COMBINATORICS OF HARISH-CHANDRA MODULES

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Abstract. These lectures survey recent work on the combinatorics of certain infinite dimensional representations of complex semisimple Lie algebras. Their focus is not on understanding the irreducible objects but rather on understanding the structure of suitable representation categories. They concentrate on the relation of these representation categories with categories of modules over the coinvariant algebra associated to the action of the Weyl group on a Cartan subalgebra. We also discuss conjectural generalizations to the representation theory of real Lie groups.

1. Introductory dream

Let k be a field. Let $G \times X \to X$ be the action of a complex algebraic group G on a complex algebraic variety X. We suppose that there are only finitely many orbits and associate to this situation a $\mathbb{Z}$-graded k-algebra $\mathrm{Ext}^\bullet_G X$, the "equivariant extension algebra" in the following way: Let $\mathcal{D}_G(X)$ be the bounded derived equivariant category, with algebraically constructible cohomology and coefficients in k, see [BL94]. Let $\mathcal{P}_G(X) \subset \mathcal{D}_G(X)$ be the abelian subcategory of equivariant perverse sheaves. Let $\mathcal{L} \in \mathcal{P}_G(X)$ be the sum of all simple objects of $\mathcal{P}_G(X)$ (take one from each isomorphism class). We put

$$\mathrm{Ext}^\bullet_G X = \bigoplus_i \mathrm{Hom}_{\mathcal{D}_G(X)}(\mathcal{L}, \mathcal{L}[i]).$$

Next let us look at respresentation theory. If G is a complex algebraic group and $\gamma : G \to G$ an antiholomorphic involution, we let $G(\mathbb{R}, \gamma)$ denote the fixed points of γ, a real Lie group. Once we fix a real form γ, all real forms of G (up to isomorphism) are parametrized by $H^1(\Gamma; \mathrm{Aut}G)$, with $\Gamma = \{1, \gamma\} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Now conjugation gives a Γ-equivariant group homomorphism $G \to \mathrm{Aut}G$. So every $\delta \in H^1(\Gamma; G)$ will give rise to a real form $G(\mathbb{R}, \delta)$ of G.

Let now G be a connected reductive complex algebraic group and $\gamma : G \to G$ an antiholomorphic involution stabilizing a Borel subgroup of G. Let $\chi \in \mathrm{Max}Z$ be a character of the center $Z = Z(\mathfrak{g}) \subset U(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of the complex Lie algebra $\mathfrak{g} = \mathrm{Lie}G$. To these data (G, γ, χ) Adams, Barbasch and Vogan [ABV92] associate a complex algebraic variety $X(\chi)$, the "modified Langlands parameter space" on which the dual group $G^\vee$, i.e. the connected reductive complex algebraic group with the dual root datum, acts with finitely many orbits. The big picture motivating this series of talks is the following

Conjecture 1.1. [Soe92b] *There is an equivalence of categories*

$$\bigoplus_{\delta \in H^1(\Gamma;G)} \mathcal{M}(G(\mathbb{R},\delta))_\chi \cong \mathrm{Ext}^\bullet_{G^\vee} X(\chi) - \mathrm{Nil}.$$

Here $\mathcal{M}(G(\mathbb{R},\delta))_\chi$ denotes the category of Harish-Chandra-modules for $G(\mathbb{R},\delta)$ killed by some power of χ, and for a graded ring $A = \oplus_{i\geq 0} A^i$ the category $A - \mathrm{Nil}$ consists of all finite dimensional A-modules killed by A^i for $i \gg 0$.

Remark 1.2. I expect the analogous conjecture to hold for p-adic groups as well. Results of Lusztig and Ginzburg point in that direction, see Ginzburg's talks [Gin98] at this school.

Evidence and Motivation 1.3.

1. The conjecture works for tori.
2. It works for $\mathrm{SL}(2,\mathbb{R})$.
3. It works for complex groups, regarded as real Lie groups.
4. The simple objects on both sides are in "natural" bijection.
5. It works for generic χ.
6. I don't know a counterexample.
7. It fits together with results of [ABV92] in a bigger conjectural framework.

In these talks I will concentrate on 3 and explain how one can get his hands on both sides sufficiently well to see they are equal. But first let's discuss some of the other points.

About 1. Choose $G = \mathbb{C}^\times$ and let γ be the complex conjugation. Then $H^1(\Gamma;\mathbb{C}^\times)$ consits just of one element δ and $G(\mathbb{R},\delta) = \mathbb{R}^\times$. The space $X(\chi)$ consists just of two points, for every χ, and this is acted upon by $G^\vee = \mathbb{C}^\times$. We find

$$\mathrm{Ext}^\bullet_{G^\vee} X(\chi) = H^\bullet_{\mathbb{C}^\times}(\text{two points}) = \mathbb{C}[t] \times \mathbb{C}[t]$$

where t is a basis of $\mathrm{Lie} G$ and is given degree $\deg t = 2$. It is clear that

$$\mathcal{M}(\mathbb{R}^\times)_\chi \cong (\mathbb{C}[t] \times \mathbb{C}[t]) - \mathrm{Nil}$$

for every χ. The case of arbitray tori is a straightforward generalization of this example.

About 4. The simple objects of $\mathcal{P}_G(X)$ are parametrized by the set $\mathrm{Par} = \mathrm{Par}(G,X) = \{(Y,\tau) \mid Y$ is a G-orbit in X and τ a G-equivariant local system on Y, alias an irreducible representation of the component group of the isotropy group G_y of some fixed $y \in Y\}$.

For $\pi = (Y, \tau) \in \mathrm{Par}$ the corresponding simple object will be denoted $\mathcal{L}^\pi = \mathcal{IC}(Y, \tau)$, so $\mathcal{L} = \bigoplus_{\pi \in \mathrm{Par}} \mathcal{L}^\pi$. Certainly the degree zero part of $A = \mathrm{Ext}^\bullet_G X$ is just $A^0 = \prod_{\pi \in \mathrm{Par}} k 1_\pi$ with $1_\pi \in \mathrm{End}\mathcal{L}$ the projection onto $\mathcal{L}^\pi$. So the irreducibles of $A - \mathrm{Nil}$ are parametrized by Par.

Let $\mathrm{Irr}\mathcal{A}$ denote the set (or class) of isomorphism classes of irreducibles in any abelian category $\mathcal{A}$. Adams, Barbasch and Vogan [ABV] construct a bijection

$$\begin{array}{ccc} \mathrm{Par}(G^\vee, X(\chi)) & \xrightarrow{\sim} & \amalg_{\delta \in H^1(\Gamma; G)} \mathrm{Irr}\mathcal{M}(G(\mathbb{R}, \delta))_\chi \\ \pi & \mapsto & L_\pi \end{array}$$

depending on additional choices, which has the following wonderful property: If we write L_π as the unique simple quotient of an induced from tempered M_π, we get a $(\mathrm{Par} \times \mathrm{Par})$-matrix of Jordan-Hölder multiplicities $JH_{\pi,\psi} = [M_\pi : L_\psi]$, which is triangular with ones on the diagonal. Dually, we have also two bases of the Grothendieck group $[\mathcal{D}_{G^\vee} X(\chi)]$, given by $[\mathcal{L}^\pi]$ and $[\mathcal{M}^\pi]$ with $\mathcal{M}^\pi = i_!(\tau[\dim_{\mathbb{C}} Y])$ for $i : Y \hookrightarrow X(\chi)$ the injection. From the geometry we then get another $(\mathrm{Par} \times \mathrm{Par})$-matrix $IC_{\pi,\psi} = [\mathcal{M}^\pi : \mathcal{L}^\psi]$ given by $[\mathcal{M}^\pi] = \sum_\psi IC_{\pi,\psi} [\mathcal{L}^\psi]$. The bijection of [ABV] is such that up to signs these matrices are inverse transpose of one another, more precisely

$$\sum_\pi (-1)^{d(\pi) + d(\psi)} IC_{\pi,\psi} JH_{\pi,\xi} = \delta_{\psi,\xi}$$

where $d(\pi) = \dim_{\mathbb{C}} Y$ for $\pi = (Y, \tau)$.

Let me finish this first lecture with some comments on the equivariant Ext-algebras. I expect these algebras to be more natural then they look like, there should be a fully faithful embedding $\mathcal{D}_G(X) \hookrightarrow \mathcal{K}^+(\mathrm{Ext}^\bullet_G X)$ of the constructible equivariant derived category $\mathcal{D}_G(X)$ into the bounded below homotopy category of differential graded modules over $\mathrm{Ext}^\bullet_G X$, where $\mathrm{Ext}^\bullet_G X$ is considered as a differential graded algebra with zero differential.

2. Endomorphisms of antidominant projectives

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra, a Borel and a Cartan. Put $\mathcal{O}^\infty = \{M \in \mathfrak{g}\text{-mod} \mid M \text{ is finitely generated over } \mathfrak{g} \text{ and locally finite for } \mathfrak{b}\}$. By work of [BG80], certain direct summands of $\mathcal{M}(G(\mathbb{C}))$ are equivalent to $\mathcal{O}^\infty$, where G is a complex reductive group with $\mathrm{Lie}G = \mathfrak{g}$ and we let $\mathcal{M}(G(\mathbb{C}))$ be the Harish-Chandra-modules for $G(\mathbb{C})$ considered as a real Lie group. So we want to understand $\mathcal{O}^\infty$.

Restrict to the easier task to understand

$$\mathcal{O} = \{M \in \mathcal{O}^\infty \mid \mathfrak{h} \text{ acts diagonally on } M\}.$$

This category is studied extensively in the literature, we will build on [BG80, Jan83] and won't give proofs for results which can be found in these two sources. For $\lambda \in \mathfrak{h}^*$, let $M(\lambda) = U \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ be the Verma module and $L(\lambda)$ its unique simple quotient. Then every irreducible object of $\mathcal{O}$ is isomorphic to exactly one of the $L(\lambda)$, every object of $\mathcal{O}$ has finite length, and we have $\mathcal{O} = \bigoplus_{\chi \in \mathrm{Max} Z} \mathcal{O}_\chi$ where $\mathcal{O}_\chi \subset \mathcal{O}$ denotes the subcategory consisting of all objects killed by some power of χ. Let $\mathrm{pr}_\chi : \mathcal{O} \twoheadrightarrow \mathcal{O}_\chi$ be the corresponding projection functors. Certainly we have a map $\xi : \mathfrak{h}^* \to \mathrm{Max} Z$ such that $L(\lambda) \in \mathcal{O}_{\xi(\lambda)}$. This gives in fact an isomorphism $\mathfrak{h}^*/(W\cdot) \xrightarrow{\sim} \mathrm{Max} Z$ with corresponding comorphism $Z \xrightarrow{\sim} S^{(W\cdot)}$, where $S = S(\mathfrak{h})$ is the symmetric algebra over $\mathfrak{h}$ alias the regular functions on $\mathfrak{h}^*$ and where the Weyl group W acts on $\mathfrak{h}^*$ by the dot action $x \cdot \lambda = x(\lambda + \rho) - \rho$, for ρ the half-sum of the roots R^+ from $\mathfrak{b}$.

Theorem 2.1. 1. *There are enough projective objects in $\mathcal{O}$.*

2. *For every $\lambda \in \mathfrak{h}^*$ there exists a unique up to isomorphism indecomposable projective $P(\lambda) \in \mathcal{O}$ surjecting onto $L(\lambda)$.*
3. *$P(\lambda)$ admits a Verma flag, and for the multiplicities $(P(\lambda) : M(\mu))$ of Vermas in such a flag we have the reciprocity formula $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$.*
4. *$M(\lambda) = P(\lambda)$ iff $\langle \lambda + \rho, \alpha^\vee \rangle \notin \{-1, -2, \dots\} \quad \forall \alpha \in R^+$.*
5. *$M(\lambda) = L(\lambda)$ iff $\langle \lambda + \rho, \alpha^\vee \rangle \notin \{1, 2, \dots\} \quad \forall \alpha \in R^+$.*
6. *If $P \in \mathcal{O}$ is projective, then so is $E \otimes P$ for every finite dimensional representation E of $\mathfrak{g}$.*
7. *$E \otimes M(\lambda)$ admits a Verma flag with $M(\nu + \lambda)$ appearing $\dim_{\mathbb{C}} E_\nu$ times, for every finite dimensional representation E of $\mathfrak{g}$.*

Proof. [Jan83] □

If $\mathcal{A}$ is an artinian category and $P \in \mathcal{A}$ is a projective generator, i.e. a projective object surjecting onto every irreducible object, then we have an equivalence of categories $\mathrm{Hom}_{\mathcal{A}}(P, \ ) : \mathcal{A} \xrightarrow{\sim} \text{mof-}\mathrm{End}_{\mathcal{A}}(P)$, denoting by mof- the category of all right modules of finite length. So to understand $\mathcal{O}_{\mathrm{triv}}$ for $\mathrm{triv} = \xi(0) \in \mathrm{Max} Z$ the trivial central character we need to understand $\mathrm{End}_{\mathfrak{g}} P$ with $P = \oplus_{x \in W} P(x \cdot 0)$. The first step here is to understand $\mathrm{End}_{\mathfrak{g}} P(-2\rho)$.

Let $S_+ \subset S$ be the elements of positive degree and let $(S_+^W) \subset S$ denote the ideal of S generated by the W-invariants in S_+. Be careful that here we consider invariants for the usual "linear" action, not for the dot-action!

Theorem 2.2. [Soe90]

1. *The multiplication induces a surjection $Z \twoheadrightarrow \mathrm{End}_{\mathfrak{g}} P(-2\rho)$.*

2. *The composition* $Z \stackrel{\sim}{\to} S^{(W\cdot)} \hookrightarrow S \twoheadrightarrow S/(S_+^W)$ *also is a surjection.*
3. *Both these surjections have the same kernel, so we have a unique isomorphism* $\mathrm{End}_{\mathfrak{g}} P(-2\rho) \cong S/(S_+^W)$ *making commute the diagram*

$$\begin{array}{ccc} Z & \twoheadrightarrow & \mathrm{End}_{\mathfrak{g}} P(-2\rho) \\ \downarrow & & \| \\ S & \twoheadrightarrow & S/(S_+^W). \end{array}$$

The proof I give due to Bernstein [Ber90]. It will take up the rest of this section. Let us abbreviate the coinvariant algebra by $S/(S_+^W) = C$. It is known [Bou81] that $\dim_{\mathbb{C}} C = |W|$. Let us check first that the algebras we want to be isomorphic at least have the same dimension, i.e. that also $\dim_{\mathbb{C}} \mathrm{End}_{\mathfrak{g}} P(-2\rho) = |W|$. But indeed $[M(x\cdot 0) : L(-2\rho)] = 1$ for all $x \in W$ by [Jan83], and then $(P(-2\rho) : M(x \cdot 0)) = 1$ by reciprocity. Thus

$$\begin{array}{rcl} \dim_{\mathbb{C}} \mathrm{End}_{\mathfrak{g}} P(-2\rho) & = & [P(-2\rho) : L(-2\rho)] \\ & = & \sum_x (P(-2\rho) : M(x \cdot 0))[M(x \cdot 0) : L(-2\rho)] \\ & = & |W|. \end{array}$$

Next we explain why our map $Z \to C$ is a surjection. This is a special case of general results in invariant theory. Namely let a finite group G act on a finite dimensional complex vector space V. Let R be the regular functions on V and for $v \in V$ let $\mathfrak{m}_v \subset R$ be the ideal of functions vanishing at v. If $v \in V$ has trivial isotropy group, then $R^G \to R/I$ is a surjection for every $\mathfrak{m}_v$-primary ideal I, i.e. for any ideal I such that $I \supset \mathfrak{m}_v^n$ for $n \gg 0$. Indeed, for n fixed and $x \in R$ given we find $f \in R$ such that $f \in x + \mathfrak{m}_v^n$ and $f \in 1 + \mathfrak{m}_{gv}^n$ if $g \in G$, $g \neq e$. Then necessarily the product of all G-conjugates of f maps to $\bar{x} \in R/I$.

From there we get easily that $Z \to C$ is a surjection, and by comparison of codimension we only need to show that $\mathrm{Ann}_Z P(-2\rho)$ is contained in the kernel of $Z \twoheadrightarrow C$. Put in other words, we should show that if $z \in Z$ doesn't go to zero under $Z \twoheadrightarrow C$, then it doesn't kill $P(-2\rho)$ either. Now it is known [Bor53] that C is the cohomology ring of a smooth connected compact complex manifold of dimension $l(w_0)$, the "flag variety" consisting of all Borel subalgebras in $\mathfrak{g}$, and thus by Poincaré duality we know that if we take a nonzero homogeneous element $\omega \in C$ of maximal possible degree $l(w_0)$, then for every nonzero $c_1 \in C$ there exists $c_2 \in C$ such that $c_1 c_2 = \omega$. Hence we are reduced to proving

Claim 2.3. All $z \in Z$ which map to ω under $Z \twoheadrightarrow C$ do not kill $P(-2\rho)$.

To prove this we need another description of $P(-2\rho)$.

Lemma 2.4. *$P(-2\rho) = \mathrm{pr}_{\mathrm{triv}}(L(\rho) \otimes M(-\rho))$.*

Proof. It is not hard to check that both sides have the same Verma flag multiplicities and are both projective in $\mathcal{O}$. □

Let us now take n so big that $\chi^n \mathrm{pr}_\chi(L(\rho) \otimes M(-\rho)) = 0$ for all $\chi \in \mathrm{Max}Z$. (It is not hard to see that $n = |W|$ will do.) To establish our Claim 2.3, it will then be sufficient to show that for some n which is sufficiently big in this sense we have

Lemma 2.5. *If $z \in Z$ maps to $\omega \in C$ and is contained in χ^n for all $\chi \in \mathrm{Max}Z$ with $\chi \neq \mathrm{triv}$ but $\mathrm{pr}_\chi(L(\rho) \otimes M(-\rho)) \neq 0$, then $z \notin \mathrm{Ann}(L(\rho) \otimes M(-\rho))$.*

So the point is to show that certain elements of Z do not kill $L(\rho) \otimes M(-\rho)$. For this we may use a beautiful construction of Bernstein. Remark that whenever V, M are vector spaces and $\dim V < \infty$, we obtain a canonical linear map

$$\mathrm{tr}_V : \mathrm{End}(V \otimes M) \to \mathrm{End}M$$

as the composition

$$\mathrm{End}(V \otimes M) \to \mathrm{End}(V^* \otimes V \otimes M) \to \mathrm{End}M$$

where the first map is tensoring with the identity on V^*, and the second is induced from the canonical maps $\mathbb{C} \to V^* \otimes V \to \mathbb{C}$. Another way to define tr_V would be to use a basis of V to identify $\mathrm{End}(V \otimes M)$ with the matrix algebra of $(\dim V \times \dim V)$ square matrices with entries in $\mathrm{End}M$, and then take the usual trace of a matrix. This is why tr_V might be called the "relative trace". It is clear that for $\dim M < \infty$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{End}(V \otimes M) & \stackrel{\mathrm{tr}_V}{\to} & \mathrm{End}M \\ \| & & \downarrow \mathrm{tr} \\ \mathrm{End}(V \otimes M) & \stackrel{\mathrm{tr}}{\to} & \mathbb{C}. \end{array}$$

If V, M are representations of a Lie algebra $\mathfrak{g}$, then certainly tr_V restricts to a linear map

$$\mathrm{tr}_V : \mathrm{End}_{\mathfrak{g}}(V \otimes M) \to \mathrm{End}_{\mathfrak{g}}M$$

which is so natural that it determines in fact a map

$$\mathrm{tr}_V : \mathrm{End}_F(V\otimes) \to \mathrm{End}_F(\mathrm{id})$$

where $V\otimes$, $\mathrm{id} : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ are considered as functors and End_F denotes the natural transformations of a functor to itself.

Now for any unitary ring A the endomorphism ring of the identity functor id on A-mod is canonically isomorphic to the center $Z(A)$ of A, in formulas

$$Z(A) \stackrel{\sim}{\to} \mathrm{End}_F(\mathrm{id}).$$

Thus in our representation theoretic context we get for any finite dimensional representation V of $\mathfrak{g}$ a linear map $\mathrm{Tr}_V : Z \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} Z & \stackrel{\mathrm{Tr}_V}{\to} & Z \\ \downarrow & & \| \\ \mathrm{End}_F(V\otimes) & \stackrel{\mathrm{tr}_V}{\to} & \mathrm{End}_F(\mathrm{id}) \end{array}$$

To prove our Lemma 2.5, it will be sufficient to show for z as in the Lemma that $\mathrm{Tr}_{L(\rho)}(z)$ doesn't kill $M(-\rho)$. We will deduce this from an explicit formula for the map Tr_V, which we explain next.

For $M \subset \mathfrak{h}^*$ a finite multiset and $f \in S = R(\mathfrak{h}^*)$ define $(M * f) \in R(\mathfrak{h}^*)$ by $(M * f)(\lambda) = \sum_{\mu \in M} f(\lambda + \mu)$. Define $\Lambda \in S$ by $\Lambda(\lambda) = \prod_{\alpha \in R^+} \langle \lambda + \rho, \alpha^\vee \rangle$. Then Λ is $(W\cdot)$-antiinvariant, and the set of all $(W\cdot)$-antiinvariant elements of S is precisely $\Lambda S^{(W\cdot)}$. Consider $Z = S^{(W\cdot)}$ as a subring of S, so that z acts on $L(\lambda)$ by the scalar $z(\lambda)$. Let P_V denote the multiset of weights of V counted with their multiplicities.

Theorem 2.6. [Ber90] $\mathrm{Tr}_V(z) = \Lambda^{-1}(P_V * \Lambda z)$ *for all* $z \in Z$.

Remark 2.7. Here z is viewed as a $(W\cdot)$-invariant element of S, then Λz is $(W\cdot)$-antiinvariant, so is $P_V * \Lambda z$, and thus we may divide by Λ to get again an element of $S^{(W\cdot)} = Z$.

Proof. Both sides are regular functions on $\mathfrak{h}^*$, thus we only need to show that they take the same values on all integral weights $\lambda \in P$ which lie in the dominant chamber far from the walls. But for those we know $V \otimes L(\lambda) \cong \bigoplus_{\mu \in P_V} L(\lambda + \mu)$, and using the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\mathrm{Tr}_V} & Z \\ \downarrow & & \downarrow \\ \mathrm{End}(V \otimes L(\lambda)) & \xrightarrow{\mathrm{tr}_V} & \mathrm{End} L(\lambda) \\ \| & & \downarrow \mathrm{tr} \\ \mathrm{End}(V \otimes L(\lambda)) & \xrightarrow{\mathrm{tr}} & \mathbb{C} \end{array}$$

we find $(\dim L(\lambda))(\mathrm{Tr}_V z)(\lambda) = \sum_{\mu \in P_V} (\dim L(\lambda + \mu)) z(\lambda + \mu)$. Now we can use the Weyl character formula $\dim L(\lambda) = \Lambda(\lambda)/\Lambda(0)$ to conclude. $\square$

Finally, we get what we really wanted.

Proof. [of Lemma 2.5 and Theorem 2.2] It will be sufficient to show that for z as in the Lemma we have $(\mathrm{Tr}_{L(\rho)} z)(-\rho) \neq 0$. To see this we have to calculate a little while. Let

$$\begin{array}{rcl} \mathrm{Sgn} & : & S \to S \\ & & f \mapsto |W|^{-1} \sum_{w \in W} (-1)^{l(w)} f^{(w \cdot)} \end{array}$$

be the antisymmetrisation with respect to the dot-action of the Weyl group. For $\mu \in \mathfrak{h}^*$ define

$$\begin{array}{rccc} \mathrm{Tr}_\mu : & S & \to & S \\ & f & \mapsto & \Lambda^{-1} \mathrm{Sgn}(\mu * \Lambda f). \end{array}$$

It is not hard to check that $\mathrm{Tr}_V = \sum_{\mu \in P_V} \mathrm{Tr}_\mu$ on Z and $\mathrm{Tr}_\mu(z) = \mathrm{Tr}_{w\mu}(z)$ for all $z \in Z$. Now if f vanishes of sufficiently high degree at $(\mu - \rho)$, then clearly $(\mathrm{Tr}_\mu f)(-\rho) = 0$. Furthermore $(\mathrm{Tr}_\rho f)(-\rho)$ depends only on the image of f in C. Indeed, for $f = gh$ with $g \in S^W$ we have $(\mathrm{Tr}_\rho f)(\lambda) = g(\lambda + \rho) \cdot (\mathrm{Tr}_\rho h)(\lambda)$ and if we put here $\lambda = -\rho$ we see that $(\mathrm{Tr}_\rho f)(-\rho) = 0$ for $f \in (S^W_+)$. Thus we can just take any preimage $f \in S$ of $\omega \in C$, why not $f = (-\rho) * \Lambda$, and need only for this f to check $(\mathrm{Tr}_\rho f)(-\rho) \neq 0$. And indeed

$$\begin{array}{rcl} (\mathrm{Tr}_\rho f) & = & \Lambda^{-1} \mathrm{Sgn}((\rho * \Lambda)\Lambda) \\ & = & |W|^{-1} \sum_{x \in W} (\rho * \Lambda)^{(x \cdot)} \end{array}$$

and thus

$$\begin{array}{rcl} (\mathrm{Tr}_\rho f)(-\rho) & = & |W|^{-1} \sum_{x \in W} (\rho * \Lambda)(-\rho) \\ & = & \Lambda(0) \\ & \neq & 0. \end{array}$$

□

3. Reduction to coinvariants

Last time we proved $C = \mathrm{End}_{\mathfrak{g}} P(-2\rho)$. Let us consider the functor

$$\begin{array}{rccc} \mathbb{V} : & \mathcal{O}_{\mathrm{triv}} & \to & C\text{-mod} \\ & M & \mapsto & \mathrm{Hom}_{\mathfrak{g}}(P(-2\rho), M). \end{array}$$

Theorem 3.1. *The functor $\mathbb{V}$ is fully faithful on projective objects, it gives even for all $M, Q \in \mathcal{O}_{\mathrm{triv}}$ with Q projective an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(M, Q) \xrightarrow{\sim} \mathrm{Hom}_C(\mathbb{V}M, \mathbb{V}Q).$$

Proof. Step 1. $Q = P(-2\rho)$. In this case both $P(-2\rho) = \mathrm{pr}_{\mathrm{triv}}(L(\rho) \otimes M(-\rho))$ and C (by Poincaré duality) are injective in $\mathcal{O}_{\mathrm{triv}}$ and C-mod respectively. Thus both sides of our presumed isomorphism are exact in $M \in \mathcal{O}_{\mathrm{triv}}$, and we only have to check equality for M simple. This is easy.

Step 2. $Q = M(0)$. Let $C_+ \subset C$ the ideal of all elements of positive degree.

Lemma 3.2. $M(0) = \{v \in P(-2\rho) \mid C_+ v = 0\}$

Proof. Let us call the right hand side N. It is not difficult to get a short exact sequence $M(0) \hookrightarrow P(-2\rho) \twoheadrightarrow K$ where K has a Verma flag. Thus the socle of K is a sum of copies of $L(-2\rho)$. Since anyhow $M(0) \subset N$, we see that $M(0) \neq N$ would imply $[N : L(-2\rho)] \geq 2$. So $I = \mathrm{Hom}_{\mathfrak{g}}(P(-2\rho), N) \subset C = \mathrm{End}_{\mathfrak{g}} P(-2\rho)$ would be an ideal with $C_+ I = 0$ and $\dim_{\mathbb{C}} I \geq 2$. But the subspace of elements C killed by C^+ is just the line through ω, and this contradiction shows that necessarily $M(0) = N$. $\square$

To complete step 2 we consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{g}}(M, M(0)) & = & \{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M, P(-2\rho)) \mid C_+\varphi = 0\} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_C(\mathbb{V}M, \mathbb{C}) & = & \{\psi \in \mathrm{Hom}_C(\mathbb{V}M, C) \mid C_+\psi = 0\}, \end{array}$$

and since here by step 1 the second vertical is an isomorphism, the first one is an isomorphism too.

Step 3. Q arbitrary. Without restriction we may assume $Q = \mathrm{pr}_{\mathrm{triv}}(E \otimes M(0))$ for some finite dimensional E. Consider the functor $F = F_E : \mathcal{O}_{\mathrm{triv}} \to \mathcal{O}_{\mathrm{triv}}$, $M \mapsto \mathrm{pr}_{\mathrm{triv}}(E \otimes M)$. Its right and left adjoint is $F_{E^*} = G$. We need some general category theory.

Call an exact functor between abelian categories

$$\mathbb{V} : \mathcal{A} \longrightarrow \mathcal{B}$$

a "quotient functor" iff it has the obvious universal property. For $\mathcal{A}$ artinian and $P \in \mathcal{A}$ projective, always $\mathrm{Hom}(P,\ ) : \mathcal{A} \to \text{mof-}\mathrm{End}_{\mathcal{A}} P$ is a quotient functor. In our case of $\mathbb{V} : \mathcal{O}_{\mathrm{triv}} \to C\text{-mof}$, the kernel

$$\ker \mathbb{V} = \{M \mid [M : L(-2\rho)] = 0\}$$

is stable under both F and G. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{triv}} & \xrightarrow{\mathbb{V}} & C\text{-mof} \\ \downarrow F, G & & \downarrow \bar{F}, \bar{G} \\ \mathcal{O}_{\mathrm{triv}} & \xrightarrow{\mathbb{V}} & C\text{-mof} \end{array}$$

for suitable $\bar{F}, \bar{G}$. One can check without much difficulty that $\bar{F}$ and $\bar{G}$ are adjoints of each other, too. And then we just calculate

$$\begin{array}{rcl} \mathrm{Hom}_{\mathfrak{g}}(M, Q) & = & \mathrm{Hom}_{\mathfrak{g}}(M, FM(0)) \text{ since } Q = FM(0) \\ & = & \mathrm{Hom}_{\mathfrak{g}}(GM, M(0)) \text{ by adjointness} \\ & = & \mathrm{Hom}_C(\mathbb{V}GM, \mathbb{V}M(0)) \text{ by step 2} \\ & = & \mathrm{Hom}_C(\bar{G}\mathbb{V}M, \mathbb{V}M(0)) \text{ since } \mathbb{V}G = \bar{G}\mathbb{V} \\ & = & \mathrm{Hom}_C(\mathbb{V}M, \bar{F}\mathbb{V}M(0)) \text{ by adjointness} \\ & = & \mathrm{Hom}_C(\mathbb{V}M, \mathbb{V}FM(0)) \text{ since } \mathbb{V}F = \bar{F}\mathbb{V} \\ & = & \mathrm{Hom}_C(\mathbb{V}M, \mathbb{V}Q) \text{ since } Q = FM(0). \end{array}$$

□

We deduce

$$\begin{array}{rcl} \mathcal{O}_{\mathrm{triv}} & \cong & \text{mof-End}_{\mathfrak{g}}(\oplus_x P(x \cdot 0)) \\ & \cong & \text{mof-End}_C(\oplus_x \mathbb{V}P(x \cdot 0)) \end{array}$$

and $\mathbb{V}P(x\cdot) \in C$-mod is indecomposable, as there are no idempotents in its endomorphism ring except 0 and 1.

So what are these strange indecomposable C-modules? Our next goal is a geometric description of the form $\mathbb{V}P(x \cdot 0) \cong \mathbb{H}^\bullet \mathcal{IC}(\overline{Bx^{-1}B/B}$. But first we need some preliminaries. Put $P^- = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \{0, -1, -2, \dots\} \quad \forall \alpha \in R^+\}$. For $\lambda \in P^-$ put $\mathcal{O}_\lambda = \mathcal{O}_{\xi(\lambda)}$. For $\lambda, \mu \in P^-$ define the translation functor $T_\lambda^\mu : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ by $T_\lambda^\mu = \mathrm{pr}_{\xi(\mu)}(L(\nu) \otimes M)$ where ν is the unique integral dominant weight in $W(\mu - \lambda)$, hence $\dim L(\nu) < \infty$. Put $W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}$.

Theorem 3.3. *For all $\lambda \in P^-$ we have $T_\lambda^{-2\rho} P(\lambda) \cong P(-2\rho)$, and the induced map $\mathrm{End}_{\mathfrak{g}} P(\lambda) \to \mathrm{End}_{\mathfrak{g}} P(-2\rho)$ is an inclusion with image $C^{W_\lambda} =: C_\lambda$.*

Proof. Omitted, see [Soe90]. □

Let us define $\mathbb{V} = \mathbb{V}_\lambda : \mathcal{O}_\lambda \to C_\lambda$-mof as $\mathbb{V}_\lambda = \mathrm{Hom}_{\mathfrak{g}}(P_\lambda, \ )$.

Corollary 3.4. *We have a commutative square*

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{triv}} & \xrightarrow{\mathbb{V}} & C\text{-mof} \\ T_{-2\rho}^\lambda \downarrow & & \downarrow \mathrm{res} \\ \mathcal{O}_\lambda & \xrightarrow{\mathbb{V}} & C_\lambda\text{-mof}. \end{array}$$

Proof. $\mathrm{Hom}_{\mathfrak{g}}(P_\lambda, T_{-2\rho}^\lambda M) = \mathrm{Hom}_{\mathfrak{g}}(T_\lambda^{-2\rho} P_\lambda, M)$. □

Corollary 3.5. *We have a commutative square*

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{triv}} & \xrightarrow{\mathbb{V}} & C\text{-mof} \\ T_\lambda^{-2\rho} \uparrow & & \uparrow C \otimes_{C_\lambda} \\ \mathcal{O}_\lambda & \xrightarrow{\mathbb{V}} & C_\lambda\text{-mof}. \end{array}$$

Proof. Take the adjoint on both verticals of the preceding diagram. □

If $W_\lambda = \{e, s\}$ with s a simple reflection, then $\Theta_s = T_\lambda^{-2\rho} T_{-2\rho}^{\lambda} : \mathcal{O}_{\mathrm{triv}} \to \mathcal{O}_{\mathrm{triv}}$ is called a "wall-crossing functor", and we get finally

Proposition 3.6. *We have a commutative square*

$$\begin{array}{ccc} \mathcal{O}_{\mathrm{triv}} & \xrightarrow{\mathbb{V}} & C\text{-mof} \\ \Theta_s \downarrow & & \downarrow C\otimes_{C^s} \\ \mathcal{O}_{\mathrm{triv}} & \xrightarrow{\mathbb{V}} & C\text{-mof}. \end{array}$$

Proof. Clear. □

Now remark that we may describe $P(x \cdot 0)$ inductively as follows: Write $x = s_1 \cdots s_l$ as a product of minimal length of simple reflections. Then $P(x \cdot 0)$ is the unique indecomposable direct summand of $\Theta_{s_l} \ldots \Theta_{s_1} M(0)$ not isomorphic to some $P(y \cdot 0)$ with $y < x$. Thus $\mathbb{V}P(x \cdot 0)$ is the unique direct summand of the C-module $C \otimes_{C^{s_l}} C \otimes \ldots \otimes_{C^{s_1}} \mathbb{C}$ which is not isomorphic to $\mathbb{V}P(y \cdot 0)$ for some $y < x$.

4. Relations with geometry

Let G be a semisimple complex algebraic group with Lie algebra $\mathfrak{g}$, and let $B \subset G$ be its Borel subgroup with Lie algebra $\mathfrak{b}$. We will explain first why $\mathbb{V}P(x \cdot 0)$ is as a C-module just the intersection cohomology group of a suitable Schubert variety, $\mathbb{V}P(x \cdot 0) \cong IC(\overline{Bx^{-1}B/B})$. Remark first that for any bounded below complex of sheaves $\mathcal{F}^\bullet$ on a topological space X the total hypercohomology $\mathbb{H}^\bullet \mathcal{F} = \mathrm{Hom}^\bullet(\mathbb{C}_X, \mathcal{F})$ is in a natural way a module over the cohomology ring $H^\bullet X = \mathrm{End}^\bullet \mathbb{C}_X$. Now let s be a simple reflection, $P_s \subset G$ the corresponding minimal parabolic subgroup containing B, and $\pi = \pi_s : G/B \to G/P_s$ the projection onto the corresponding partial flag variety.

Lemma 4.1. *Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves on G/B. Then $\mathbb{H}^\bullet(\pi^*\pi_*\mathcal{F}^\bullet) \cong C \otimes_{C^s} \mathbb{H}^\bullet \mathcal{F}$ are isomorphic as C-modules.*

Proof. [Soe90]. □

To get our argument running, we still need another fact. Let $\mathcal{IC}_x$ denote the intersection cohomology complex of $\overline{(Bx^{-1}B/B)}$.

Theorem 4.2. *For $x, y \in W$ the hypercohomology functor induces an isomorphism*

$$\mathrm{Hom}^\bullet(\mathcal{IC}_x, \mathcal{IC}_y) \xrightarrow{\sim} \mathrm{Hom}^\bullet_C(\mathbb{H}^\bullet \mathcal{IC}_x, \mathbb{H}^\bullet \mathcal{IC}_y).$$

Proof. [Gin91]. □

Let now $x \in W$ be given and let $x = s_1 \dots s_l$ be a reduced decomposition. Then $\pi_{s_l}^* \pi_{s_r *} \dots \pi_{s_1}^* \pi_{s_1 *} \mathcal{IC}_e$ decomposes into a direct sum of objects of the form $\mathcal{IC}_y[i]$ for suitable y and i, here the biggest y occuring is x itself, and it occurs just once. So $\mathbb{H}^\bullet \mathcal{IC}_x$ can be described inductively as the one direct summand of the C-module

$$\mathbb{H}^\bullet \pi_{s_l}^* \pi_{s_l *} \dots \pi_{s_1}^* \pi_{s_1 *} \mathcal{IC}_e \cong C \otimes_{C^{s_l}} \dots C \otimes_{C^{s_1}} \mathbb{C}$$

that is not isomorphic to $\mathbb{H}^\bullet \mathcal{IC}_y$ for $y < x$. Comparing this description with the description of $\mathbb{V}P(x \cdot 0)$ given at the end of the preceding section shows that indeed $\mathbb{H}^\bullet \mathcal{IC}_x \cong \mathbb{V}P(x \cdot 0)$.

Let us put $\mathcal{L} = \bigoplus_{x \in W} \mathcal{IC}_x$. The following theorem was a conjecture of Beilinson and Ginzburg [BG86]. Its ramifications are explored in [BGS96].

Theorem 4.3. *There is an equivalence of categories*

$$\mathcal{O}_{\text{triv}} \cong \text{mof-End}^\bullet(\mathcal{L}, \mathcal{L}).$$

Proof. Let $P = \bigoplus_{x \in W} P(x \cdot 0)$ be a minimal projective generator of $\mathcal{O}_{\text{triv}}$. We get

$$\begin{array}{rcl} \mathcal{O}_{\text{triv}} & \cong & \text{mof-End}_{\mathfrak{g}} P \text{ by category theory} \\ & \cong & \text{mof-End}_C \mathbb{V}P \text{ by Theorem 3.1} \\ & \cong & \text{mof-End}_C \mathbb{H}^\bullet \mathcal{L} \text{ since } \mathbb{V}P \cong \mathbb{H}^\bullet \mathcal{L} \\ & \cong & \text{mof-End}^\bullet(\mathcal{L}, \mathcal{L}) \text{ by Theorem 4.2.} \end{array}$$

□

This theorem is almost the statement of Conjecture 1.1 in the case of complex groups. A proof of the conjecture in the case of trivial central character is given in [Soe92b] based on combinatorial work from [Soe92a].

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