Exercise sheet 1

12.05.2020

Exercise 1.1. Let R be a Noetherian and commutative integral domain; for each $f \in R[x_1, ..., x_n]$, denote $U_f := \{x \in R^n | f(x) \neq 0\}$. Show that each Zariski open set in R^n can be written as a finite union of such subsets U_f.

Exercise 1.2. Let k be a field. Show that the proper Zariski closed subsets of k^2 are union of a finite subset and a single $Z(f)$, for some polynomial $f \in k[x_1, ..., x_n] \setminus \{0\}$.
(Hint: use the fact that two coprime polynomials in two variables have at most finitely many common zeros. The proof of this fact can be found in [AL] 2.7.16.)

Exercise 1.3. A subset of a topological space X is called **dense** if its closure is the whole X.

1. Show that a subset of X is dense if and only if its intersection with any nonempty open subset of X is nonempty.
2. Show that, if k is an infinite field, every nonempty Zariski open set in k^n is dense. Deduce that k^n is not Hausdorff with respect to the Zariski topology.
   What about finite fields?
3. Show that, if $k = \mathbb{R},\mathbb{C}$, every nonempty Zariski open set in k^n is dense also in the euclidean topology.
4. Show that, for any field k of characteristic 0, $\mathbb{Z}^n$ is Zariski dense in k^n.

Exercise 1.4. Let k be a field and $\gamma : k \tilde{\to} k$ a field automorphism. Show that the induced map

$$\begin{aligned} k^n &\to k^n \\ (x_1, ..., x_n) &\mapsto (\gamma(x_1), ..., \gamma(x_n)) \end{aligned}$$

is a homeomorphism in the Zariski topology.

Exercise 1.5. Let M be a Noetherian module over a ring R. Show that a surjective R-module endomorphism $\phi : M \to M$ is necessarily an isomorphism. Which "dual" property to Noetherianity should we require on M in order to get the "dual" statement, i.e. that an injective endomorphism $\psi : M \to M$ is necessarily an isomoprhism?