Euklid's plane through Symmetry

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- **Incidence geometry:** Pair (X, G) with X a set of "points", $G \subset \mathcal{P}(X)$ a set of "lines", each line has at least two points, through every two distinct points there goes exactly one line.
- **Betweeness:** Subset $Z \subset X^3$ of collinear tripels giving two opposite orders on every line, such that a line never meets only one segment of a triangle.
- **Congruence group:** A subgroup $K \subset \mathrm{Aut}(X, G, Z)$ such that for any two halflines $A, B \subset X$ there exist exactly two $k, h \in K$ with $k(A) = B = h(A)$.
- **Supremum property:** With respect to a Z-order every nonempty bounded above subset on a line has a supremum.
- **Parallel Axiom:** $\forall g \in G, p \in X \backslash g$ there exists uniquely $h \in G$ with $p \in h$ and $h \cap g = \emptyset$.

- **Theorem:** There is up to isomorphism a unique quadrupel (X, G, Z, K) of an incidence geometry with betweeness relation and congruence group that satisfies supremum and parallel axioms and has at least one line. [Soergel: Elementargeometrie]
- A **Congruence group** $K \subset \mathrm{Aut}(X, G, Z)$ is a subgroup such that for any two halflines $A, B \subset X$ exist exactly two $k, h \in K$ with $k(A) = B = h(A)$.
- If we ask instead congruences to act free and transitive on the set of halflines, there are is a “bad” model for every nontrivial group homomorphism $\mathrm{SO}(2) \to \mathbb{R}_{>0}$.
- If we ask the parallel axiom to be false, there should be also a unique model. I would like an easy proof.

Now I am going make lots of claims and if you don't believe one of them, you are welcome to speak up and I will try to explain the proof on the blackboard.

Let (X, G, Z) be an incidence geometry with betweeness.

- A line meeting no vertex of a triangle meets exactly to segments or none.
- The complement of a line is the disjoint union of at most two equivalence classes under the relation "joinable by a segment".

Let (X, G, Z, K) be an incidence geometry with betweeness and congruences.

- Every halfline is infinite.
- For every line g there is a unique nontrivial congruence s_g fixing it pointwise, the **reflection along** g.
- Every segment is infinite.
- For every line g there are exactly two halfspaces. They are exchanged by the reflection s_g.

Let (X, G, Z, K) be an incidence geometry with betweeness and congruences.

- Let $h \perp g$ mean $g \neq h = s_g(h)$.
- For every line g and every point x there is a unique perpendicular $h \perp g$ with $x \in h$.
- $h \perp g \Leftrightarrow s_h s_g = s_g s_h$
- Two perpendiculars to a line g are disjoint.
- Under the parallel axiom perpendiculars to a line are perpendicular to its parallels.

Let (X, G, Z, K) be incidence geometry with betweeness and congruences and let $g \in G$ a line.

- Denote by $K_{|g} \subset K$ the stabilizer of a line and its halfspaces.
- Denote by $\vec{g} \subset K_{|g}$ the subgroup stabilizing both Z-orders on the line g.
- $\vec{g}$ acts free and transitive on g.

Let (X, G, Z, K) be an incidence geometry with betweeness and congruences and supremum axiom and let $g \in G$ be a line.

- Given $v \in \vec{g} \backslash e_K$ we have $v(x) > x$ for all x and some Z-order on g.
- All elements of $K_{|g} \backslash \vec{g}$ are involutions.
- Any two different points $x \neq y$ can be exchanged by a unique reflection. It stabilizes the $\overline{xy}$-halfplanes.

Let (X, G, Z, K) be an incidence geometry with betweeness and congruences and supremum axiom and let $g \in G$ be a line.

- All elements of $K_{|g} \backslash \vec{g}$ are reflections. These elements generate $K_{|g}$.
- Every element $v \in \vec{g}$ has a square root. Conjugating v by an element of $K_{|g} \backslash \vec{g}$ we get its inverse.
- $\vec{g}$ is commutative.

Let (X, G, Z, K) be an incidence geometry with betweeness and congruences and supremum and parallels axiom.

- $g \parallel h \Rightarrow \vec{g} = \vec{h}$
- Assume there exists a line. Then all translations form a commutative subgroup $\vec{X} := \bigcup_{g \in G} \vec{g} \subset K$ acting free and transitive on X.

- Let A be an ordered group such that no nontrivial cyclic subgroup has an upper bound in A and every element has a root. Then for every $a > e_A$ there exists a unique order preserving group homomorphism $A \to (\mathbb{R}, +)$ with $a \mapsto 1$ and it is injective.

Let (X, G, Z, K) be incidence geometry with betweeness and congruences and supremum axiom.

- For every line g, there is a unique structure on $\vec{g}$ as a real vector space compatible with Z.
- If the parallel axiom holds and there is a least one line, there is a unique structure of real vector space on $\vec{X}$ such that $\vec{g}$ is a subspace for all $g \in G$. Furthermore the space $\vec{X}$ has dimension two, so that X acquires the structure of a twodimensional real affine space.

Let (X, G, Z, K) be incidence geometry with betweeness and congruences and supremum and parallel axioms and a line.

- The congruence group K consists of affinities and contains all translations.
- The isotropy groups K_x all have the same image $D \subset \mathrm{GL}(\vec{X})$.
- The subgroup $D \subset \mathrm{GL}(\vec{X})$ has the property that for any two rays $A, B \subset \vec{X}$ there are exactly two elements $r, s \in D$ with $r(A) = B = s(A)$.

Let V be a twodimensinal real vector space. Let $D \subset \mathrm{GL}(V)$ a subgroup such that for any two rays $A, B \subset V$ there are exactly two elements $r, s \in D$ with $r(A) = B = s(A)$.

- There exists a D-invariant scalar product on V.
- Any two D-invariant scalar products are scalar multiples of one another.
- The group D is the orthogonal group for one and any of these scalar products.

This proves that an incidence geometry with betweeness and congruences (X, G, Z, K) satisfying supremum and parallel axioms and having a line is isomorphic to

$$(\mathbb{R}^2, \text{affine lines}, \text{obvious } Z, \mathrm{O}(2)_{\mathsf{aff}})$$

Thanks!