THE COMPOSITION SERIES OF MODULES INDUCED FROM WHITTAKER MODULES

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Abstract. We study a category of representations over a semisimple Lie algebra, which contains category $\mathcal{O}$ as well as the so-called Whittaker modules, and prove a generalization of the Kazhdan-Lusztig conjectures in this context.

1. Introduction

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{n}$ be a semisimple complex Lie algebra, a Borel subalgebra, and its nilradical. Let $U(\mathfrak{g}) = U \supset Z$ be the enveloping algebra of $\mathfrak{g}$ and its center. Let $\chi \subset Z$ be a maximal ideal and $f \in (\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])^* =: \mathrm{ch}\mathfrak{n}$ a character of $\mathfrak{n}$, giving rise to a one-dimensional $\mathfrak{n}$-module $\mathbb{C}_f$. By [McD85] the $\mathfrak{g}$-module

$$Y(\chi, f) = U/\chi U \otimes_{U(\mathfrak{n})} \mathbb{C}_f$$

is of finite length. For any ring A let $\mathrm{Max}A$ denote its set of maximal ideals. We are interested in the following

Problem 1.1. *Compute the composition factors of $Y(\chi, f)$ with their multiplicities, for all $\chi \in \mathrm{Max}Z$ and $f \in \mathrm{ch}\mathfrak{n}$.*

We will solve this problem completely for integral χ and partially for other χ as well[1]. Let us first consider the two extreme cases. Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra and $\mathfrak{h}^* \supset R \supset R^+ \supset \Delta$ its dual, the roots, the roots of $\mathfrak{b}$ and the simple roots. So $\mathfrak{g} = \oplus_{\alpha \in R}\, \mathfrak{g}_\alpha \oplus \mathfrak{h}$. Call $f \in \mathrm{ch}\mathfrak{n}$ regular if and only if $f|_{\mathfrak{g}_\alpha} \neq 0$ for all $\alpha \in \Delta$. For regular f our problem is solved completely by the following theorem of Kostant [Kos78], (see also [MS95] for a geometric proof).

Theorem 1.2. *If $f \in \mathrm{ch}\mathfrak{n}$ is regular, then $Y(\chi, f)$ is irreducible for all $\chi \in \mathrm{Max}Z$.*

These irreducible $Y(\chi, f)$ are the so-called Whittaker modules. In the other extreme, i.e. for $f = 0$, we have (see 2.5)

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[1]The general case was recently settled by E. Backelin [Bac97]

Proposition 1.3. *If $\chi \in \mathrm{Max}Z$ is regular, then $Y(\chi, 0)$ is the direct sum of all Verma modules for $\mathfrak{g} \supset \mathfrak{b}$ with central character χ. For χ singular $Y(\chi, 0)$ still has a Verma flag such that each Verma module with central character χ appears with the same multiplicity and the length of the flag is the cardinality of the Weyl group.*

Thus for $f = 0$ our problem is solved by the Kazhdan-Lusztig conjectures, which describe the composition series of Verma modules. The general case will be a mixture of these two. We will partially solve it by reducing to the Kazhdan-Lusztig conjectures. To explain how this is done, let us put our problem in a different perspective.

Consider the full subcategory $\mathcal{N} = \mathcal{N}(\mathfrak{g}, \mathfrak{b}) \subset \mathfrak{g}$-mod of all $\mathfrak{g}$-modules M which are (1) finitely generated over $\mathfrak{g}$, (2) locally finite over $\mathfrak{n}$ and (3) locally finite over Z. By [McD85] all objects of $\mathcal{N}$ have finite length. From a geometric perspective [MS95] this is evident, they just correspond to holonomic $\mathcal{D}$-modules. The action of Z decomposes $\mathcal{N}$ into a direct sum $\mathcal{N} = \oplus_\chi \mathcal{N}(\chi)$ where χ runs over $\mathrm{Max}Z$. The action of $\mathfrak{n}$ also decomposes $\mathcal{N}$ into a direct sum $\mathcal{N} = \oplus_f \mathcal{N}(f)$ over all $f \in \mathrm{ch}\mathfrak{n}$, where

$$\mathcal{N}(f) = \{M \in \mathcal{N} \mid X - f(X) \text{ acts locally nilpotently on } M, \forall X \in \mathfrak{n}\}.$$

In total, we have $\mathcal{N} = \oplus_{\chi,f} \mathcal{N}(\chi, f)$ with $\mathcal{N}(\chi, f) = \mathcal{N}(\chi) \cap \mathcal{N}(f)$ and all these categories are stable under subquotients and extensions in $\mathfrak{g}$-mod. Certainly $Y(\chi, f) \in \mathcal{N}(\chi, f)$. To solve our problem, we have to study the categories $\mathcal{N}(\chi, f)$.

Again the two extreme cases are more or less well understood. For regular f there is an equivalence of categories

$$\mathcal{N}(f) \cong \{M \in Z\text{-mod} \mid \dim_{\mathbb{C}} M < \infty\}$$

as was shown by Kostant [Kos78]. For $f = 0$ our $\mathcal{N}(f) = \mathcal{N}(0)$ consists just of all finite length $\mathfrak{g}$-modules with only highest weight modules as composition factors. For general f, the situation was investigated by McDowell [McD85]. In fact, McDowell's results as well as Kostant's results cited above admit very natural geometric proofs if one uses localization. We worked this out in our joint paper [MS95].

Let $(\mathcal{W}, \mathcal{S})$ be the Coxeter system of $\mathfrak{g} \subset \mathfrak{b}$. There is a bijection $\Delta \xrightarrow{\sim} \mathcal{S}, \alpha \mapsto s_\alpha$. Put $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ as usual and set $w \cdot \lambda = w(\lambda + \rho) - \rho$ for all $w \in \mathcal{W}, \lambda \in \mathfrak{h}^*$. Let the Harish-Chandra homomorphism $\xi^\sharp : Z \to S = S(\mathfrak{h})$ be normalized by the condition $\xi^\sharp(z) - z \in U\mathfrak{n}$. For the corresponding $\xi : \mathfrak{h}^* \to \mathrm{Max}Z$ we have $\xi(\lambda) = \xi(\mu) \Leftrightarrow \mathcal{W}\cdot\lambda = \mathcal{W}\cdot\mu$. For $f \in \mathrm{ch}\mathfrak{n}$ put $\Delta_f = \{\alpha \in \Delta \mid f|_{\mathfrak{g}_\alpha} \neq 0\}$, $\mathcal{S}_f = \{s_\alpha \mid \alpha \in \Delta_f\}$ and $\mathcal{W}_f = \langle \mathcal{S}_f \rangle \subset \mathcal{W}$.

Now for any $\lambda \in \mathfrak{h}^*, f \in \operatorname{ch}\mathfrak{n}$ one constructs a "standard module" $M(\lambda, f) \in \mathcal{N}(\xi(\lambda), f)$. It has a unique simple quotient $L(\lambda, f)$ and $M(\lambda, f) = M(\mu, f)$ if and only if $\mathcal{W}_f \cdot \lambda = \mathcal{W}_f \cdot \mu$. Finally any simple object $L \in \mathcal{N}(f)$ has the form $L \cong L(\lambda, f)$ for a unique $\lambda \in \mathfrak{h}^*/(W_f \cdot)$. All this is due to McDowell. The definitions are set up in such a way that $M(\lambda, 0)$ is just the usual Verma module $M(\lambda) = U \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. For simplicity we state the analog of our first Proposition 1.3 only for regular χ. The general case is contained in Corollary 2.5.

Proposition 1.4. *Suppose $\chi \in \operatorname{Max} Z$ is regular. Then $Y(\chi, f) \cong \bigoplus_\lambda M(\lambda, f)$ where λ runs over $\xi^{-1}(\chi)/(\mathcal{W}_f \cdot)$.*

In this way our original problem of computing the composition series of the $Y(\chi, f)$ reduces to the following

Problem 1.5. *Compute the multiplicities $[M(\lambda, f) : L(\mu, f)]$ for all $\lambda, \mu \in \mathfrak{h}^*$.*

Let us just explain how we solve this problem for regular integral central character χ. Let $\mu \in \mathfrak{h}^*$ be integral dominant (i.e. $\langle \mu + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R^+$) and such that $\mathcal{W}_f = \{x \in \mathcal{W} \mid x \cdot \mu = \mu\}$. Let $\lambda \in \mathfrak{h}^*$ be integral dominant such that $\chi = \xi(\lambda)$. At the end of Section 5 we establish an equivalence of categories

$$\mathcal{N}(\chi, f) \cong \mathcal{N}(\xi(\mu), 0)$$

under which $M(x \cdot \lambda, f)$ corresponds to $M(x^{-1} \cdot \mu)$. This reduces our problem to the Kazhdan-Lusztig conjectures, which by now are a theorem.

2. Standard modules and simple modules

Remember $f \in \operatorname{ch}\mathfrak{n}$ determines a subset $\Delta_f = \{\alpha \in \Delta \mid f|_{\mathfrak{g}_\alpha} \neq 0\}$ of simple roots. Let $\mathfrak{p}_f \subset \mathfrak{g}$ be the corresponding parabolic subalgebra containing $\mathfrak{b}$ and $\mathfrak{p}_f = \mathfrak{g}_f \oplus \mathfrak{n}^f$ its ad$\mathfrak{h}$-stable Levi decomposition. Remark that $\mathfrak{g}_f$ is not semisimple in general, but only reductive. For example $\Delta_0 = \emptyset$, $\mathfrak{p}_0 = \mathfrak{b}$, $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{n}^0 = \mathfrak{n}$. Let $U(\mathfrak{g}_f) = U_f \supset Z_f$ be the enveloping algebra of $\mathfrak{g}_f$ and its center. Put $\mathfrak{b}_f = \mathfrak{b} \cap \mathfrak{g}_f$ and let $\mathfrak{n}_f \subset \mathfrak{b}_f$ be its nilradical, so that $\mathfrak{n} = \mathfrak{n}_f \oplus \mathfrak{n}^f$. Let $\xi_f^\sharp : Z_f \to S$ be the Harish-Chandra homomorphism of $\mathfrak{g}_f$, normalized as before by the condition $\xi_f^\sharp(z) - z \in U_f \mathfrak{n}_f$. It induces on the maximal ideals a map $\xi_f : \mathfrak{h}^* \to \operatorname{Max} Z_f$. For any ideal $I \subset Z_f$ define the $\mathfrak{g}_f$-module

$$Y_f(I, f) = U_f / I U_f \otimes_{U(\mathfrak{n}_f)} \mathbb{C}_f.$$

If I is a maximal ideal, then this is an irreducible $\mathfrak{g}_f$-module, since the restriction of f to $\mathfrak{n}_f$ is nondegenerate by definition of $\mathfrak{n}_f$. We define

$$M(\lambda, f) = U \otimes_{U(\mathfrak{p}_f)} Y_f(\xi_f(\lambda), f)$$

where we extend the $\mathfrak{g}_f$-action on Y_f to an action of $\mathfrak{p}_f$ letting $\mathfrak{n}^f$ act by zero.

Proposition 2.1. (1) *We have $M(\lambda, f) \cong M(\mu, f)$ if and only if $\mathcal{W}_f \cdot \lambda = \mathcal{W}_f \cdot \mu$.*

(2) *$M(\lambda, f)$ has a unique simple quotient $L(\lambda, f)$. We have $L(\lambda, f) \cong L(\mu, f)$ if and only if $\mathcal{W}_f \cdot \lambda = \mathcal{W}_f \cdot \mu$.*

(3) $\mathrm{Ann}_U M(\lambda, f) = \xi(\lambda)U$.

Proof. All this is in fact contained in [McD85] and, from a geometric point of view, in [MS95]. However for us it is not a great detour, so we will give complete arguments. We start with 3. This is easily reduced to the case of Verma modules by some general considerations: Let $\mathfrak{a} \to \mathfrak{b}$ be a morphism of Lie algebras. For any $\mathfrak{b}$-module M let $\mathrm{res}_{\mathfrak{b}}^{\mathfrak{a}} M$ denote the $\mathfrak{a}$-module obtained by restriction. For any $\mathfrak{a}$-module N let $\mathrm{ind}_{\mathfrak{a}}^{\mathfrak{b}} N$ denote the $\mathfrak{b}$-module $\mathrm{ind}_{\mathfrak{a}}^{\mathfrak{b}} N = U(\mathfrak{b}) \otimes_{U(\mathfrak{a})} N$ obtained by induction. For any module M over a Lie algebra let $\mathrm{Ann} M$ denote its annihilator in the enveloping algebra.

Lemma 2.2. (1) *Let M, M' be $\mathfrak{b}$-modules. Then from the inclusion $\mathrm{Ann} M \subset \mathrm{Ann} M'$ follows the inclusion $\mathrm{Ann}(\mathrm{res}_{\mathfrak{b}}^{\mathfrak{a}} M) \subset \mathrm{Ann}(\mathrm{res}_{\mathfrak{b}}^{\mathfrak{a}} M')$.*

(2) *Let N, N' be $\mathfrak{a}$-modules. Then from the inclusion $\mathrm{Ann} N \subset \mathrm{Ann} N'$ follows the inclusion $\mathrm{Ann}(\mathrm{ind}_{\mathfrak{a}}^{\mathfrak{b}} N) \subset \mathrm{Ann}(\mathrm{ind}_{\mathfrak{a}}^{\mathfrak{b}} N')$.*

Proof. Omitted. □

From this we deduce part 3 of the proposition, as follows: Let $M_f(\lambda)$ denote the Verma module with highest weight λ for $\mathfrak{g}_f$. Then

$$\mathrm{Ann}_{U_f} Y_f(\xi_f(\lambda), f) = \xi_f(\lambda) U_f = \mathrm{Ann}_{U_f} M_f(\lambda)$$

by a theorem of Kostant [Kos78] and of Duflo [Dix74] respectively. Now we apply part 1 of our lemma with the surjection $\mathfrak{p}_f \twoheadrightarrow \mathfrak{g}_f$ and then part 2 with the inclusion $\mathfrak{p}_f \hookrightarrow \mathfrak{g}$ and deduce $\mathrm{Ann}_U M(\lambda, f) = \mathrm{Ann}_U(U \otimes_{U(\mathfrak{p}_f)} M_f(\lambda))$. But certainly $U \otimes_{U(\mathfrak{p}_f)} M_f(\lambda) = M(\lambda)$, hence $\mathrm{Ann}_U M(\lambda, f) = \mathrm{Ann}_U M(\lambda) = U\xi(\lambda)$ by the theorem of Duflo once again, and 3 is proved.

Next we prove part 1 of the proposition. Let us decompose $\mathfrak{h} = \mathfrak{h}^f \oplus \mathfrak{h}_f$ with $\mathfrak{h}_f = \mathfrak{h} \cap [\mathfrak{g}_f, \mathfrak{g}_f]$ and $\mathfrak{h}^f = \cap_{\alpha \in \Delta_f} \ker\alpha$ the centralizer of f alias the center of $\mathfrak{g}_f$. The action of $\mathcal{W}_f$ as well as the dot-action of $\mathcal{W}_f$ on $\mathfrak{h}^*$ respect this decomposition, and they are trivial on $(\mathfrak{h}^f)^*$.

Remark that a priori there are two dot-actions of $\mathcal{W}_f$ on $\mathfrak{h}^*$, fixing the halfsum of positive roots of $\mathfrak{g}$ and of $\mathfrak{g}_f$ respectively, but one checks that they coincide. Now it is clear that $\mathcal{W}_f \cdot \lambda = \mathcal{W}_f \cdot \mu$ implies $\xi_f(\lambda) = \xi_f(\mu)$ and hence $M(\lambda, f) = M(\mu, f)$. To prove the reverse implication, denote for any $\lambda \in \mathfrak{h}^*$ by $\lambda^f \in (\mathfrak{h}^f)^*$ its restriction to $\mathfrak{h}^f$. Clearly $\mathfrak{h}^f$ acts via λ^f on $Y_f(\xi_f(\lambda), f)$. Hence $M(\lambda, f)$ decomposes under $\mathfrak{h}^f$ into weight spaces $M(\lambda, f)_\mu$ with $\mu \in \lambda^f - \sum_{\alpha \in R^+} \mathbb{Z}_{\geq 0}\alpha^f$. Clearly all the $M(\lambda, f)_\mu$ are $\mathfrak{g}_f$-submodules, and $M(\lambda, f)_{\lambda^f} = Y_f(\xi_f(\lambda), f)$. Thus $M(\lambda, f) \cong M(\mu, f)$ implies $\xi_f(\lambda) = \xi_f(\mu)$, hence $\mathcal{W}_f \cdot \lambda = \mathcal{W}_f \cdot \mu$ and we proved 1.

Finally we go for 2. We have to show that $M(\lambda, f)$ has a unique maximal proper submodule. But any submodule $N \subset M(\lambda, f)$ is the sum of its $\mathfrak{h}^f$-weight spaces $N = \oplus N_\mu$, and these are $\mathfrak{g}_f$-submodules of the $M(\lambda, f)_\mu$. Since $M(\lambda, f)_{\lambda^f}$ is irreducible over $\mathfrak{g}_f$ and generates $M(\lambda, f)$ over $\mathfrak{g}$, any proper submodule has to be contained in $\oplus_{\mu \neq \lambda^f} M(\lambda, f)_\mu$. Hence the sum of all proper submodules is itself a proper submodule, necessarily the biggest one. Thus $M(\lambda, f)$ has a unique simple quotient $L(\lambda, f)$. Again $L(\lambda, f) \cong L(\mu, f)$ implies $\lambda^f = \mu^f$ and $L(\lambda, f)_{\lambda^f} \cong L(\mu, f)_{\lambda^f}$, hence $\xi_f(\lambda) = \xi_f(\mu)$ and finally $\mathcal{W}_f \cdot \lambda = \mathcal{W}_f \cdot \mu$. □

Next we establish Proposition 1.4 from the introduction. Let us define the "relative Harish-Chandra homomorphism" $\theta^\sharp : Z \to Z_f$ by the condition $\xi_f^\sharp \theta^\sharp = \xi^\sharp : Z \to S$.

Lemma 2.3. *For all $z \in Z$ we have $\theta^\sharp(z) - z \in U\mathfrak{n}^f$.*

Proof. Consider U_f as a $\mathfrak{p}_f$-module with $\mathfrak{n}^f$ acting by zero and form the induced module $U \otimes_{U(\mathfrak{p}_f)} U_f$. Then $\mathrm{Ann}_U(1 \otimes 1) = U\mathfrak{n}^f$ and $1 \otimes U_f \subset U \otimes_{U(\mathfrak{p}_f)} U_f$ is just the space of invariants of $\mathfrak{h}^f$. This means that there is a map $\tilde{\theta} : Z \to U_f$ such that $z(1 \otimes 1) = 1 \otimes \tilde{\theta}(z)$ for all $z \in Z$, and we see easily that this defines an algebra homomorphism $\tilde{\theta} : Z \to Z_f$ and that furthermore $\tilde{\theta}(z) - z \in U\mathfrak{n}^f$ for all $z \in Z$. From there we find that $\xi_f^\sharp \tilde{\theta} : Z \to S$ is an algebra homomorphism such that $\xi_f^\sharp \tilde{\theta}(z) - z \in U\mathfrak{n}$, thus $\theta_f^\sharp \tilde{\theta} = \xi^\sharp$, thus $\theta^\sharp = \tilde{\theta}$ and the Lemma is proved. □

Proposition 2.4. *Let $I \subset Z$ be an ideal. We have an isomorphism*

$$Y(I, f) \cong U \otimes_{U(\mathfrak{p}_f)} Y_f(\theta^\sharp(I) Z_f, f).$$

Proof. Recall that we defined $Y(I, f) = U/IU \otimes_{U(\mathfrak{n})} \mathbb{C}_f$. Certainly this object represents the functor

$$\begin{array}{ccc} U/IU\text{-mod} & \to & \mathbb{C}\text{-mod} \\ M & \mapsto & \mathrm{Hom}_{\mathfrak{n}}(\mathbb{C}_f, M). \end{array}$$

Since $\theta^\sharp(z) - z \in U\mathfrak{n}^f$, the right hand side of our future isomorphism is annihilated by I also. It follows furthermore that for any $M \in U/IU$-mod we have $\theta^\sharp(I)M^{\mathfrak{n}^f} = 0$. Thus for any $M \in U/IU$-mod we have

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{n}}(\mathbb{C}_f, M) &= \mathrm{Hom}_{\mathfrak{n}_f}(\mathbb{C}_f, M^{\mathfrak{n}^f}) \\ &= \mathrm{Hom}_{\mathfrak{g}_f}(Y_f(\theta^\sharp(I)Z_f, f),\, M^{\mathfrak{n}^f}) \\ &= \mathrm{Hom}_{\mathfrak{p}_f}(Y_f(\theta^\sharp(I)Z_f, f),\, M) \\ &= \mathrm{Hom}_{\mathfrak{g}}(U \otimes_{U(\mathfrak{p}_f)} Y_f(\theta^\sharp(I)Z_f, f), M). \end{aligned}$$

Our isomorphism follows now from the fact that both sides share a universal property, i.e. represent the same functor. □

Corollary 2.5. (1) *For all $\lambda \in \mathfrak{h}^*$ the module $Y(\xi(\lambda), f)$ admits a filtration with subquotients $M(x \cdot \lambda, f)$, where x runs over $\mathcal{W}_f \setminus \mathcal{W}$.*

(2) *If λ is regular, then we have even a decomposition into a direct sum $Y(\xi(\lambda), f) \cong \bigoplus_x M(x \cdot \lambda, f)$ with x running our $\mathcal{W}_f \setminus \mathcal{W}$.*

Proof. We start with some generalities. For any commutative ring A let $A\text{-mod}^{fl}$ be the category of finite length A-modules and $[A\text{-mod}^{fl}]$ its Grothendieck group. Any finite flat ring extension $j : A \to B$ gives a homomorphism $[j] : [A\text{-mod}^{fl}] \to [B\text{-mod}^{fl}]$, $[M] \mapsto [B \otimes_A M]$. Remark that in case $M = A/I$ we have $B \otimes_A M \cong B/BI$. We remark further that $[A\text{-mod}^{fl}]$ can be identified with the free abelian group $\mathbb{Z}\mathrm{Max}A$ over $\mathrm{Max}A$ via $\mathfrak{m} \mapsto [A/\mathfrak{m}]$, so we can view $[j]$ as a morphism $[j] : \mathbb{Z}\mathrm{Max}A \to \mathbb{Z}\mathrm{Max}B$. Now all our ring extensions $Z \subset Z_f \subset S$ given by $\theta^\sharp, \xi_f^\sharp$ and $\xi^\sharp$ are finite and flat. Let us identify as usual $\mathfrak{h}^* \xrightarrow{\sim} \mathrm{Max}S$ via $\lambda \mapsto \langle\lambda\rangle$. Now we know from invariant theory, say, that for every $\lambda \in \mathfrak{h}^*$ we have in $\mathbb{Z}\mathrm{Max}S$ the equalities

$$[\xi^\sharp]\xi(\lambda) = \sum_{x \in \mathcal{W}} \langle x \cdot \lambda \rangle$$

and

$$[\xi_f^\sharp]\xi_f(\lambda) = \sum_{x \in \mathcal{W}_f} \langle x \cdot \lambda \rangle.$$

Since $[\xi^\sharp] = [\xi_f^\sharp] \circ [\theta^\sharp]$ we deduce $[\theta^\sharp]\xi(\lambda) = \sum_{x \in \mathcal{W}_f \setminus \mathcal{W}} \xi_f(x \cdot \lambda)$. In other words, $Z_f/\theta^\sharp(\xi(\lambda))Z_f$ admits a filtration with subquotients $Z_f/\xi_f(x \cdot \lambda)$ where x runs over $\mathcal{W}_f \setminus \mathcal{W}$. Certainly this is in fact a direct sum decomposition when the $\xi_f(x \cdot \lambda)$ are pairwise different, e.g. for λ regular. Now U_f is a free Z_f-module, and we deduce that $U_f/\theta^\sharp(\xi(\lambda))U_f$ admits a filtration with subquotients $U_f/\xi_f(x \cdot \lambda)U_f$, $x \in \mathcal{W}_f \setminus \mathcal{W}$, which splits for regular λ to give a direct sum decomposition. But U_f is known to be even a free right module over $Z \otimes U(\mathfrak{n}_f)$, thus we find that $Y_f(\theta^\sharp(\xi(\lambda))Z_f, f)$ admits a filtration with subquotients $Y_f(\xi_f(x \cdot \lambda), f)$, $x \in \mathcal{W}_f \setminus \mathcal{W}$, which splits for regular λ to give a direct sum decomposition. We now apply $U \otimes_{U(\mathfrak{p}_f)}$ and the Corollary follows from the Proposition. □

We now reprove McDowell's results.

Theorem 2.6. (1) *Any $M \in \mathcal{N}$ is of finite length.*
(2) *The $L(\lambda, f)$ with $\lambda \in \mathfrak{h}^*/(\mathcal{W}_f \cdot)$ represent the isomorphism classes of simple objects in $\mathcal{N}(f)$.*

Proof. We start with 1. Put $I = \mathrm{Ann}_Z M$. By definition of $\mathcal{N}$ this is an ideal of finite codimension in Z. Without restriction of generality we can assume $I \in \mathrm{Max} Z$. Using the definition of $\mathcal{N}$ once more we find a finite dimensional $\mathfrak{n}$-stable subspace $E \subset M$ that generates M as a $\mathfrak{g}$-module. Thus M is a quotient of $U/UI \otimes_{U(\mathfrak{n})} E$ and we may restrict our attention to such M. We can filter E by $\mathfrak{n}$-submodules with one-dimensional subquotients. This way we reduce our problem to showing that the $Y(I, f)$ are of finite length. Using Corollary 2.5 we further reduce to showing that all $M(\lambda, f)$ are of finite length. By [Kos78] we know that for any finite dimensional $\mathfrak{g}_f$-module E and $\eta \in \mathrm{Max} Z_f$ the $\mathfrak{g}_f$-module $E \otimes Y_f(\eta, f)$ is of finite length and has its composition factors among the $Y_f(\eta', f)$ with $\eta' \in \mathrm{Max} Z_f$. Let $\bar{\mathfrak{n}}^f \subset \mathfrak{g}$ be the ad$\mathfrak{h}$-stable complement of $\mathfrak{p}_f$. Certainly

$$M(\lambda, f) \cong U(\bar{\mathfrak{n}}^f) \otimes_{\mathbb{C}} Y_f(\xi_f(\lambda), f)$$

as $\mathfrak{g}_f$-modules, and we deduce that all $M(\lambda, f)_\mu$ with $\mu \in (\mathfrak{h}^f)^*$ are finite length modules over $\mathfrak{g}_f$ with their composition factors among the $Y_f(\eta', f)$.

Now any simple subquotient L of $M(\lambda, f)$ has to have a "highest" weight $\mu \in (\mathfrak{h}^f)^*$ such that $L_\mu \neq 0$ and $\mathfrak{n}^f L_\mu = 0$. We then find an $\eta \in \mathrm{Max} Z_f$ such that $\mathrm{Hom}_{\mathfrak{g}_f}(Y_f(\eta, f), L_\mu) \neq 0$, thus $\mathrm{Hom}_{\mathfrak{g}}(M(\nu, f), L) \neq 0$ if $\nu \in \mathfrak{h}^*$ is such that $\xi_f(\nu) = \eta$, thus $L \cong L(\nu, f)$ since L is simple. Modulo the things we saw already this proves 2.

All simple subquotients of $M(\lambda, f)$ have the same central character, hence are among the $L(x \cdot \lambda, f), x \in \mathcal{W}$ by Proposition 2.1. We deduce that the length of $M(\lambda, f)$ is bounded by the sum of the lengths of the $\mathfrak{g}_f$-modules $M(\lambda, f)_{(x \cdot \lambda)^f}, x \in \mathcal{W}$. □

3. Equivalences between categories of Harish-Chandra bimodules and of representations

In this section we recall results of [BG80] in a form suitable for our purposes. Let for the moment $\mathfrak{g}$ be any complex Lie algebra and $U = U(\mathfrak{g})$ its enveloping algebra. On any U-bimodule $X \in U$-mod-U we can define a third $\mathfrak{g}$-action $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}} X$ by the formula $(\mathrm{ad}A)x = Ax - xA$ for any $A \in \mathfrak{g}$, $x \in X$. This is called the adjoint action on a bimodule. We get a functor U-mod-$U \to \mathfrak{g}$-mod, $M \mapsto M^{\mathrm{ad}}$ considering any U-bimodule as a representation of $\mathfrak{g}$ for the adjoint action. For $M, N \in \mathfrak{g}$-mod we make $\mathrm{Hom}_{\mathbb{C}}(M, N)$ into a U-bimodule in the obvious way and consider on $M \otimes N$ the standard $\mathfrak{g}$-module structure. For $M \in \mathfrak{g}$-mod, $X \in U$-mod-U we define $M \otimes X \in U$-mod-U by the prescriptions $A(m \otimes x) = (Am) \otimes x + m \otimes (Ax)$, $(m \otimes x)A = m \otimes (xA)$ for all $A \in \mathfrak{g}$, $m \in M$, $x \in X$. With these definitions we find that

(1) For any $N, M, E \in \mathfrak{g}$-mod the canonical isomorphism
$$\mathrm{Hom}_{\mathbb{C}}(E, \mathrm{Hom}_{\mathbb{C}}(M, N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(E \otimes M, N)$$
induces an isomorphism
$$\mathrm{Hom}_{\mathfrak{g}}(E, \mathrm{Hom}_{\mathbb{C}}(M, N)^{\mathrm{ad}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(E \otimes M, N).$$

(2) Consider U as an object of U-mod-U. For any $E \in \mathfrak{g}$-mod and $X \in U$-mod-U we obtain a canonical isomorphism
$$\mathrm{Hom}_{U-U}(E \otimes U, X) \to \mathrm{Hom}_{\mathfrak{g}}(E, X^{\mathrm{ad}})$$
by composing a morphism $E \otimes U \to X$ with the obvious map $E \to E \otimes U$, $e \mapsto e \otimes 1$.

For any $X \in U$-mod-U let $X_{\mathrm{adf}} \subset X$ denote the subspace of ad$\mathfrak{g}$-finite vectors, $X_{\mathrm{adf}} = \{x \in X \mid$ There exists a finite dimensional ad$\mathfrak{g}$-stable subspace of X containing $x\}$. For $M \in \mathfrak{g}$-mod the subspace $(\mathrm{End}_{\mathbb{C}} M)_{\mathrm{adf}} \subset \mathrm{End}_{\mathbb{C}} M$ is actually a subring.

If $\mathfrak{g}$ is finite dimensional, $X_{\mathrm{adf}} \subset X$ is a sub-U-bimodule. Let us define the category
$$\mathcal{H} = \{X \in U\text{-mod-}U \mid X = X_{\mathrm{adf}} \text{ and } X \text{ is finitely generated}\}.$$
It is of no importance here whether X is supposed to be finitely generated as left module, right module, or bimodule: For bimodules consisting of ad$\mathfrak{g}$-finite vectors all these properties are equivalent.

Let us return now to our semisimple Lie algebra $\mathfrak{g}$. Let us denote by $\mathcal{F}(\mathfrak{g}) = \mathcal{F}$ the category of all finite dimensional representations of $\mathfrak{g}$. To $M \in \mathfrak{g}$-mod we associate two full subcategories of $\mathfrak{g}$-mod:

(1) The category $\langle \mathcal{F} \otimes M \rangle$ consisting of all subquotients of objects of the form $E \otimes M$ with $E \in \mathcal{F}$.

(2) The category $\mathrm{coker}(\mathcal{F}\otimes M)$ consisting of all $N\in\mathfrak{g}$-mod that admit a two-step resolution $E\otimes M\to F\otimes M\twoheadrightarrow N$ with $E,F\in\mathcal{F}$.

On the other hand define for any subset $I\subset U$ the category

$$\mathcal{H}(I)=\{X\in\mathcal{H}\mid XI=0\}.$$

We are now ready to state the result of [BG80] in the form in which we need it.

Theorem 3.1. *Let $I\subset U$ be a twosided ideal, $\bar{U}=U/I$ the quotient and $M\in\mathfrak{g}$-mod a representation with $IM=0$. Suppose that*

(1) *The multiplication $U\to\mathrm{End}_{\mathbb{C}}M$ induces an isomorphism $\bar{U}\xrightarrow{\sim}(\mathrm{End}_{\mathbb{C}}M)_{\mathrm{adf}}$ and*
(2) *M is a projective object in $\langle\mathcal{F}\otimes M\rangle$.*

Then the functor $\otimes_U M:U\text{-mod-}U\to\mathfrak{g}\text{-mod}$ induces an equivalence of categories

$$\mathcal{H}(I)\xrightarrow{\sim}\mathrm{coker}(\mathcal{F}\otimes M).$$

Proof. From the preceding considerations it is clear that for all $E\in\mathcal{F}$, $X\in\mathcal{H}(I)$ we have

$$\mathrm{Hom}_{U-U}(E\otimes\bar{U},X)\cong\mathrm{Hom}_{\mathfrak{g}}(E,X^{\mathrm{ad}}).$$

Whence the $E\otimes\bar{U}$ with $E\in\mathcal{F}$ are projective in $\mathcal{H}(I)$ and generate $\mathcal{H}(I)$. Since $\otimes_U M$ is right exact, we see that it induces indeed a functor from $\mathcal{H}(I)$ to $\mathrm{coker}(\mathcal{F}\otimes M)$.

Next we claim that for all $E,F\in\mathcal{F}$ the functor $\otimes_U M$ induces a bijection

$$\mathrm{Hom}_{U-U}(E\otimes\bar{U},F\otimes\bar{U})\to\mathrm{Hom}_{\mathfrak{g}}(E\otimes M,F\otimes M).$$

To see this, remark first that for any three vectorspaces V,W,F with $\dim F<\infty$ there is a canonical isomorphism $\mathrm{Hom}_{\mathbb{C}}(V,F\otimes W)\cong\mathrm{Hom}_{\mathbb{C}}(F^*\otimes V,W)$. This is compatible with all our actions, so we need only prove our displayed bijection in case $F=\mathbb{C}$. But now we identified the left hand side with $\mathrm{Hom}_{\mathfrak{g}}(E,\bar{U}^{\mathrm{ad}})$ and the right hand side with $\mathrm{Hom}_{\mathfrak{g}}(E,(\mathrm{End}_{\mathbb{C}}M)^{\mathrm{ad}})$ and the claim follows from assumption 1.

Remark next that from assumption 2 actually follows that all $E\otimes M$ with $E\in\mathcal{F}$ are projectives in $\langle\mathcal{F}\otimes M\rangle$. Indeed, $\mathrm{Hom}_{\mathfrak{g}}(E\otimes M,N)=\mathrm{Hom}_{\mathfrak{g}}(M,E^*\otimes N)$ is an exact functor in $N\in\langle\mathcal{F}\otimes M\rangle$. So we proved that our functor goes from $\mathcal{H}(I)$ to $\langle\mathcal{F}\otimes M\rangle$, is fully faithful on a system of projective generators of $\mathcal{H}(I)$ and maps those to projective objects in $\langle\mathcal{F}\otimes M\rangle$. The theorem now follows by standard arguments, see for example [BG80], 5.10. □

4. Action of the center

Let $\mathfrak{g}$ for this section be any reductive complex Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let $S = S(\mathfrak{h})$ be the symmetric algebra over $\mathfrak{h}$ and $\hat{S}$ its completion at the maximal ideal generated by $\mathfrak{h}$. This is acted upon by the Weyl group $\mathcal{W}$ and we consider the invariants $\hat{S}^{\mathcal{W}}$.

Let $\mathcal{M} = \mathcal{M}(\mathfrak{g})$ be the category of all representations of $\mathfrak{g}$ that are locally finite over the center Z of $U = U(\mathfrak{g})$. If $E \in \mathfrak{g}$-mod is finite dimensional and $M \in \mathcal{M}$, then $E \otimes M \in \mathcal{M}$. Let $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ be the identity functor. In this section we are going to define a ring homomorphism $\theta : \hat{S}^{\mathcal{W}} \to \mathrm{End}(\mathrm{id}_{\mathcal{M}})$. This gives rise, for every $M \in \mathcal{M}$, to a ring homomorphism $\vartheta_M : \hat{S}^{\mathcal{W}} \to \mathrm{End}_{\mathfrak{g}} M$ and we will prove:

Theorem 4.1. *Let $E \in \mathfrak{g}$-mod be semisimple and finite dimensional, $M \in \mathcal{M}$ arbitrary, $s \in \hat{S}^{\mathcal{W}}$. Then $\mathrm{id}_E \otimes \vartheta_M(s) = \vartheta_{E\otimes M}(s)$ as endomorphisms of $E \otimes M$.*

Remark 4.2. Since $\mathfrak{g}$ was only supposed reductive, there may be finite dimensional representations E that are not semisimple.

We first construct ϑ and then prove the theorem. Certainly $\mathcal{M} = \times_{\chi \in \mathrm{Max}Z} \mathcal{M}_\chi$ where

$$\mathcal{M}_\chi = \{M \in \mathfrak{g}\text{-mod} \mid \forall v \in M\ \exists n > 0 \text{ such that } \chi^n v = 0\}.$$

The notation $\mathcal{M} = \times_\chi \mathcal{M}_\chi$ can be spelled out as follows: For any $M \in \mathcal{M}$ let $M_\chi \subset M$ be the maximal submodule contained in $\mathcal{M}_\chi$. Then $M = \oplus_\chi M_\chi$ and furthermore $\mathrm{Hom}_{\mathfrak{g}}(M, M') = 0$ if $M \in \mathcal{M}_\chi, M' \in \mathcal{M}_{\chi'}$ and $\chi \neq \chi'$. Certainly the completion $Z_\chi^\wedge$ of Z at χ acts on $\mathcal{M}_\chi$. Now we have our bijection $\mathfrak{h}^* \to \mathrm{Max}S$, $\lambda \mapsto \langle\lambda\rangle$. We let $S_\lambda^\wedge$ denote the completion of S at $\langle\lambda\rangle$, so that $\hat{S} = S_0^\wedge$. The Harish-Chandra homomorphism $\xi^\sharp : Z \to S$ induces an inclusion $\xi^\sharp : Z_{\xi(\lambda)}^\wedge \hookrightarrow S_\lambda^\wedge$ for every $\lambda \in \mathfrak{h}^*$. Then translation by λ induces an isomorphism $T_\lambda : S_\lambda^\wedge \xrightarrow{\sim} \hat{S}$. It is clear that $\hat{S}^{\mathcal{W}}$ lies in the image of the composition $T_\lambda \circ \xi^\sharp : Z_{\xi(\lambda)}^\wedge \hookrightarrow \hat{S}$, thus we get an inclusion $\vartheta_\lambda : \hat{S}^{\mathcal{W}} \hookrightarrow Z_{\xi(\lambda)}^\wedge$. It is clear as well that this inclusion depends only on $\xi(\lambda)$, not on λ itself, so for any $\chi \in \mathrm{Max}Z$ we defined an inclusion

$$\vartheta_\chi : \hat{S}^{\mathcal{W}} \hookrightarrow Z_\chi^\wedge.$$

Now for any $M \in \mathcal{M}_\chi$ we define $\vartheta_M : \hat{S}^{\mathcal{W}} \to \mathrm{End}_{\mathfrak{g}} M$ by the prescription that $\vartheta_M(s)$ should be multiplication with $\vartheta_\chi(s)$, and then define ϑ_M for arbitrary $M \in \mathcal{M} = \times_\chi \mathcal{M}_\chi$ in the obvious way. This completes the construction of ϑ. Remark that for more convenience in other parts of our paper we define the Harish-Chandra homomorphism $\xi^\sharp : Z \to S$ in such a way that it actually depends on the choice of a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. However it is clear that ϑ does not depend on this choice.

Proof[Theorem.] Any $M \in \mathcal{M}$ is a quotient of a (possibly infinite) direct sum of objects of the form $U/\chi^n U$, $\chi \in \mathrm{Max}Z$. Thus we only need to prove the theorem for all $M = U/\chi^n U$. Now put $M^i(\lambda) = U \otimes_{U(\mathfrak{b})} S/\langle\lambda\rangle^i$ and $\xi^i(\lambda) = \mathrm{Ann}_Z M^i(\lambda)$. The system of all $\xi^i(\lambda)$ is cofinal to the system of all $(\xi(\lambda))^n$. Thus we only need to prove the theorem for $M = U/\xi^i(\lambda)U$. In [Soe86] it is shown that $U/\xi^i(\lambda)U$ acts faithfully on $M^i(\lambda)$. Hence $U/\xi^i(\lambda)U$ injects into an (infinite) direct product of copies of $M^i(\lambda)$. Thus we only need to show the theorem for $M = M^i(\lambda)$.

Consider more generally any $M \in \tilde{\mathcal{O}} = \{M \in \mathfrak{g}\text{-mod} \mid M$ is finitely generated over $\mathfrak{g}$ and locally finite over $\mathfrak{b}\}$. (This category $\tilde{\mathcal{O}}$ coincides with our $\mathcal{N}(0)$, but we won't use this fact.) The "nilpotent part of the $\mathfrak{h}$-action" gives rise to an action of $\hat{S}$ on M to be denoted $n = n_M : \hat{S} \to \mathrm{End}_{\mathfrak{g}} M$. This is just the ϑ constructed above when we consider M as a module over the reductive Lie algebra $\mathfrak{h}$. More explicitely we decompose M into generalized weight spaces under the action of $\mathfrak{h}$, $M = \oplus_{\mu\in\mathfrak{h}^*} M_\mu$, and define $n(H)(m) = (H - \mu(H))m$ for any $\mu \in \mathfrak{h}^*$, $m \in M_\mu$, $H \in \mathfrak{h}$. For all λ, i the diagram

$$\begin{array}{ccc} \hat{S}^{\mathcal{W}} & \stackrel{\vartheta}{\to} & \mathrm{End}_{\mathfrak{g}} M^i(\lambda) \\ \downarrow & & \| \\ \hat{S} & \stackrel{n}{\to} & \mathrm{End}_{\mathfrak{g}} M^i(\lambda) \end{array}$$

commutes, as one sees by comparing the actions of $n(s)$ and $\vartheta(s)$ on the highest weight space $M^i(\lambda)_\lambda$, for $s \in \hat{S}^{\mathcal{W}}$. On the other hand it is clear that for $M \in \tilde{\mathcal{O}}$, $s \in \hat{S}$ we have $id_E \otimes n_M(s) = n_{E\otimes M}(s) : E \otimes M \to E \otimes M$, since E is semisimple over $\mathfrak{g}$ or, equivalently, over $\mathfrak{h}$. Now for generic $\lambda \in \mathfrak{h}^*$ we have an isomorphism

$$E \otimes M^i(\lambda) \cong \bigoplus_{\nu\in P(E)} M^i(\lambda + \nu)$$

where $P(E) \subset \mathfrak{h}^*$ is the multiset of weights of E. Indeed the tensor identity gives us an isomorphism $E \otimes M^i(\lambda) \cong U \otimes_{U(\mathfrak{b})} (E \otimes S/\langle\lambda\rangle^i)$ and then a filtration of E as a $\mathfrak{b}$-module with subquotients the weight spaces E_ν induces a filtration of the $\mathfrak{g}$-module $E \otimes M^i(\lambda)$ with subquotients the $U \otimes_{U(\mathfrak{b})} (E_\nu \otimes S/\langle\lambda\rangle^i) \cong E_\nu \otimes M^i(\lambda + \nu)$. Since λ is generic, these subquotients have pairwise distinct central character, whence the filtration splits step by step to give the desired direct sum decomposition. Hence for generic λ we have in $\mathrm{End}_{\mathfrak{g}}(E \otimes M^i(\lambda))$ the equalities (abbreviating $M^i(\lambda) = M$)

$$\mathrm{id}_E \otimes \vartheta_M(s) = \mathrm{id}_E \otimes n_M(s) = n_{E\otimes M}(s) = \vartheta_{E\otimes M}(s)$$

for all $s \in \hat{S}^{\mathcal{W}}$. We extend this result by Zariski continuity to all λ. Namely we identify all $S/\langle\lambda\rangle^i$ by translation with $S/(\mathfrak{h})^i$ and then identify all $M^i(\lambda)$ with the vector space $U(\bar{\mathfrak{n}}) \otimes S/(\mathfrak{h})^i$. So all the $E \otimes M^i(\lambda)$ get identified canonically with the vector space $E \otimes U(\bar{\mathfrak{n}}) \otimes S/(\mathfrak{h})^i$, and for $s \in \hat{S}^{\mathcal{W}}$ the endomorphisms $id_E \otimes \vartheta_M(s)$ and $\vartheta_{E\otimes M}(s)$ of $E \otimes M^i(\lambda)$ get identified with certain endomorphisms

$$\phi(\lambda), \psi(\lambda) \text{ of } E \otimes U(\bar{\mathfrak{n}}) \otimes S/(\mathfrak{h})^i.$$

But one sees that $\phi(\lambda)$, $\psi(\lambda)$ are algebraic in $\lambda \in \mathfrak{h}^*$, and since they coincide for generic λ they have to coincide for all λ. □

For later use we record the following fact from folklore.

Lemma 4.3. *Let E be a finite dimensional $\mathfrak{g}$-module, $P(E) \subset \mathfrak{h}^*$ its weights, $\lambda \in \mathfrak{h}^*$, $M \in \mathcal{M}_{\xi(\lambda)}$. Then $E \otimes M \in \bigoplus_{\nu\in P(E)} \mathcal{M}_{\xi(\lambda+\nu)}$.*

Proof. With obvious arguments we reduce to the case $\xi(\lambda)M = 0$. Then $\mathrm{Ann}_U M \supset \mathrm{Ann}_U M(\lambda)$ by Duflo's theorem, hence $\mathrm{Ann}_U(E \otimes M) \supset \mathrm{Ann}_U(E \otimes M(\lambda))$ and the lemma follows. □

5. The main results

Let now again $\mathfrak{g}$ be our semisimple Lie algebra, $\mathcal{H} \subset U$-mod-U its category of Harish-Chandra bimodules. For $\chi \in \mathrm{Max}Z$ put

$$\mathcal{H}_\chi = \{X \in \mathcal{H} \mid X\chi^n = 0 \text{ for } n \gg 0\}.$$

On the other hand put $\mathcal{N}(P, f) = \bigoplus_\eta \mathcal{N}(\eta, f)$ where η runs over all integral elements of $\mathrm{Max}Z$, i.e. over the image $\xi(P)$ under ξ of the lattice of integral weights $P \subset \mathfrak{h}^*$. We will establish an equivalence of categories $\mathcal{H}_{\xi(\mu)} \cong \mathcal{N}(P, f)$ for all dominant $\mu \in P$ such that $\mathcal{W}_f = \{w \in \mathcal{W} \mid w \cdot \mu = \mu\}$.

We even want to prove a more general statement and have to introduce a finer decomposition of our category $\mathcal{N}(f)$. Namely remark that $\mathcal{N}(f) \subset \mathcal{M}(\mathfrak{g}_f)$, hence any $M \in \mathcal{N}(f)$ decomposes into (generalized) eigenspaces under the action of Z_f, say $M = \oplus_\chi M_\chi$ with χ running over $\mathrm{Max}Z_f$. For any coset $\Lambda \in \mathfrak{h}^*/P$ put $\mathcal{N}(\Lambda, f) = \{M \in \mathcal{N}(f) \mid M_\chi \neq 0$ only for $\chi \in \xi_f(\Lambda)\}$. This category is stable under taking tensor products with finite dimensional $\mathfrak{g}$-modules, as follows from Lemma 4.3. For $\mu \in \mathfrak{h}^*$ put $\mathcal{W}_\mu = \{w \in \mathcal{W} \mid w \cdot \mu = \mu\}$. We call μ dominant if and only if $\langle \mu + \rho, \alpha^\vee \rangle \notin \{-1, -2, \dots\}$ for all $\alpha \in R^+$. For $\mu \in \mathfrak{h}^*$ let us define $M^n(\mu, f) = U \otimes_{U(\mathfrak{p}_f)} Y_f(\xi_f(\mu)^n, f)$. These form a projective system in an obvious way. We will prove:

Theorem 5.1. *Suppose $\mu \in \mathfrak{h}^*$ is dominant with $\mathcal{W}_\mu = \mathcal{W}_f$. Then the functor $X \mapsto \varprojlim_n X \otimes_U M^n(\mu, f)$ determines an equivalence of categories $T : \mathcal{H}_{\xi(\mu)} \cong \mathcal{N}(\mu + P, f)$.*

Remark 5.2. The case $f = 0$ is treated in [Soe86].

To prove this theorem, we reduce it to a special case of the main result from Section 3, stated below as Theorem 5.3. Certainly $\mathcal{N} \subset \mathcal{M}(\mathfrak{g}_f)$, hence applying the results of the previous section to $\mathfrak{g} = \mathfrak{g}_f$ we find for any $M \in \mathcal{N}$ a canonical morphism

$$\vartheta = \vartheta_M : \hat{S}^{\mathcal{W}_f} \to \mathrm{End}_{\mathfrak{g}_f} M.$$

Now $\mathfrak{g}$ is a semisimple $\mathfrak{g}_f$-module, and the equality $\vartheta_{\mathfrak{g}\otimes M}(s) = \mathrm{id}_{\mathfrak{g}} \otimes \vartheta_M(s)$ from Theorem 4.1 along with naturality tells us that in fact we constructed a homomorphism

$$\vartheta : \hat{S}^{\mathcal{W}_f} \to \mathrm{End}_{\mathfrak{g}} M.$$

Let $\mathfrak{m} \subset \hat{S}^{\mathcal{W}_f}$ be the maximal ideal and define $\mathcal{N}(f)^n = \{M \in \mathcal{N}(f) \mid \vartheta(\mathfrak{m}^n)M = 0\}$ and $\mathcal{N}(\Lambda, f)^n = \mathcal{N}(\Lambda, f) \cap \mathcal{N}(f)^n$. Our first Theorem 5.1 will follow easily from

Theorem 5.3. *Suppose $\mu \in \mathfrak{h}^*$ is dominant with $\mathcal{W}_\mu = \mathcal{W}_f$. Then the functor $X \mapsto X \otimes_U M^n(\mu, f)$ determines an equivalence of categories $\mathcal{H}(\xi(\mu)^n) \cong \mathcal{N}(\mu + P, f)^n$.*

Remark 5.4. The case $f = 0, n = 1$ is treated in [BG80].

Proof. We will apply Theorem 3.1 with $I = \xi(\mu)^n, M = M^n(\mu, f)$. For this we need

Proposition 5.5. *Suppose $\mu \in \mathfrak{h}^*$ is dominant with $\mathcal{W}_\mu = \mathcal{W}_f$. Then*

1. *$M^n(\mu, f)$ is a projective object in $\mathcal{N}(f)^n$.*
2. *$U/\xi(\mu)^n U \xrightarrow{\sim} (\mathrm{End}_{\mathbb{C}} M^n(\mu, f))_{\mathrm{adf}}$.*

Proof. Postponed. □

Now by Theorem 4.1 and Lemma 4.3 we know that $\mathcal{N}(\mu + P, f)^n$ is stable under tensoring with finite dimensional $\mathfrak{g}$-modules. Hence part 1 of the proposition implies that $\langle \mathcal{F} \otimes M^n(\mu, f)\rangle \subset \mathcal{N}(\mu + P, f)^n$ and that $M^n(\mu, f)$ is projective in $\langle \mathcal{F} \otimes M^n(\mu, f)\rangle$. Using also part two of the proposition we can now apply Theorem 3.1 and deduce that

$$\otimes_U M^n(\mu, f) : \mathcal{H}(\xi(\mu)^n) \to \mathcal{N}(\mu + P, f)^n$$

is a fully faithful functor. It only remains to show that it is essentially surjective. We will do this by counting indecomposable projectives.

In both our categories the indecomposable objects are precisely those with a local endomorphism ring. Our functor being fully faithful, it maps indecomposables to indecomposables and defines an injection of isomorphism classes of objects. Since $\mathcal{H}(\xi(\mu)^n)$ has enough projectives, the (isomorphism classes of) indecomposable projectives and of simple objects correspond bijectively. Recalling the classification of simple objects from [BG80], we see that all indecomposable projectivs in $\mathcal{H}(\xi(\mu)^n)$ are annihilated by a power of $\xi(\lambda)$ for some $\lambda \in \mu + P$, and the isomorphism classes of those are parametrized by the $(\mathcal{W}_\mu)$-orbits in $(\mathcal{W} \cdot \lambda) \cap (\mu + P)$.

Now recall that the projective objects in any $\mathcal{H}(I)$ are just the direct sums of objects of the form $E \otimes U/IU$, $E \in \mathcal{F}$. By the proposition $M^n(\mu, f)$ is projective in $\mathcal{N}(\mu+P, f)^n$, hence so are all the $E \otimes M^n(\mu, f)$ with $E \in \mathcal{F}$, hence our functor maps projective objects to projective objects. But now the simple objects in $\mathcal{N}(\mu + P, f)^n$ annihilated by $\xi(\lambda)$ are parametrized by the $(\mathcal{W}_f\cdot)$-orbits in $(\mathcal{W}\cdot\lambda) \cap (\mu+P)$. (Remark this space is $(\mathcal{W}_f\cdot)$-stable, since $\mathcal{W}_f = \mathcal{W}_\mu$). Just counting we see that $\mathcal{N}(\mu + P, f)^n$ has enough projectives and they are all in the image of our functor. Thus indeed our functor gives an equivalence of categories $\mathcal{H}(\xi(\mu)^n) \xrightarrow{\sim} \mathcal{N}(\mu + P, f)^n$. □

We now prepare the proof of Proposition 5.5. We begin with some lemmas on invariant theory. Recall ϑ_μ from Section 4. We now use it for $\mathfrak{g} = \mathfrak{g}_f$.

Lemma 5.6. *Let $\mu \in \mathfrak{h}^*$ be given with $\mathcal{W}_f \cdot \mu = \mu$. Then $\vartheta_\mu : \hat{S}^{\mathcal{W}_f} \to (Z_f)^\wedge_{\xi_f(\mu)}$ is an isomorphism.*

Proof. This is clear from the definitions. □

Lemma 5.7. *Suppose stronger $\mathcal{W}_f = \mathcal{W}_\mu$. Consider Z as a subring of Z_f via the relative Harish-Chandra homomorphism $\theta^\sharp$. Then $\xi_f(\mu)^n \cap Z = \xi(\mu)^n$.*

Proof. We have to show that $\theta : \mathrm{Spec} Z_f \to \mathrm{Spec} Z$ is étale at $\xi_f(\mu)$. But this is clear from the condition on μ. □

Proposition 5.8. *Let $\mu \in \mathfrak{h}^*$ be dominant and suppose $\mathcal{W}_f \cdot \mu = \mu$. Then $M^n(\mu, f)$ is projective in $\mathcal{N}(f)^n$.*

Proof. We need just to show that it is projective in $\mathcal{N}(\xi(\mu), f)^n$. Choose $N \in \mathcal{N}(\xi(\mu), f)^n$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(M^n(\mu, f), N) &= \mathrm{Hom}_{\mathfrak{p}_f}(Y_f(\xi_f(\mu)^n, f), N) \\ &= \mathrm{Hom}_{\mathfrak{p}_f}(Y_f(\xi_f(\mu)^n, f), N_{\mu^f}) \\ &= \mathrm{Hom}_{\mathfrak{g}_f}(Y_f(\xi_f(\mu)^n, f), N_{\mu^f}), \end{aligned}$$

the first equality by definition of $M^n(\mu, f)$, the second since $\mathfrak{h}^f$ acts via μ^f on Y_f, the third since μ is dominant, thus the weight μ^f is highest possible for $N \in \mathcal{N}(\xi(\mu), f)$, hence N_{μ^f} is annihilated by $\mathfrak{n}^f$.

Let us put $\mathcal{N}_f = \mathcal{N}(\mathfrak{g}_f, \mathfrak{b}_f)$ and for $\eta \in \mathrm{Max} Z_f$ define $\mathcal{N}_f(\eta), \mathcal{N}_f(\eta, f)$ as subcategories of $\mathfrak{g}_f$-mod in the obvious way. Since μ is dominant, N_{μ^f} lies in $\mathcal{N}_f(\xi_f(\mu), f)$. By 5.7 we see that $N \in \mathcal{N}(f)^n$ implies already $\xi_f(\mu)^n N_{\mu^f} = 0$. Now a theorem in [Kos78] tells us that for any $\eta \in \mathrm{Max} Z_f$ we have an equivalence of categories

$$\{M \in (Z_f/\eta^n) - \mathrm{mod} \mid \dim_{\mathbb{C}} M < \infty\} \to \{H \in \mathcal{N}_f(\eta, f) \mid \eta^n H = 0\}$$

given by the functor $M \mapsto (U_f \otimes_{Z_f} M) \otimes_{U(\mathfrak{n}_f)} \mathbb{C}_f$. Thus $Y_f(\eta^n, f)$ is a projective object on the right hand side, and thus $\mathrm{Hom}_{\mathfrak{g}}(M^n(\mu, f), N) = \mathrm{Hom}_{\mathfrak{g}_f}(Y_f(\xi_f(\mu)^n, f), N_{\mu^f})$ is an exact functor when restricted to $N \in \mathcal{N}(f)^n$. Thus it only remains to be shown that $M^n(\mu, f) \in \mathcal{N}(f)^n$. But this is clear from Lemma 5.6. □

We now prove the second part of Proposition 5.5. We begin with

Lemma 5.9. *Let $I \subset Z_f$ be an ideal, $N \in \mathfrak{g}_f$-mod a representation such that $\mathrm{Ann}_{U_f} N = U_f I$. Then $\mathrm{Ann}_U(U \otimes_{U(\mathfrak{p}_f)} N) = U(I \cap Z)$ when we view Z as a subring of Z_f via $\theta^\sharp$.*

Remark 5.10. We extend here the $\mathfrak{g}_f$-action on N to a $\mathfrak{p}_f$-action via the surjection $\mathfrak{p}_f \to \mathfrak{g}_f$ with kernel $\mathfrak{n}^f$.

Proof. Using Lemma 2.2, we may just check on one N. Let us consider Z_f as a subring of S via $\xi_f^\sharp$. By Duflo's theorem (more precisely, its generalization from [Soe86]) we may take $N = U_f \otimes_{U(\mathfrak{b}_f)} (S/IS)$. Then $U \otimes_{U(\mathfrak{p}_f)} N = U \otimes_{U(\mathfrak{b})} S/IS$ and by the generalized Duflo theorem again we conclude $\mathrm{Ann}_U(U \otimes_{U(\mathfrak{p}_f)} N) = U(Z \cap IS)$. But S is faithfully flat over Z_f, hence $Z_f \cap IS = I$, hence $Z \cap IS = Z \cap I$. □

We deduce

Lemma 5.11. *Let $\mu \in \mathfrak{h}^*$ be given with $\mathcal{W}_\mu = \mathcal{W}_f$. Then we have $\mathrm{Ann}_U M^n(\mu, f) = U\xi(\mu)^n$.*

Proof. Apply the previous lemma to $N = Y_f(\xi_f(\mu)^n, f)$ and use Lemma 5.7 to see that $\xi_f(\mu)^n \cap Z = \xi(\mu)^n$. □

So we already have an injection $U/\xi(\mu)^n U \hookrightarrow (\mathrm{End}_{\mathbb{C}} M^n(\mu, f))_{\mathrm{adf}}$. To prove that it is a surjection we compare multiplicities under the adjoint $\mathfrak{g}$-action on both sides. For this we study how our standard modules behave under translation. For $E \in \mathcal{F}$ let $P(E) \subset \mathfrak{h}^*$ be the multiset of weights of E (counted with their multiplicities).

Lemma 5.12. *$E \otimes M(\lambda, f)$ has a filtration with subquotients $M(\lambda + \nu, f), \nu \in P(E)$.*

Proof. If f is regular, thus $M(\lambda, f) = Y(\xi(\lambda), f)$, this was proved by Kostant [Kos78]. In general write

$$E \otimes M(\lambda, f) = U \otimes_{U(\mathfrak{p}_f)} (E \otimes Y_f(\xi_f(\lambda), f)).$$

Now filter $E|_{\mathfrak{p}_f}$ in such a way that $\mathfrak{n}^f$ annihilates the subquotients, and then apply Kostant's result to the Lie algebra $\mathfrak{g}_f$. □

Finally we can prove what we were after.

Proposition 5.13. *Let $\mu \in \mathfrak{h}^*$ be dominant with $\mathcal{W}_\mu = \mathcal{W}_f$. Then the multiplication*

$$U/\xi(\mu)^n U \to (\mathrm{End}_{\mathbb{C}} M^n(\mu, f))_{\mathrm{adf}}$$

is an isomorphism.

Proof. This map is injective by Lemma 5.11. Let E be a finite dimensional simple $\mathfrak{g}$-module and E_0 its zero weight space. We only have to check that E occurs with the same multiplicity on both sides, regarded as $\mathfrak{g}$-modules via the adjoint action. We have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(E, (U/\xi(\mu)^n U)^{\mathrm{ad}}) = (\dim_{\mathbb{C}} Z/\xi(\mu)^n) \cdot (\dim_{\mathbb{C}} E_0)$$

by Kostant's theorem describing U^{ad}. On the other hand

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(E, (\mathrm{End}_{\mathbb{C}} M^n(\mu, f))^{\mathrm{ad}}) = \\ &\qquad = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(M^n(\mu, f), E^* \otimes M^n(\mu, f)) \\ &\qquad = [E^* \otimes M^n(\mu, f) : L(\mu, f)] \end{aligned}$$

since $M^n(\mu, f)$ is the projective cover of $L(\mu, f)$ in $\mathcal{N}(f)^n$. Now $M^n(\mu, f)$ has a filtration with $\dim_{\mathbb{C}}(Z_f/\xi_f(\mu)^n)$ steps where all subquotients are copies of $M(\mu, f)$. Certainly $\dim_{\mathbb{C}}(Z_f/\xi_f(\mu)^n) = \dim_{\mathbb{C}}(Z/\xi(\mu)^n)$. Thus we only have to check the equality

$$[E^* \otimes M(\mu, f) : L(\mu, f)] = \dim_{\mathbb{C}} E_0.$$

This however is clear from Lemma 5.12 since μ is dominant and $\mathcal{W}_f \subset \mathcal{W}_\mu$. □

The proof of Theorem 5.3 is now complete. To deduce Theorem 5.1 we just have to check

Lemma 5.14. *Let $\mu \in \mathfrak{h}^*$ be given with $\mathcal{W}_f = \mathcal{W}_\mu$. For $n > m$ the canonical surjection $M^n(\mu, f) \to M^m(\mu, f)$ has kernel $\xi(\mu)^m M^n(\mu, f)$.*

Proof. Omitted. □

So we finally get for any dominant $\mu \in \mathfrak{h}^*$ with $\mathcal{W}_\mu = \mathcal{W}_f$ our equivalence of categories

$$T : \mathcal{H}_{\xi(\mu)} \xrightarrow{\sim} \mathcal{N}(\mu + P, f).$$

We now investigate the effect of our equivalence on standard modules and simple modules. Recall from [BG80, Jan83] the description of simple objects in $\mathcal{H}_{\xi(\mu)}$. For any $\lambda, \mu \in \mathfrak{h}^*$ one forms the U-bimodule $\mathcal{L}(\lambda, \mu) = \mathrm{Hom}_{\mathbb{C}}(M(\mu), M(\lambda))_{\mathrm{adf}}$. It can be shown that $\mathcal{L}(\lambda, \mu)$ is actually finitely generated, i.e. it is an object of $\mathcal{H}$. It is nonzero if and only if $\lambda + P = \mu + P$. Assume now that μ is dominant. Then $\mathcal{L}(\lambda, \mu) \cong \mathcal{L}(\lambda', \mu)$ if and only if $\mathcal{W}_\mu \cdot \lambda = \mathcal{W}_\mu \cdot \lambda'$, the $\mathcal{L}(\lambda, \mu)$ have unique simple quotients $\bar{\mathcal{L}}(\lambda, \mu)$ for every $\lambda \in \mu + P$, and the $\bar{\mathcal{L}}(\lambda, \mu)$ with λ running over the $(\mathcal{W}_\mu \cdot)$-orbits in $\mu + P$ form a system of representatives for the simple isomorphism classes in $\mathcal{H}_{\xi(\mu)}$.

Proposition 5.15. *Let $\mu \in \mathfrak{h}^*$ be dominant with $\mathcal{W}_\mu = \mathcal{W}_f$. For any $\lambda \in \mu + P$ we have $T\mathcal{L}(\lambda, \mu) \cong M(\lambda, f)$ and $T\bar{\mathcal{L}}(\lambda, \mu) \cong L(\lambda, f)$.*

Proof. Let for any abelian category $\mathcal{A}$ denote $[\mathcal{A}]$ its Grothendieck group. Any object $M \in \mathcal{A}$ determines an element $[M] \in [\mathcal{A}]$. Any exact functor $T : \mathcal{A} \to \mathcal{B}$ to another abelian category gives rise to a group homomorphism $T : [\mathcal{A}] \to [\mathcal{B}]$. In our situation we know already that $T\mathcal{L}(\mu, \mu) \cong U/\xi(\mu)U \otimes_U M(\mu, f) \cong M(\mu, f)$. We know furthermore that our functor T commutes with all functors $E\otimes$ for $(E \in \mathcal{F})$ and with the (left) Z-action on our categories. Now $E \otimes \mathcal{L}(\mu, \mu)$ has a filtration with subquotients $\mathcal{L}(\mu + \nu, \mu)$, ν running over the multiset $P(E)$ of weights of E. By 5.12 we know that $E \otimes M(\mu, f)$ similarily has a filtration with subquotients $M(\mu + \nu, f)$, $\nu \in P(E)$. Since this holds for all E, we deduce for all integral weights $\nu \in P$ the equality

$$T \sum_{w \in \mathcal{W}} [\mathcal{L}(w\nu + \mu, \mu)] = \sum_{w \in \mathcal{W}} [M(w\nu + \mu, f)].$$

If we split it up according to central character and use the isomorphisms $\mathcal{L}(v \cdot \lambda, \mu) \cong \mathcal{L}(\lambda, \mu)$, $M(v \cdot \lambda, f) \cong M(\lambda, f)$ for $v \in \mathcal{W}_\mu = \mathcal{W}_f$, we deduce $|\mathcal{W}_\mu|\ T\ [\mathcal{L}(\lambda, \mu)] = |\mathcal{W}_f|\ [M(\lambda, f)]$ and thus $T[\mathcal{L}(\lambda, \mu)] = [M(\lambda, f)]$ for all $\lambda \in \mu + P$. Choose now representatives $\lambda_1, \ldots, \lambda_n$ of the $(\mathcal{W}_\mu \cdot)$-orbits in $(\mathcal{W} \cdot \lambda) \cap (\mu + P)$ such that $\lambda_i \in \lambda_j - \mathbb{R}_{\geq 0} R^+$ implies $i \geq j$. Then the multiplicity matrices $[\mathcal{L}(\lambda_i, \mu) : \bar{\mathcal{L}}(\lambda_j, \mu)]$ and $[M(\lambda_i, f) : L(\lambda_j, f)]$ are upper triangular with ones on the diagonal, thus the equations $T[\mathcal{L}(\lambda_i, \mu)] = [M(\lambda_i, f)]$ imply $T[\bar{\mathcal{L}}(\lambda_i, \mu)] = [L(\lambda_i, f)]$ and the effect of T on simples is as asserted.

Next we determine the effect of T on standard objects. We claim that for any $N \in \mathcal{N}(\mu + P, f)$ annihilated by some power of $\xi(\lambda)$ and such that $[N : L(\lambda_i, f)] \neq 0$ and $[N : L(\lambda_j, f)] = 0$ for $j < i$ there is a nonzero morphism $M(\lambda_i, f) \to N$. Indeed the conditions on N imply that its $\mathfrak{h}^f$-weight space of weight λ_i^f is not zero, annihilated by $\mathfrak{n}^f$ and isomorphic to $Y_f(\xi_f(\lambda_i), f)$ as a $\mathfrak{g}_f$-module. We apply this to $N = T\mathcal{L}(\lambda_i, \mu)$ and find a nonzero morphism $\varphi : M(\lambda_i, \mu) \to T\mathcal{L}(\lambda_i, \mu)$. By construction this morphism φ has to induce a surjection onto the unique simple quotient $L(\lambda_i, f)$ of $T\mathcal{L}(\lambda_i, \mu)$, thus φ is a surjection itself. Since we know already $[M(\lambda_i, f)] = [T\mathcal{L}(\lambda_i, \mu)]$, this surjection φ has even to be an isomorphism. □

Let us finally fulfill our promise from the introduction. Let us define for $\chi, \eta \in \mathrm{Max} Z$ the category

$$_\chi\mathcal{H}_\eta = \{X \in \mathcal{H} \mid \chi^n X = 0,\ X\eta^n = 0 \text{ for } n \gg 0\}.$$

Remark that for an integral weight $\lambda \in P$ actually

$$\mathcal{N}(\xi(\lambda), f) \subset \mathcal{N}(\lambda + P, f).$$

So if λ, μ are dominant integral weights with λ regular and $\mathcal{W}_\mu = \mathcal{W}_f$, then we find equivalences of categories

$$\begin{array}{rcl} _{\xi(\mu)}\mathcal{H}_{\xi(\lambda)} & \cong & \mathcal{N}(\xi(\mu), 0) \\ _{\xi(\lambda)}\mathcal{H}_{\xi(\mu)} & \cong & \mathcal{N}(\xi(\lambda), f), \end{array}$$

and since the two categories of bimodules can be identified by interchanging the left and the right action via the Chevalley antiautomorphism of $\mathfrak{g}$, we finally find an equivalence

$$\mathcal{N}(\xi(\mu), 0) \cong \mathcal{N}(\xi(\lambda), f).$$

Using the proposition and [Jan83], 6.34 it can be checked that under this equivalence $M(x \cdot \lambda, f)$ corresponds to $M(x^{-1} \cdot \mu)$.

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