Exercise sheet 10

23.07.2020

In all the following exercises, k will always denote an algebraically closed field.

Exercise 10.1. Consider $M = k[x_1, ..., x_n]$ as a graded module over the same polynomial ring $k[x_1, ..., x_n]$. Compute $\mathrm{hdim}(M)$ and $\mathrm{mult}(M)$.

Exercise 10.2. Let $X \subseteq \mathbb{A}^n_k$ be a closed subset.

1. Show that the subset $X^{\mathrm{sm}} \subseteq X$ of smooth points is always an open subset of X.

2. If $k = \mathbb{C}$ show that, if X is of pure dimension d, X^{sm} carries also a natural structure of $2d$-dimensional real submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$.

Exercise 10.3. Let d be an integer, $d \geq 3$.

1. Show that an irreducible curve of degree d in $\mathbb{P}^2_k$ has at most $\binom{d-1}{2}$ singular points.

   (**Hint:** by contradiction, assume there are $\binom{d-1}{2}+1$ distinct singular points and pick $d-3$ further arbitrary distinct points on the curve. Then find another curve of degree at most $d-2$ passing through all the previous points...and use Bezout's theorem.)

2. Show that a curve of degree d in $\mathbb{P}^2_k$ has at most $\binom{d}{2}$ singular points.
   Then, for each d, find a curve of degree d in $\mathbb{P}^2_k$ having exactly $\binom{d}{2}$ singular points.

Note: for your info, the previous two facts are true also for $d = 1, 2$ (interpreting $\binom{n}{k} = 0$ for $k > n$); the case $d = 1$ is pretty simple, but the case $d = 2$ would need the knowledge of the classification of affine conics in $\mathbb{A}^2_k$ up to isomorphism.

Exercise 10.4. Let $(A, \mathfrak{m})$ be a Noetherian local commutative ring of Krull dimension d, and consider $f_1 \in \mathfrak{m}^{r_1}, \dots, f_d \in \mathfrak{m}^{r_d}$.

1. Show that
$$l(A/\langle f_1, \dots, f_d\rangle) \geq r_1 \cdot ... \cdot r_d \cdot \mathrm{mult}(A).$$

2. If $d = 2$ and A is also regular, show that the above inequality gives $l(A/\langle f_1, f_2\rangle) \geq r_1 r_2$, and the equality holds if and only if $\bar{f}_1 \in \mathfrak{m}^{r_1}/\mathfrak{m}^{r_1+1}$ and $\bar{f}_2 \in \mathfrak{m}^{r_2}/\mathfrak{m}^{r_2+1}$ are coprime in $\mathrm{gr}_{\mathfrak{m}} A$.

   (**Hint:** realize that, thanks to Nakayama's lemma, $\bar{f}_1$ and $\bar{f}_2$ are coprime if and only if $\mathfrak{m}^{r_1+r_2} \subseteq \langle f_1, f_2\rangle$...and use also that, in the coprime case, one has a short exact sequence $A/\mathfrak{m}^{r_1} \oplus A/\mathfrak{m}^{r_2} \hookrightarrow A/\mathfrak{m}^{r_1+r_2} \to A/\langle f_1, f_2\rangle$, where the first map is given by $(a, b) \mapsto af_1 + bf_2$...)