping every Verma to the same Verma. Rather the equivalence constructed there will map $M(x \cdot \lambda)$ to $M(x^{-1} \cdot \lambda)$ for λ the highest weight of a projective Verma alias the dominant weight of the corresponding Weyl group orbit. This means that Proposition 3.9 breaks down and instead of Theorem 3.2 we only get

$$\mathrm{H}^i(\mathfrak{n}, T_\psi^\chi N)_\lambda \cong \mathrm{Ext}^i_{U_\psi}(T_\chi^\psi(M(\tilde{\lambda})) \otimes_Z Z/\psi, N)$$

However I claim that instead of Proposition 4.1 we have, now with $\psi = Z^+$, an isomorphism

$$\Delta(T_\chi^\psi(M(\tilde{\lambda})) \otimes_Z Z/\psi) \cong j_! \mathcal{A}$$

and thus there is no effect of this error on the final result. For this we need to alter the proof of Proposition 4.1 and show in the notation used there rather that $(T_\chi^\psi(M(\tilde{\lambda})) \otimes_Z Z/\psi)$ is a projective cover of M_y in the subcategory $\langle M_x \mid x \in D \rangle$ of $\mathcal{O}'_\psi$. Well, for N in this subcategory we find

$$
\begin{aligned}
\mathrm{Hom}_\mathfrak{g}(T_\chi^\psi(M(\tilde{\lambda})) \otimes_Z Z/\psi, N) &= \mathrm{Hom}_\mathfrak{g}(T_\chi^\psi(M(\tilde{\lambda})), N) \\
&= \mathrm{Hom}_\mathfrak{g}(M(\tilde{\lambda}), T_\psi^\chi N)
\end{aligned}
$$

Since $T_\psi^\chi N$ has to belong to the subcategory $\langle M(\lambda) \rangle$ of $\mathcal{O}'_\chi$ generated in the same way as above, by definition of $M(\tilde{\lambda})$ this is exact as a functor of N. In addition it maps each M_x for $x \in D$ to a one-dimensional space and thus has to be the projective cover of M_y as claimed. Now it is clear how to rewrite the proof of Corollary 4.2, and from there on one can continue as written in the article.

Let me add a sample calculation. Take the real form $\mathrm{SU}(2,2) \subset \mathrm{SL}(4; \mathbb{C}) = G$ and take as a first Borel B_std the upper triangular matrices. Let r, s, t be the standard generators of the Weyl group, so that r, t generate the compact Weyl group. The $K_\mathbb{C}$-orbit Y of str in the flag variety X is closed. If we now take instead the Borel B fixing this point $str B_\mathrm{std}$, then as the B-orbits meeting Y we find those with parameters 1, tst, rsr and $strs$. So this is our set $A = A(Y)$. Let us concentrate on χ the most singular central character, so $D = \mathcal{W}$. Now from our B-orbits meeting Y, the first meets Y in codimension zero and the other three in codimension two. So if $\mathfrak{n}$ is the nilradical of $\mathrm{Lie}\, B$ and we compute the $\mathfrak{n}$-cohomology of the most degenerate limit of principal series $\mathrm{H}^n(T^\chi; i_* \underline{Y})$ for χ the most singular central character, we find nonvanishing complexes $\mathcal{V}(A, D, c)$ only for $c = 0$ and $c = 2$. For $c = 0$ the complex has only one entry $\mathbb{C}$ sitting in degree zero and gives a one-dimensional contribution to H^4. For $c = 2$ the complex has a one-dimensional entry in degree four, coming from to $strs$, and

a two-dimensional entry in degree three, coming from to rsr and tst, and the differential is not zero. This gives a one-dimensional contribution to H^5, and we find no more cohomology than that.

References

[Soe89] Wolfgang Soergel, $\mathfrak{n}$-*cohomology of simple highest weight modules on walls and purity*, Inventiones **98** (1989), 565–580.

[Soe97] _____ , *On the $\mathfrak{n}$-cohomology of limits of discrete series representations*, Representation Theory (An electronic Journal of the AMS) **1** (1997), 69–82.