The Lusztig Conjecture

Wolfgang Soergel

Mathematisches Insitut
Universität Freiburg

28. März 2013

The Lusztig Conjecture
on
on irreducible characters
of algebraic groups

- Dimensions of irreducible representations of a given connected affine algebraic group?
- Dimensions of its weight spaces for a maximal torus?

- Dimensions of irreducible representations of a given connected affine algebraic group?
- Dimensions of its weight spaces for a maximal torus?
- Characteristic Zero: Weyl character formula

- Dimensions of irreducible representations of a given connected affine algebraic group?
- Dimensions of its weight spaces for a maximal torus?
- Characteristic Zero: Weyl character formula
- Characteristic positive: Steinberg tensor product formula and Lusztig conjecture

- Dimensions of irreducible representations of a given connected affine algebraic group?
- Dimensions of its weight spaces for a maximal torus?
- Characteristic Zero: Weyl character formula
- Characteristic positive: Steinberg tensor product formula and Lusztig conjecture

$\mathrm{char} k = 0$:

$$\left\{ \begin{array}{c} \text{irreducible representations} \\ \text{of } \mathrm{SL}(2;k) \end{array} \right\} \quad \xleftrightarrow{\sim} \quad \mathbb{N}$$

$$\begin{array}{ccc} L & \mapsto & (\dim L) - 1 \\ k[X,Y]^{(n)} & \leftarrow\!\shortmid & n \end{array}$$

In case $\mathrm{char}\, k = p > 0$ the $k[X,Y]^{(n)}$ are rarely irreducible, for example $kX^p + kY^p \subsetneq k[X,Y]^{(p)}$ is a subrepresentation.

$\mathrm{char} k = 0$:

$$\left\{ \begin{array}{c} \text{irreducible representations} \\ \text{of } \mathsf{SL}(2; k) \end{array} \right\} \begin{array}{c} \tilde{\leftrightarrow} \\ \\ \mapsto \\ \leftarrow\!\shortmid \end{array} \begin{array}{c} \mathbb{N} \\ \\ (\dim L) - 1 \\ n \end{array}$$

$$\begin{array}{ccc} L & \mapsto & (\dim L) - 1 \\ k[X, Y]^{(n)} & \leftarrow\!\shortmid & n \end{array}$$

In case $\mathsf{char}\, k = p > 0$ the $k[X, Y]^{(n)}$ are rarely irreducible, for example $kX^p + kY^p \subsetneq k[X, Y]^{(p)}$ is a subrepresentation.

Arbitrary characteristic:

$$\left\{ \begin{matrix} \text{irreducible representations} \\ \text{of } \mathrm{SL}(2; k) \end{matrix} \right\} \quad \tilde{\leftrightarrow} \quad \mathbb{N}$$

$$\mathrm{soc}\, k[X, Y]^{(n)} \quad \leftarrow\!\shortmid \quad n$$

But what are the dimensions of those socles? And what happens for more general groups?

Arbitrary characteristic:

$$\left\{ \begin{array}{c} \text{irreducible representations} \\ \text{of } \mathrm{SL}(2;k) \end{array} \right\} \quad \tilde{\leftrightarrow} \quad \mathbb{N}$$

$$\textcolor{red}{\mathrm{soc}}\; k[X,Y]^{(n)} \quad \leftarrow\!\shortmid \quad n$$

But what are the dimensions of those socles? And what happens for more general groups?

For affine algebraic groups $G \supset B$ the restriction admits a right adjoint, induction

$$G\text{-Mod} \underset{\text{ind}}{\overset{\text{res}}{\rightleftarrows}} B\text{-Mod}$$

$$\begin{aligned} \text{ind}_B^G V &= \{f : G \to V \mid f \text{ algebraic } B\text{-equivariant}\} \\ &= \{ \text{ algebraic sections in } G \times_B V \twoheadrightarrow G/B\} \end{aligned}$$

From now on:

- $k = \bar{k}$ algebraically closed field

From now on:

- $k = \bar{k}$ algebraically closed field
- $G \supset B$ connected affine algebraic group over k with a Borel, for example $G = \mathrm{GL}(r; k) \supset B$ upper triangular matrices

From now on:

- $k = \bar{k}$ algebraically closed field
- $G \supset B$ connected affine algebraic group over k with a Borel, for example $G = \mathrm{GL}(r; k) \supset B$ upper triangular matrices
- $\mathfrak{X} = \mathfrak{X}(B) := \{\lambda : B \to k^\times \mid \lambda \text{ homomorphism}\}$ the weight lattice

From now on:

- $k = \bar{k}$ algebraically closed field
- $G \supset B$ connected affine algebraic group over k with a Borel, for example $G = \mathrm{GL}(r; k) \supset B$ upper triangular matrices
- $\mathfrak{X} = \mathfrak{X}(B) := \{\lambda : B \to k^\times \mid \lambda \text{ homomorphism}\}$ the weight lattice
- $\nabla(\lambda) := \mathrm{ind}_B^G k_\lambda$ induced representation of $\lambda \in \mathfrak{X}$

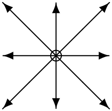
From now on:

- $k = \bar{k}$ algebraically closed field
- $G \supset B$ connected affine algebraic group over k with a Borel, for example $G = \mathrm{GL}(r; k) \supset B$ upper triangular matrices
- $\mathfrak{X} = \mathfrak{X}(B) := \{\lambda : B \to k^{\times} \mid \lambda \text{ homomorphism}\}$ the weight lattice
- $\nabla(\lambda) := \mathrm{ind}_B^G k_\lambda$ induced representation of $\lambda \in \mathfrak{X}$
- $\mathfrak{X}^+ := \{\lambda \in \mathfrak{X} \mid \nabla(\lambda) \neq 0\}$ dominant weights

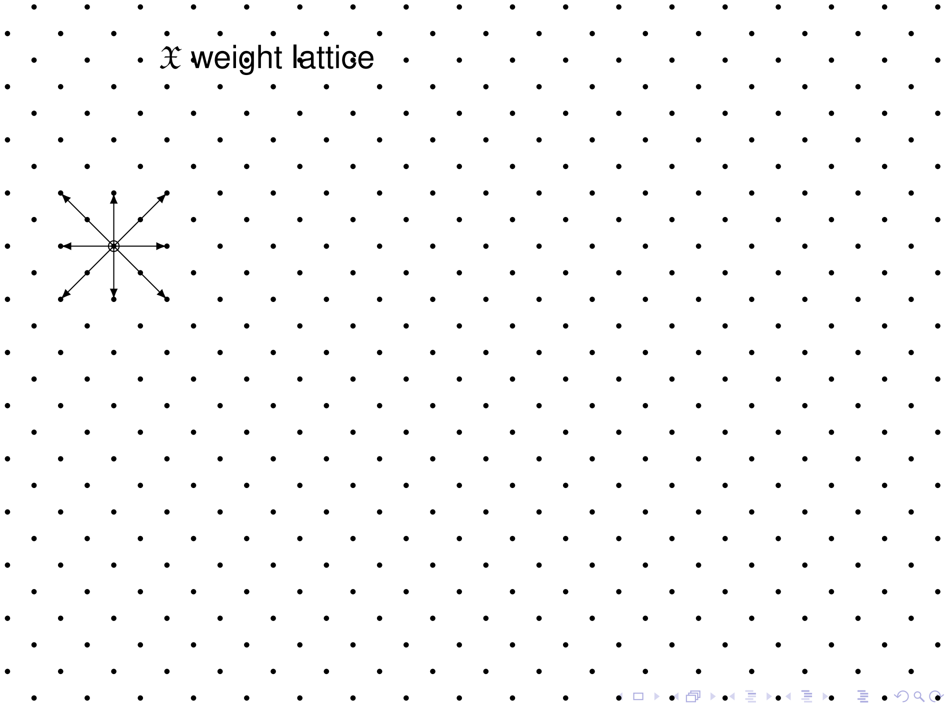
From now on:

- $k = \bar{k}$ algebraically closed field
- $G \supset B$ connected affine algebraic group over k with a Borel, for example $G = \mathrm{GL}(r; k) \supset B$ upper triangular matrices
- $\mathfrak{X} = \mathfrak{X}(B) := \{\lambda : B \to k^\times \mid \lambda \text{ homomorphism}\}$ the weight lattice
- $\nabla(\lambda) := \mathrm{ind}_B^G k_\lambda$ induced representation of $\lambda \in \mathfrak{X}$
- $\mathfrak{X}^+ := \{\lambda \in \mathfrak{X} \mid \nabla(\lambda) \neq 0\}$ dominant weights

Example $G = \mathrm{Sp}(4; k)$



$\mathfrak{X}$ weight lattice



$\mathfrak{X}^+$ dominant weights

$$\mathfrak{X}^+ \xrightarrow{\sim} \{\text{irreducible representations of } G\}$$
$$\lambda \mapsto L(\lambda) := \operatorname{soc} \nabla(\lambda)$$

simple module with highest weight λ

$$\begin{array}{ccl} \mathfrak{X}^+ & \xrightarrow{\sim} & \{\text{irreducible representations of } G\} \\ \lambda & \mapsto & L(\lambda) := \operatorname{soc} \nabla(\lambda) \end{array}$$

simple module with highest weight λ

$\nabla(\lambda)$ described by the Weyl character formula

$\mathfrak{X}^+$ dominant weights

$$\begin{aligned} \mathfrak{X}^+ &\xrightarrow{\sim} \{\text{irreducible representations of } G\} \\ \lambda &\mapsto \quad L(\lambda) := \mathrm{soc}\, \nabla(\lambda) \end{aligned}$$

simple module with highest weight λ

$\nabla(\lambda)$ described by the Weyl character formula

For $\mathrm{char}\, k = 0$ we have $L(\lambda) = \nabla(\lambda)$

$\mathfrak{X}^+$ dominant weights

$$\begin{array}{ccl} \mathfrak{X}^+ & \xrightarrow{\sim} & \{\text{irreducible representations of } G\} \\ \lambda & \mapsto & L(\lambda) := \operatorname{soc} \nabla(\lambda) \\ & & \quad \text{simple module with highest weight } \lambda \end{array}$$

$\nabla(\lambda)$ described by the Weyl character formula

For $\operatorname{char} k = 0$ we have $L(\lambda) = \nabla(\lambda)$

For $G = \mathrm{SL}(2; k)$:

- $B = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$ ist Borel
- $\mathfrak{X} = \mathbb{Z}\varepsilon$ with $\varepsilon : B \to k^\times$ given by $\begin{pmatrix} t & 0 \\ * & * \end{pmatrix} \mapsto t$
- $\nabla(n\varepsilon) = k[X, Y]^{(n)}$

For $G = \mathrm{SL}(2;k)$:

- $B = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$ ist Borel
- $\mathfrak{X} = \mathbb{Z}\varepsilon$ with $\varepsilon : B \to k^\times$ given by $\begin{pmatrix} t & 0 \\ * & * \end{pmatrix} \mapsto t$
- $\nabla(n\varepsilon) = k[X, Y]^{(n)}$

In case $\operatorname{char} k = p > 0$:

- $L(n\varepsilon) = \nabla(n\varepsilon)$ if $n < p$

For $G = \mathrm{SL}(2; k)$:

- $B = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$ ist Borel
- $\mathfrak{X} = \mathbb{Z}\varepsilon$ with $\varepsilon : B \to k^\times$ given by $\begin{pmatrix} t & 0 \\ * & * \end{pmatrix} \mapsto t$
- $\nabla(n\varepsilon) = k[X, Y]^{(n)}$

In case $\operatorname{char} k = p > 0$:

- $L(n\varepsilon) = \nabla(n\varepsilon)$ if $n < p$
- $L(p\varepsilon) \subsetneq \nabla(p\varepsilon)$ alias $kX^p + kY^p \subsetneq k[X, Y]^{(p)}$

For $G = \mathrm{SL}(2; k)$:

- $B = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$ ist Borel
- $\mathfrak{X} = \mathbb{Z}\varepsilon$ with $\varepsilon : B \to k^\times$ given by $\begin{pmatrix} t & 0 \\ * & * \end{pmatrix} \mapsto t$
- $\nabla(n\varepsilon) = k[X, Y]^{(n)}$

In case $\operatorname{char} k = p > 0$:

- $L(n\varepsilon) = \nabla(n\varepsilon)$ if $n < p$
- $L(p\varepsilon) \subsetneq \nabla(p\varepsilon)$ alias $kX^p + kY^p \subsetneq k[X, Y]^{(p)}$
- $L(p\varepsilon) = L(\varepsilon)^{[1]}$ Frobenius-Twist of $L(\varepsilon)$

For $G = \mathrm{SL}(2; k)$:

- $B = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$ ist Borel
- $\mathfrak{X} = \mathbb{Z}\varepsilon$ with $\varepsilon : B \to k^\times$ given by $\begin{pmatrix} t & 0 \\ * & * \end{pmatrix} \mapsto t$
- $\nabla(n\varepsilon) = k[X, Y]^{(n)}$

In case $\operatorname{char} k = p > 0$:

- $L(n\varepsilon) = \nabla(n\varepsilon)$ if $n < p$
- $L(p\varepsilon) \subsetneq \nabla(p\varepsilon)$ alias $kX^p + kY^p \subsetneq k[X, Y]^{(p)}$
- $L(p\varepsilon) = L(\varepsilon)^{[1]}$ Frobenius-Twist of $L(\varepsilon)$
- description of all irreduzible characters by Steinberg tensor product theorem

For $G = \mathrm{SL}(2; k)$:

- $B = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}$ ist Borel
- $\mathfrak{X} = \mathbb{Z}\varepsilon$ with $\varepsilon : B \to k^\times$ given by $\begin{pmatrix} t & 0 \\ * & * \end{pmatrix} \mapsto t$
- $\nabla(n\varepsilon) = k[X, Y]^{(n)}$

In case $\mathrm{char}\, k = p > 0$:

- $L(n\varepsilon) = \nabla(n\varepsilon)$ if $n < p$
- $L(p\varepsilon) \subsetneq \nabla(p\varepsilon)$ alias $kX^p + kY^p \subsetneq k[X, Y]^{(p)}$
- $L(p\varepsilon) = L(\varepsilon)^{[1]}$ Frobenius-Twist of $L(\varepsilon)$
- description of all irreduzible characters by Steinberg tensor product theorem

Steinberg tensor product theorem:

$G \supset B$ and $\mathfrak{X} \supset \mathfrak{X}^+$ general again.
For $\lambda \in \mathfrak{X}^+$ consider the p-adic expansion

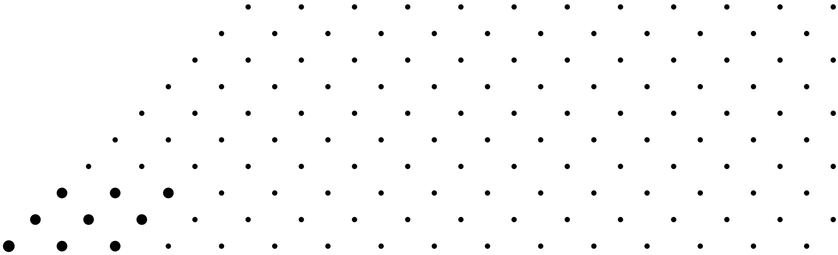
$$\lambda = p^d \lambda_d + \ldots + p^2 \lambda_2 + p \lambda_1 + \lambda_0$$

with digits λ_i in the fundamental box, given by
Box$:= \{\mu \in \mathfrak{X}^+ \mid \langle \mu, \alpha^\vee \rangle < p$ for all simple roots $\alpha\}$

Then we have

$$L(\lambda) \cong L(\lambda_d)^{[d]} \otimes \ldots \otimes L(\lambda_2)^{[2]} \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_0)$$

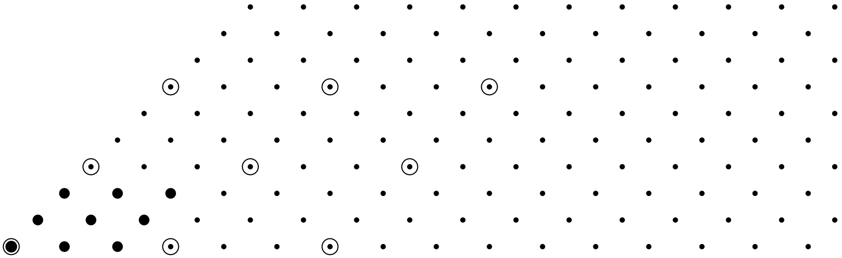
Here $L^{[i]}$ is the twist of L by the i-th power of the Frobenius automorphism of $\mathrm{GL}(L)$.



The 9 elements of box in case $p = 3$ and $G = \mathrm{Sp}(4; k)$

$$L(\lambda) \cong L(\lambda_d)^{[d]} \otimes \ldots \otimes L(\lambda_2)^{[2]} \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_0)$$

But what are the characters, even the dimensions of the $L(\lambda)$ for $\lambda \in \mathrm{Box}$? Lusztig conjecture from $p = 5$ on.

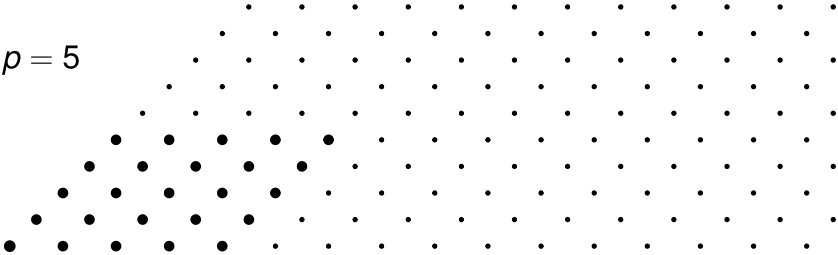


The 9 elements of box in case $p = 3$ and $G = \mathrm{Sp}(4; k)$ along with the $p\lambda_1$ for $\lambda_1 \in \mathrm{Box}$

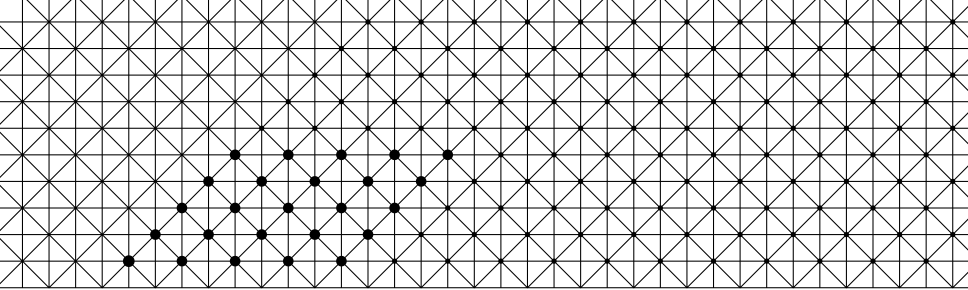
$$L(\lambda) \cong L(\lambda_d)^{[d]} \otimes \ldots \otimes L(\lambda_2)^{[2]} \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_0)$$

But what are the characters, even the dimensions of the $L(\lambda)$ for $\lambda \in \mathrm{Box}$? Lusztig conjecture from $p = 5$ on.

$p = 5$

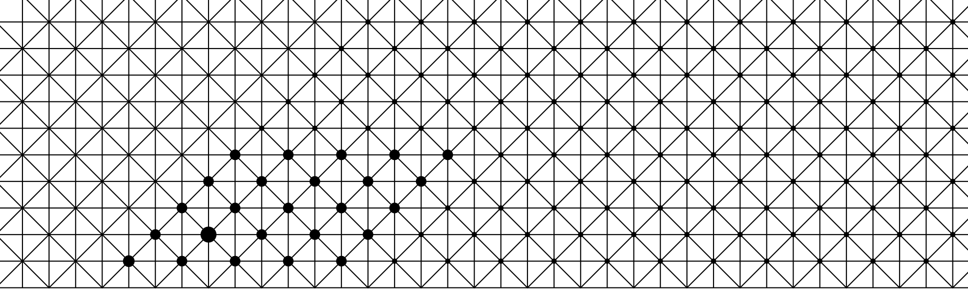


Consider affine Weyl group $\mathcal{W} = W \ltimes \langle R \rangle$

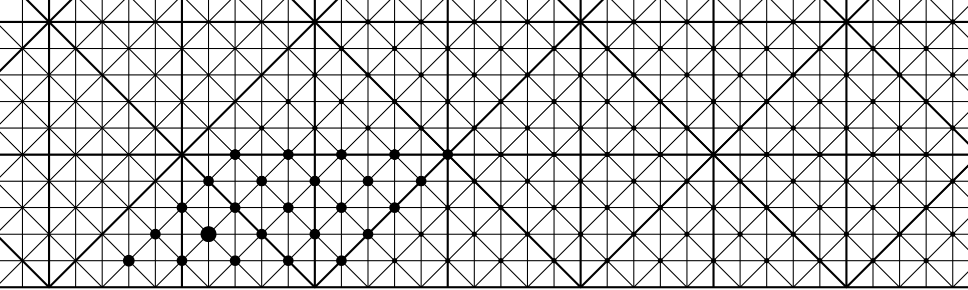


Consider affine Weyl group $\mathcal{W} = W \ltimes \langle R \rangle$

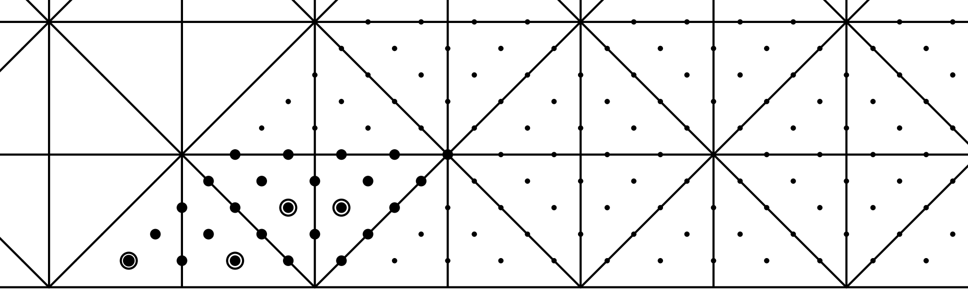
ρ half the sum of positive roots



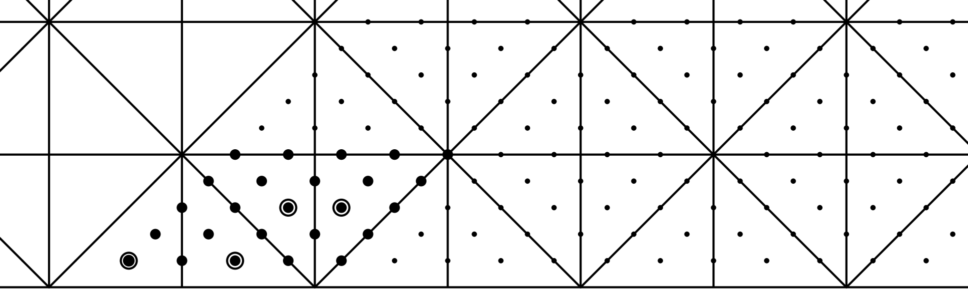
Consider affine Weyl group $\mathcal{W} = W \ltimes \langle R \rangle$
ρ half the sum of positive roots
New $\mathcal{W}$-action $x \cdot_p \lambda := pxp^{-1}(\lambda + \rho) - \rho$



Consider affine Weyl group $\mathcal{W} = W \ltimes \langle R \rangle$
ρ half the sum of positive roots
New $\mathcal{W}$-action $x \cdot_p \lambda := pxp^{-1}(\lambda + \rho) - \rho$
The weights $x \cdot_p 0$ from the box



Consider affine Weyl group $\mathcal{W} = W \ltimes \langle R \rangle$
ρ half the sum of positive roots
New $\mathcal{W}$-action $x \cdot_p \lambda := pxp^{-1}(\lambda + \rho) - \rho$
The weights $x \cdot_p 0$ from the box

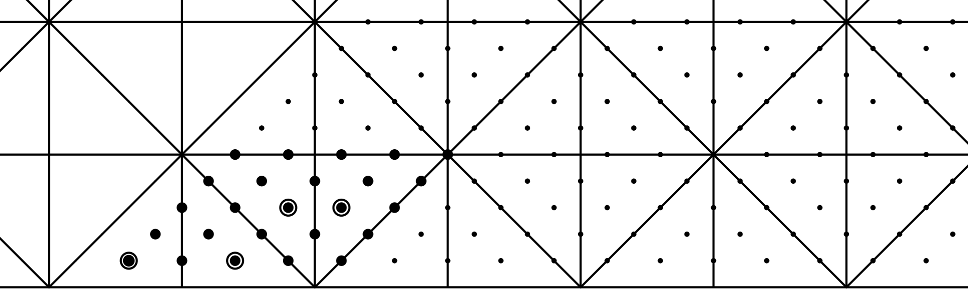


Lusztig conjecture

For $x \in \mathcal{W}$ with $x \cdot_p 0 \in \mathrm{Box}$ and p so big, that $z \cdot_p 0 = 0 \Rightarrow z = 1$ we should have:

$$[L(x \cdot_p 0)] = \sum_y (-1)^{l(x)+l(y)} P_{w_\circ y, w_\circ x}(1)\ [\nabla(y \cdot_p 0)]$$

Translation principle: These $[L(x \cdot_p 0)]$ give all $[L(\lambda)]$ for $\lambda \in \mathrm{Box}$



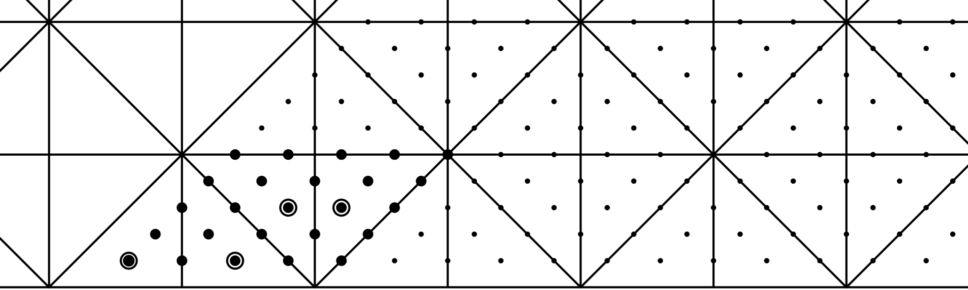
Lusztig conjecture

For $x \in \mathcal{W}$ with $x \cdot_p 0 \in \mathrm{Box}$ and p so big, that $z \cdot_p 0 = 0 \Rightarrow z = 1$ we should have:

$[L(x \cdot_p 0)] = \sum_y (-1)^{l(x)+l(y)} P_{w_\circ y, w_\circ x}(1)\ [\nabla(y \cdot_p 0)]$

Translation principle: These $[L(x \cdot_p 0)]$ give all $[L(\lambda)]$ for $\lambda \in \mathrm{Box}$

But what are the Kazhdan-Lusztig polynomials $P_{w_\circ y, w_\circ x}$?

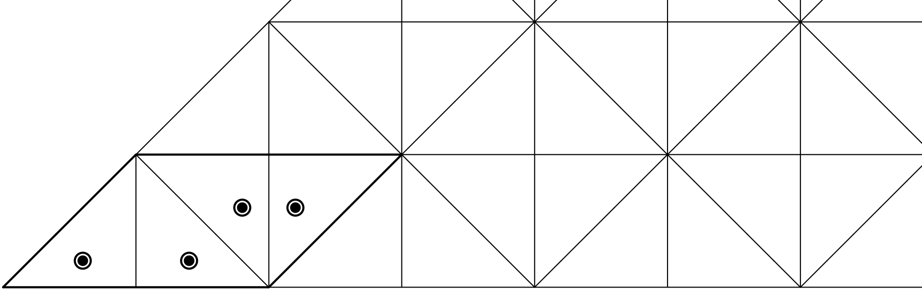


Lusztig conjecture

For $x \in \mathcal{W}$ with $x \cdot_p 0 \in \mathrm{Box}$ and p so big,
that $z \cdot_p 0 = 0 \Rightarrow z = 1$ we should have:

$$[L(x \cdot_p 0)] = \sum_y (-1)^{l(x)+l(y)} P_{w_\circ y, w_\circ x}(1)\ [\nabla(y \cdot_p 0)]$$

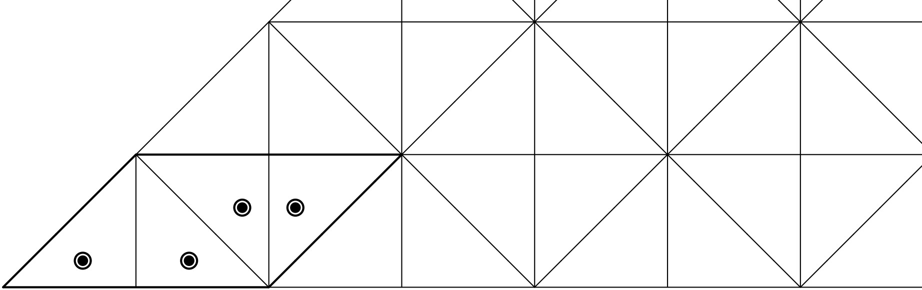
Translation principle: These $[L(x \cdot_p 0)]$ give all $[L(\lambda)]$ for $\lambda \in \mathrm{Box}$
But what are the Kazhdan-Lusztig polynomials $P_{w_\circ y, w_\circ x}$?



For A, B the alcoves of $x \cdot_p 0,\ y \cdot_p 0$ put
$L_A := L(x \cdot_p 0)$, $\nabla_B := \nabla(y \cdot_p 0)$ and $m_{B,A} := P_{w_\circ y, w_\circ x}$

Lusztig conjecture: For A in the fundamental box should have

$$[L_A] = \sum_B (-1)^{d(A,B)} m_{B,A}(1)\ [\nabla_B]$$

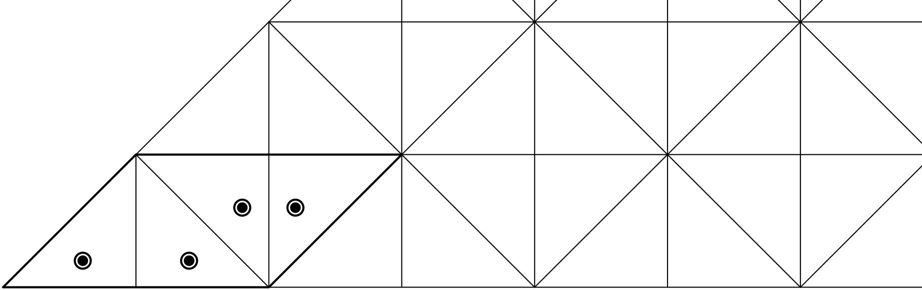


For A, B the alcoves of $x \cdot_p 0$, $y \cdot_p 0$ put
$L_A := L(x \cdot_p 0)$, $\nabla_B := \nabla(y \cdot_p 0)$ and $m_{B,A} := P_{w_\circ y, w_\circ x}$

Lusztig conjecture: For A in the fundamental box should have

$$[L_A] = \sum_B (-1)^{d(A,B)} m_{B,A}(1)\ [\nabla_B]$$

$d(A, B)$ number of reflecting hyperplanes separating A from B

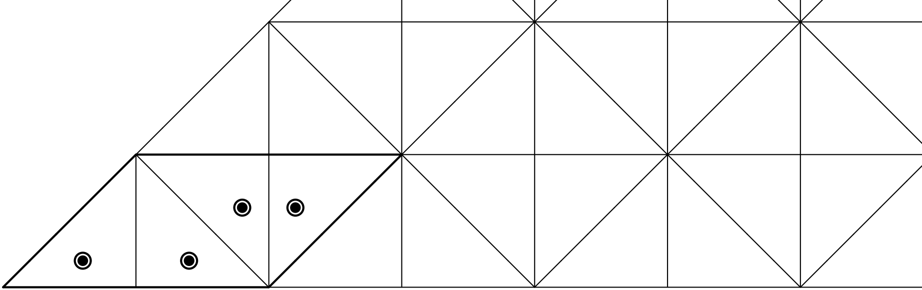


For A, B the alcoves of $x \cdot_p 0$, $y \cdot_p 0$ put
$L_A := L(x \cdot_p 0)$, $\nabla_B := \nabla(y \cdot_p 0)$ and $m_{B,A} := P_{w_\circ y, w_\circ x}$

Lusztig conjecture: For A in the fundamental box should have

$$[L_A] = \sum_B (-1)^{d(A,B)} m_{B,A}(1)\ [\nabla_B]$$

$d(A, B)$ number of reflecting hyperplanes separating A from B
But what are the Kazhdan-Lusztig polynomials $m_{B,A} \in \mathbb{Z}[v, v^{-1}]$?

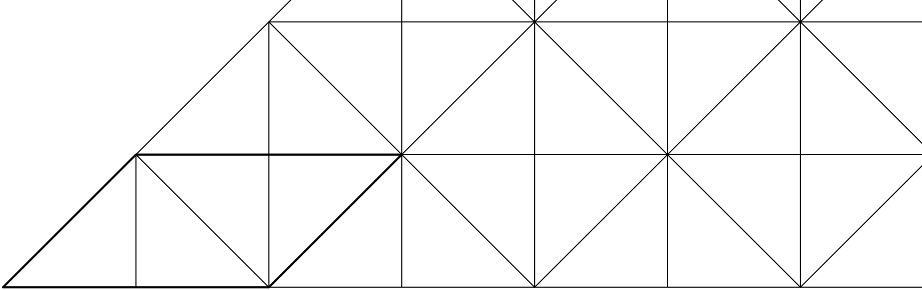


For A, B the alcoves of $x \cdot_p 0$, $y \cdot_p 0$ put
$L_A := L(x \cdot_p 0)$, $\nabla_B := \nabla(y \cdot_p 0)$ and $m_{B,A} := P_{w_\circ y, w_\circ x}$

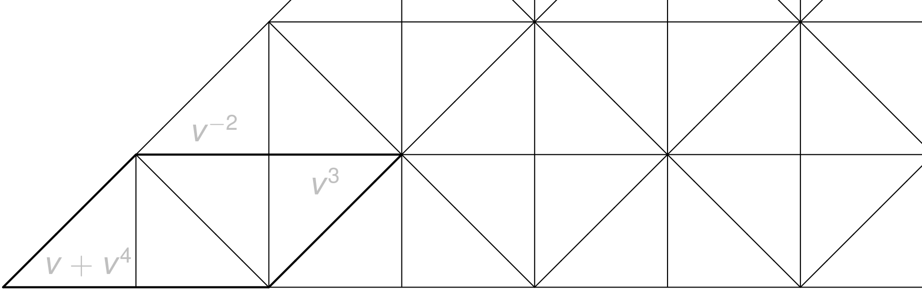
Lusztig conjecture: For A in the fundamental box should have

$$[L_A] = \sum_B (-1)^{d(A,B)} m_{B,A}(1)\ [\nabla_B]$$

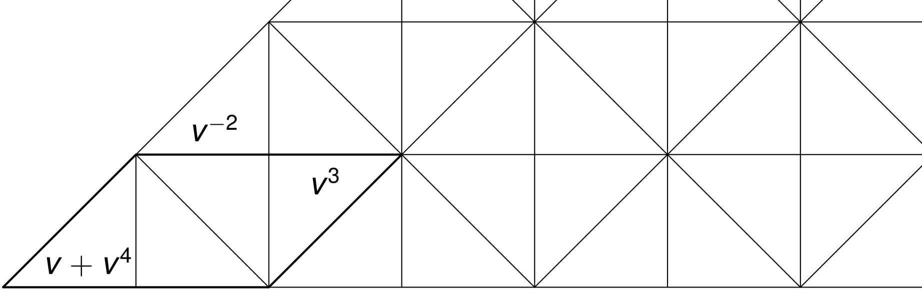
$d(A, B)$ number of reflecting hyperplanes separating A from B
But what are the Kazhdan-Lusztig polynomials $m_{B,A} \in \mathbb{Z}[v, v^{-1}]$?



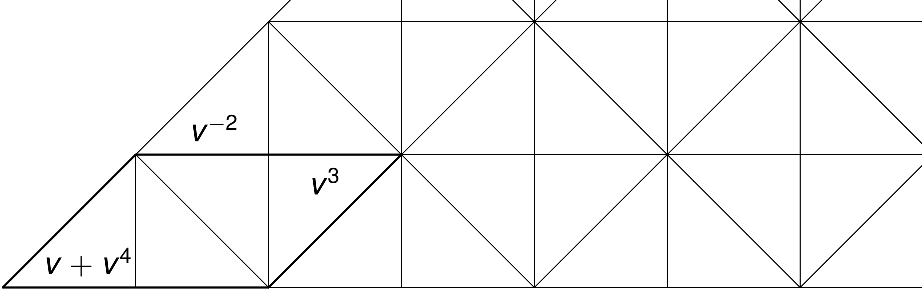
- Consider the free module $\mathcal{M} := \mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ over the set $\mathcal{A}^+$ of all alcoves in the dominant chamber



- Consider the free module $\mathcal{M} := \mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ over the set $\mathcal{A}^+$ of all alcoves in the dominant chamber
- Notation: Write coefficients in their alcoves

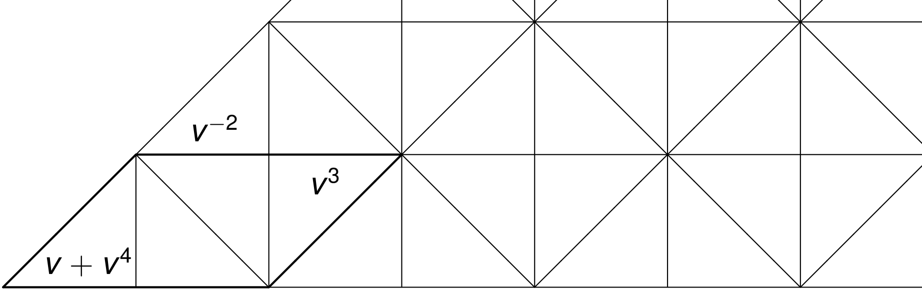


- Consider the free module $\mathcal{M} := \mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ over the set $\mathcal{A}^+$ of all alcoves in the dominant chamber
- Notation: Write coefficients in their alcoves
- Elements of $\mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ called „Patterns“



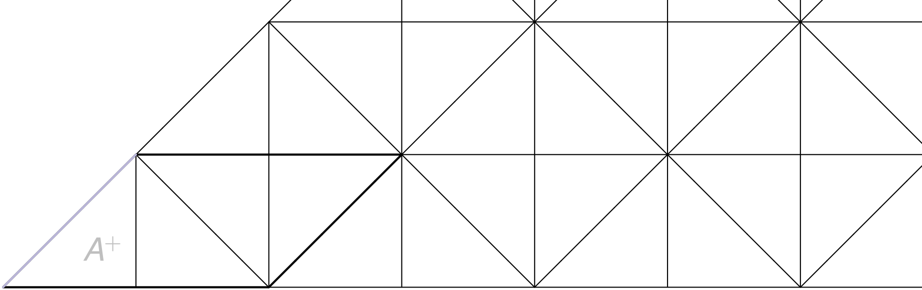
- Consider the free module $\mathcal{M} := \mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ over the set $\mathcal{A}^+$ of all alcoves in the dominant chamber
- Notation: Write coefficients in their alcoves
- Elements of $\mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ called „Patterns"
- Define distinguished patterns

$$\underline{M}_A = \sum_B m_{B,A} B \in A + v\mathbb{Z}[v]\mathcal{A}^+$$

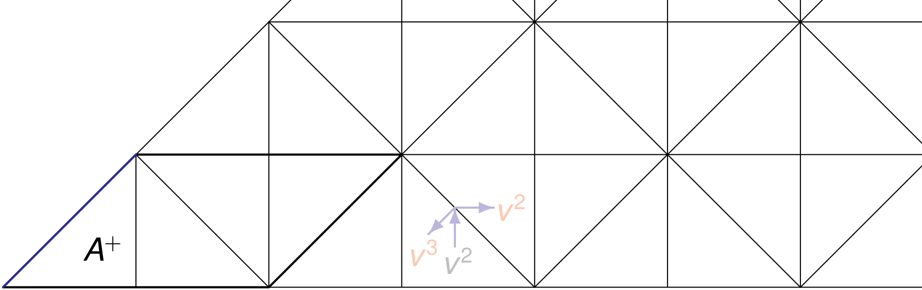


- Consider the free module $\mathcal{M} := \mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ over the set $\mathcal{A}^+$ of all alcoves in the dominant chamber
- Notation: Write coefficients in their alcoves
- Elements of $\mathbb{Z}[v, v^{-1}]\mathcal{A}^+$ called „Patterns“
- Define distinguished patterns

$$\underline{M}_A = \sum_B m_{B,A} B \in A + v\mathbb{Z}[v]\mathcal{A}^+$$

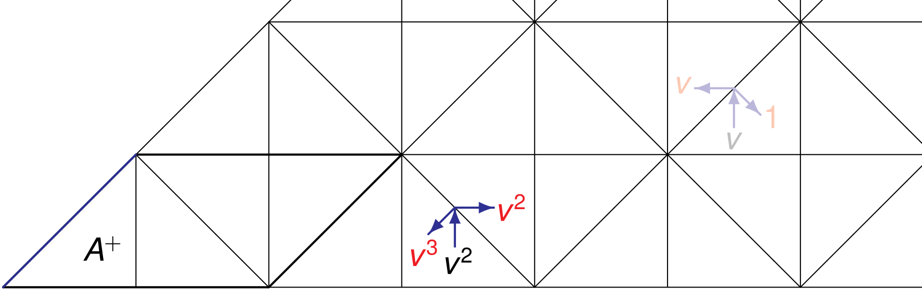


- Given s a wall of A^+ define $[s] : \mathcal{M} \to \mathcal{M}$ by



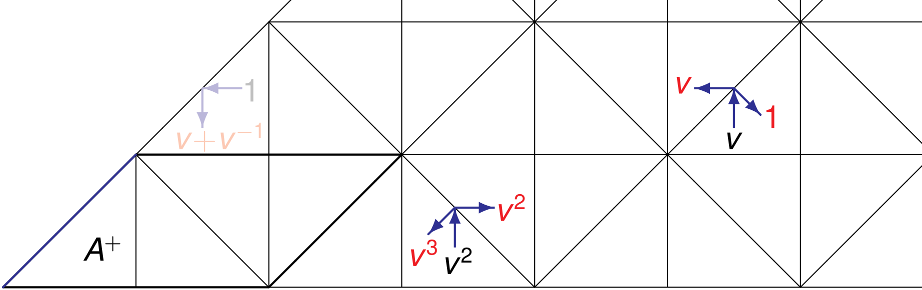
- Given s a wall of A^+ define $[s] : \mathcal{M} \to \mathcal{M}$ by

  $[s] : A \mapsto As + vA$ in case $As > A$ and $As \in \mathcal{A}^+$;



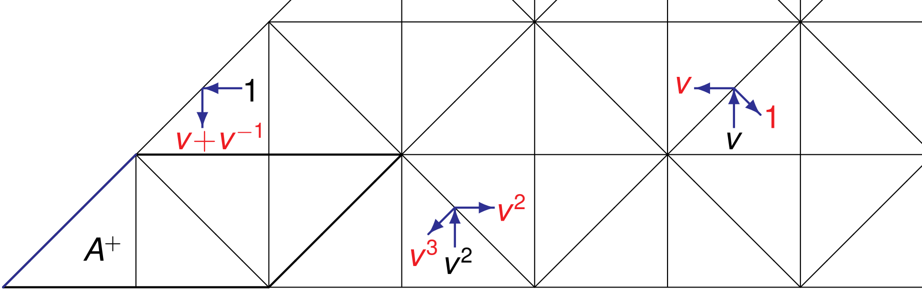
- Given s a wall of A^+ define $[s] : \mathcal{M} \to \mathcal{M}$ by

$$[s] : A \mapsto As + vA \qquad \text{in case } As > A \text{ and } As \in \mathcal{A}^+;$$
$$[s] : A \mapsto As + v^{-1}A \qquad \text{in case } As < A \text{ and } As \in \mathcal{A}^+;$$



- Given s a wall of A^+ define $[s] : \mathcal{M} \to \mathcal{M}$ by

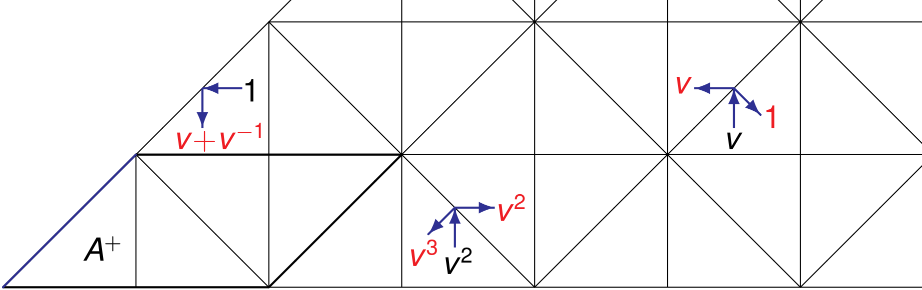
$$
\begin{array}{ll}
[s] : A \mapsto As + vA & \text{in case } As > A \text{ and } As \in \mathcal{A}^+; \\
[s] : A \mapsto As + v^{-1}A & \text{in case } As < A \text{ and } As \in \mathcal{A}^+; \\
[s] : A \mapsto (v + v^{-1})A & \text{in case } As \notin \mathcal{A}^+;
\end{array}
$$



- Given s a wall of A^+ define $[s] : \mathcal{M} \to \mathcal{M}$ by

$$
\begin{aligned}
[s] &: A \mapsto As + vA && \text{in case } As > A \text{ and } As \in \mathcal{A}^+;\\
[s] &: A \mapsto As + v^{-1}A && \text{in case } As < A \text{ and } As \in \mathcal{A}^+;\\
[s] &: A \mapsto (v + v^{-1})A && \text{in case } As \notin \mathcal{A}^+;
\end{aligned}
$$

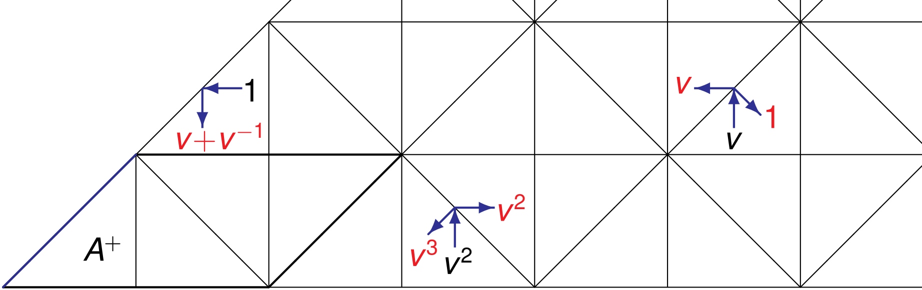
- Let $\mathcal{M}^{sd} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$



- Given s a wall of A^+ define $[s] : \mathcal{M} \to \mathcal{M}$ by

$$[s] : A \mapsto As + vA \quad \text{in case } As > A \text{ and } As \in \mathcal{A}^+;$$
$$[s] : A \mapsto As + v^{-1}A \quad \text{in case } As < A \text{ and } As \in \mathcal{A}^+;$$
$$[s] : A \mapsto (v + v^{-1})A \quad \text{in case } As \notin \mathcal{A}^+;$$

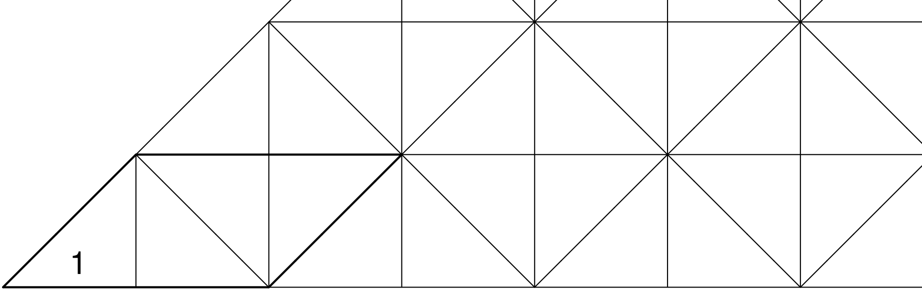
- Let $\mathcal{M}^{sd} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{sd}$ uniquely determined for $A \in \mathcal{A}^+$



- Given s a wall of A^+ define $[s] : \mathcal{M} \to \mathcal{M}$ by

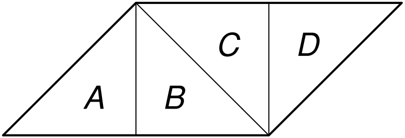
$$
\begin{aligned}
[s] : A &\mapsto As + vA && \text{in case } As > A \text{ and } As \in \mathcal{A}^+;\\
[s] : A &\mapsto As + v^{-1}A && \text{in case } As < A \text{ and } As \in \mathcal{A}^+;\\
[s] : A &\mapsto (v + v^{-1})A && \text{in case } As \notin \mathcal{A}^+;
\end{aligned}
$$

- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$

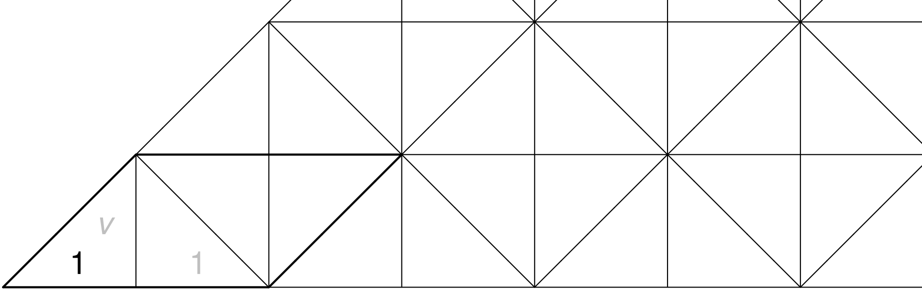


$\underline{M}_{A^+} = A^+$ alias $\underline{M}_A = A$

$[L_A] = [\nabla_A]$ the trivial representation

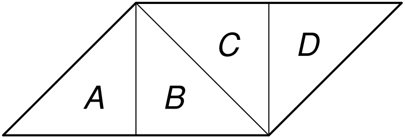


- Let $\mathcal{M}^{\text{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\text{sd}}$ uniquely determined for $A \in \mathcal{A}^+$

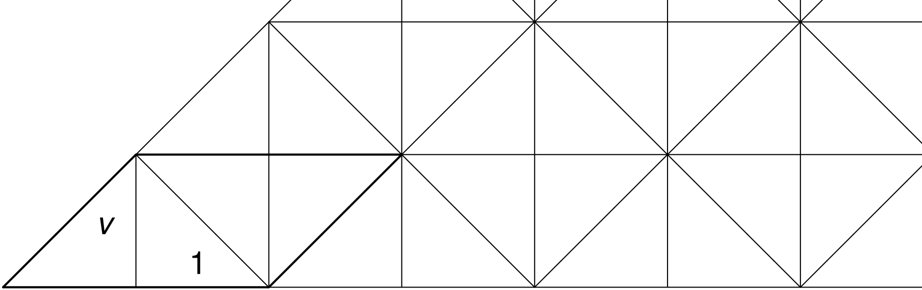


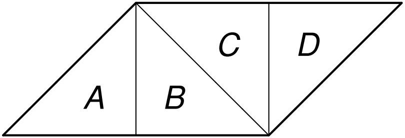
$\underline{M}_{A^+} = A^+$ alias $\underline{M}_A = A$

$[L_A] = [\nabla_A]$ the trivial representation

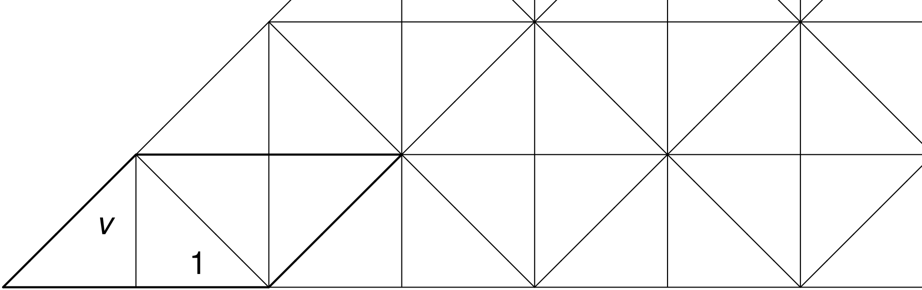


- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



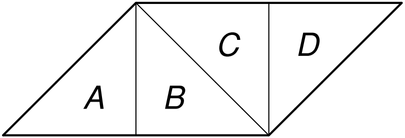


- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$

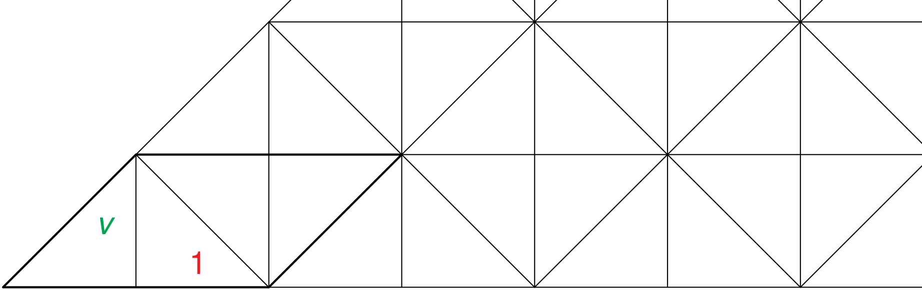


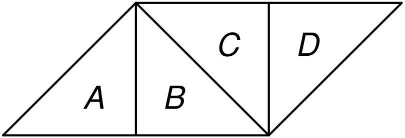
$\underline{M}_B = B + vA$

$[L_B] = [\nabla_B] - [\nabla_A]$

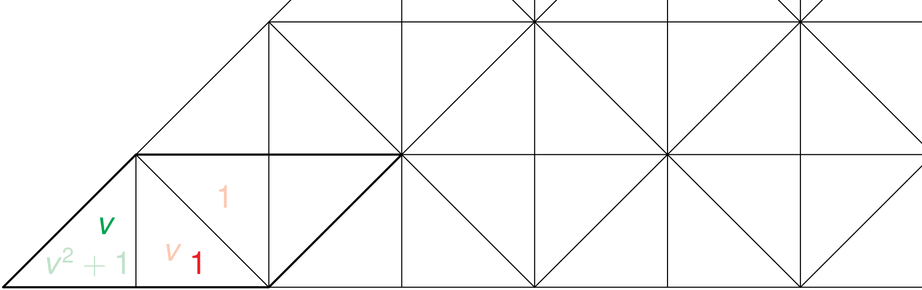


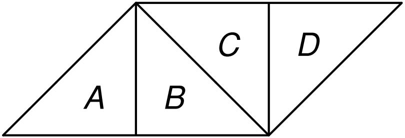
- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



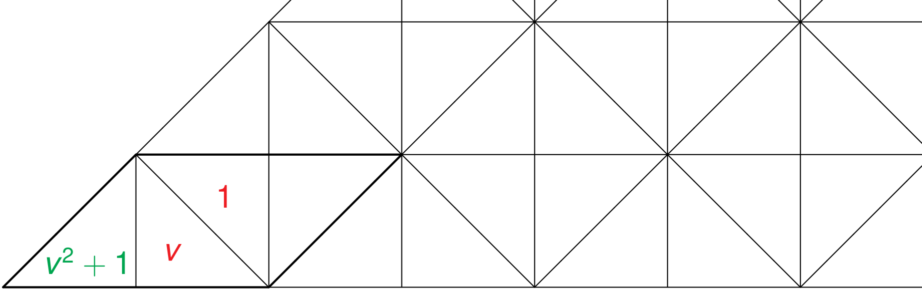


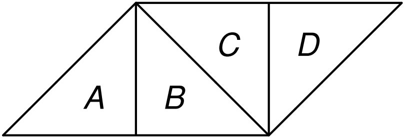
- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



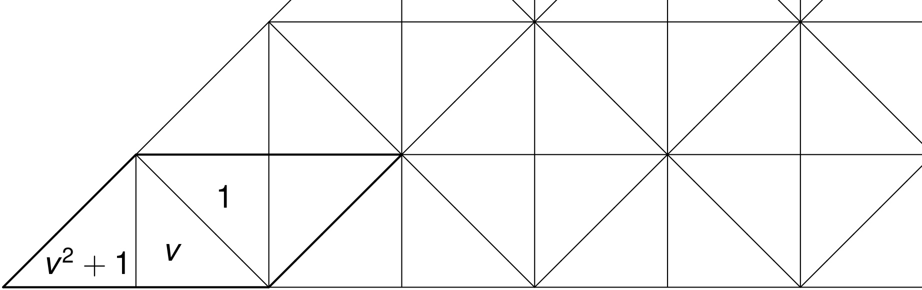


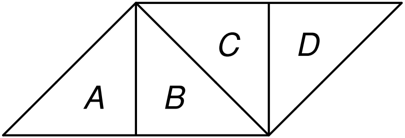
- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



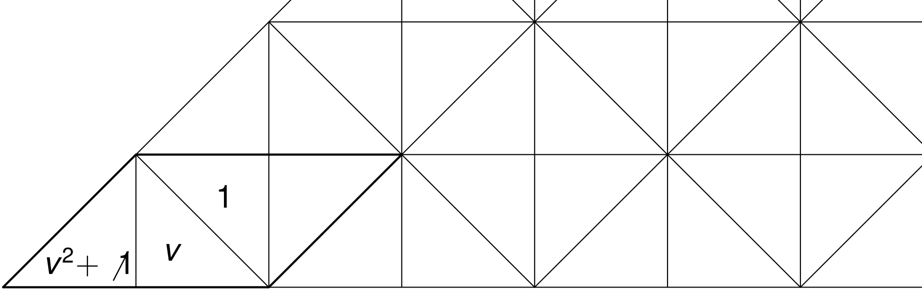


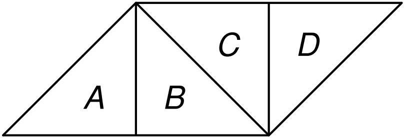
- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



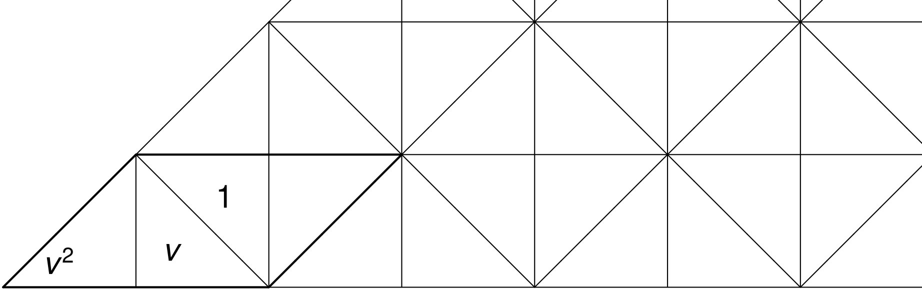


- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



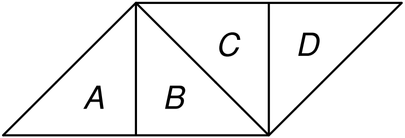


- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$

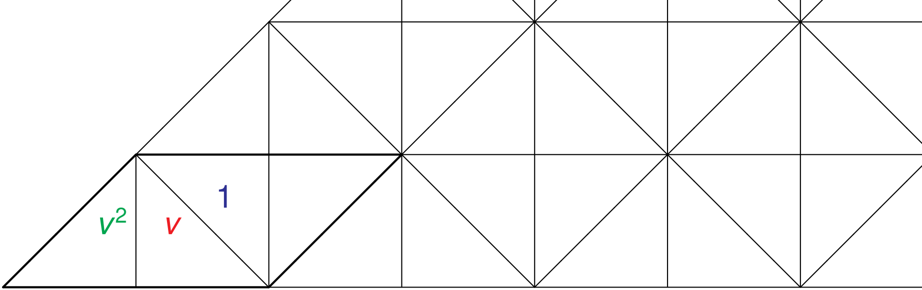


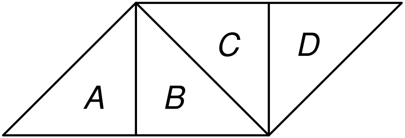
$\underline{M}_C = C + vB + v^2 A$

$[L_C] = [\nabla_C] - [\nabla_B] + [\nabla_A]$

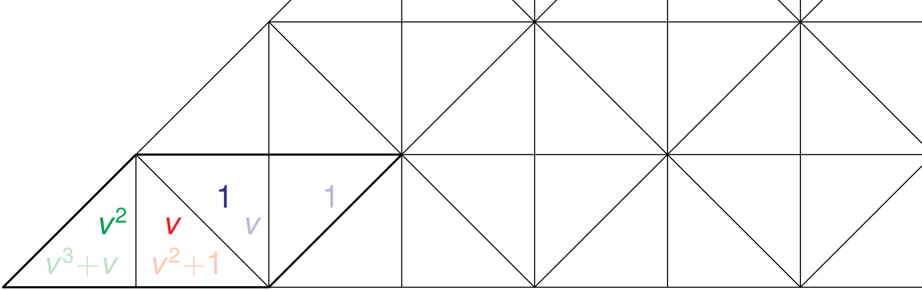


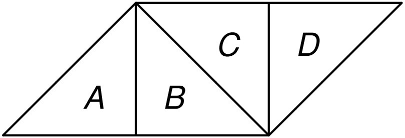
- Let $\mathcal{M}^{\text{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\text{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



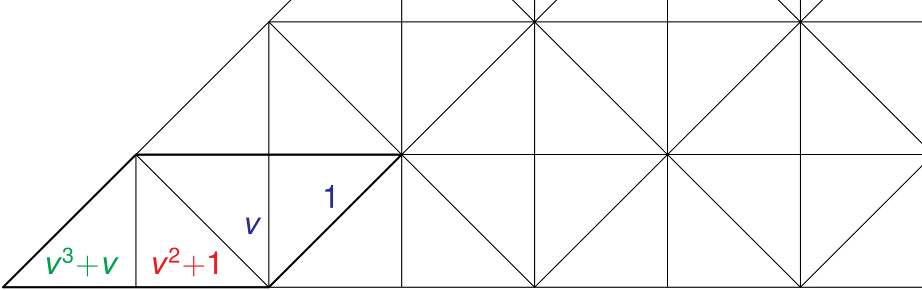


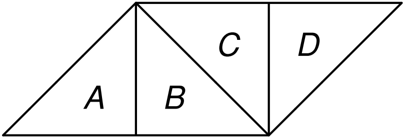
- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



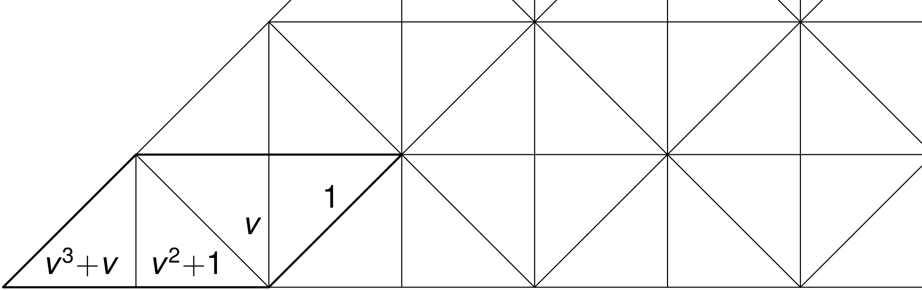


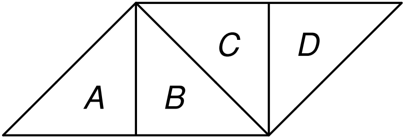
- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



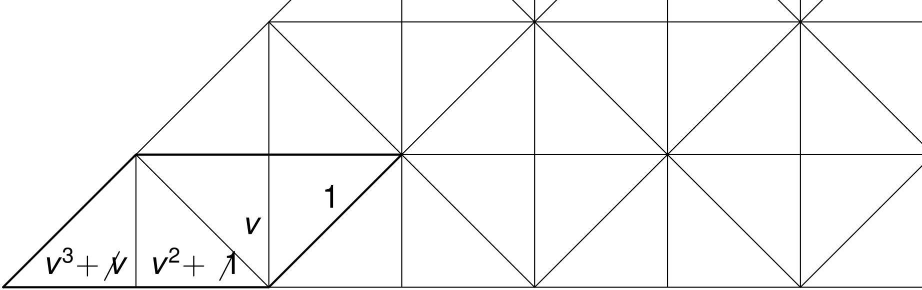


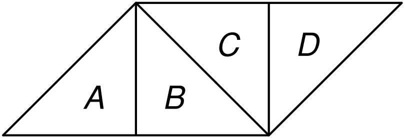
- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



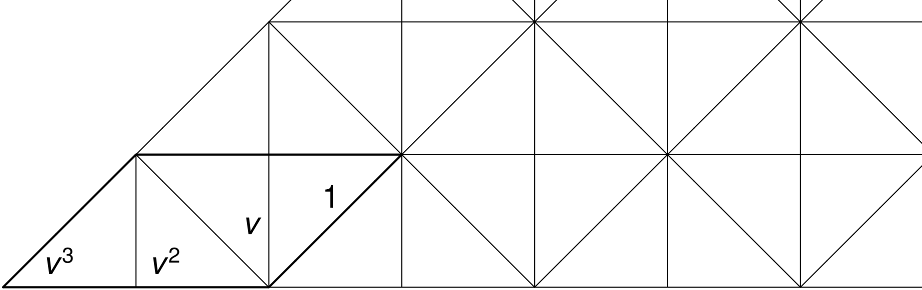


- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



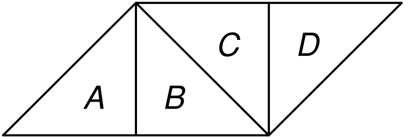


- Let $\mathcal{M}^{\text{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\text{sd}}$ uniquely determined for $A \in \mathcal{A}^+$

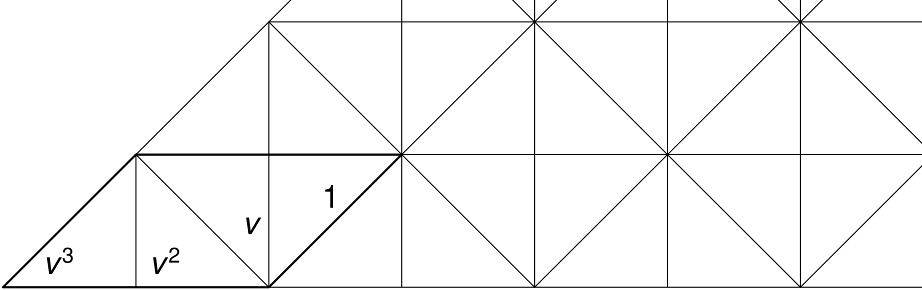


$$\underline{M}_D = D + vC + v^2B + v^3A$$

$$[L_D] = [\nabla_D] - [\nabla_C] + [\nabla_B] - [\nabla_A]$$

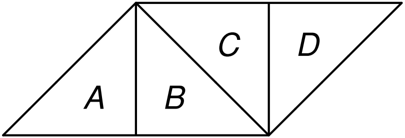


- Let $\mathcal{M}^{\text{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\text{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



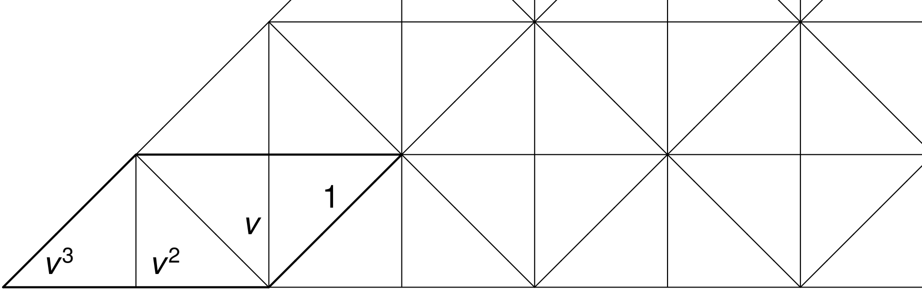
$\underline{M}_D = D + vC + v^2 B + v^3 A$

$[L_D] = [\nabla_D] - [\nabla_C] + [\nabla_B] - [\nabla_A]$



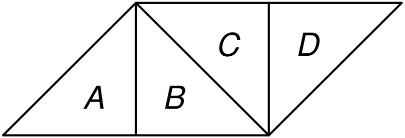
THANK YOU!

- Let $\mathcal{M}^{\mathrm{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\mathrm{sd}}$ uniquely determined for $A \in \mathcal{A}^+$



$\underline{M}_D = D + vC + v^2B + v^3A$

$[L_D] = [\nabla_D] - [\nabla_C] + [\nabla_B] - [\nabla_A]$



THANK YOU!

- Let $\mathcal{M}^{\text{sd}} \subset \mathcal{M}$ be the smallest subgroup containing A^+ and is stable under all the $[s]$
- $\underline{M}_A \in (A + v\mathbb{Z}[v]\mathcal{A}^+) \cap \mathcal{M}^{\text{sd}}$ uniquely determined for $A \in \mathcal{A}^+$