Exercise sheet 2

20.05.2020

Exercise 2.1. Let $k = \bar{k}$ be an algebraically closed field and $X \subseteq k^n$ a closed subspace in the Zariski topology. Generalize the Hilbert's Nullstellensatz to $\mathcal{O}(X)$, i.e. show that for each ideal $J \subseteq \mathcal{O}(X)$ it holds $I(Z(J)) = \sqrt{J}$.
Does that hold if X is not closed?

(Of course here I and Z are intended with respect to X, i.e. $Z(J) := \{x \in X \mid f(x) = 0\ \forall f \in J\}$ and $I(S) := \{f \in \mathcal{O}(X) \mid f|_S \equiv 0\}$, for $J \subseteq \mathcal{O}(X)$ and $S \subseteq X$ resp.).

Exercise 2.2. Let $k = \bar{k}$ be an algebraically closed field and let $X \subseteq k^n$ be a closed subset, regarded as topological subspace with respect to the Zariski topology:

- show that any polynomial map from $X \to k^m$ is continuous with respect to the Zariski topology.
- is any continuous function with respect to the Zariski topology a polynomial function?

Exercise 2.3. Let $k = \bar{k}$ be an algebraically closed field; let $A \in Mat_{n\times n}(k)$, $B \in Mat_{n\times 1}(k)$ and look at $X = (x_1, ..., x_n) \in k^n$ as a column vector. Show that:

1. an affine map
$$\begin{array}{ccc} k^n & \longrightarrow & k^n \\ X & \longmapsto & A \cdot X + B \end{array}$$
is a polynomial map. Furthermore it is an automorphism (i.e. a polynomial invertible map with the inverse being polynomial as well) if and only if $A \in GL_n(k)$.
2. all the automorphisms of k are given by invertible affine maps.
3. the previous point is false for k^n, $\forall n \geq 2$.

Exercise 2.4. Let $k = \bar{k}$ be an algebraically closed field, and $X, Y \subseteq k^n$ closed subsets.

1. Show that it is not always true that given $f \in \mathcal{O}(X)$ and $g \in \mathcal{O}(Y)$ s.t. $f|_{X\cap Y} = g|_{X\cap Y}$ there exists a regular function $h \in \mathcal{O}(X \cup Y)$, extending f and g.
2. Which algebraic/geometric hypotheses should one make on X and Y (or better, on their intersection...) in order to ensure that there exists a regular extension for all f, g with the aforementioned property?

(**Hint/Guide**: the keyword to approach this exercise is "(generalized) Chinese remainder theorem": if R is a commutative ring, $I, J \subseteq R$ ideals, then the map

$$\begin{aligned} R &\to R/I \times R/J \\ r &\mapsto ([r]_I, [r]_J) \end{aligned}$$

has as image exactly the set of pairs $([a]_I, [b]_J) \in R/I \times R/J$ such that $[a]_{I+J} = [b]_{I+J} \in R/(I+J)$.
Prove this result and then apply it to the case $R = k[x_1, ..., x_n]$...)

Exercise 2.5. Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(X, Y)$ be a morphism between two objects of that category. Show that:

1. f is an isomorphism if and only if there exist $g, h \in \mathcal{C}(Y, X)$ such that $f \circ g = id_Y$ and $h \circ f = id_X$, and under these assumptions one necessarily has $g = h$.

2. if there exist $g \in \mathcal{C}(W, X)$ and $h \in C(Y, Z)$ such that $f \circ g$ and $h \circ f$ are isomorphisms, then f is an isomorphism.