Exercise sheet 3

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In all the following exercises, k will always denote an algebraically closed field.

Exercise 3.1.

1. Consider the map
$$\begin{aligned} k &\to k^2 \\ t &\mapsto (t^2, t^3) \end{aligned}$$
Show that this map induces a bijective morphism between k and $Z(x^3 - y^2) \subseteq k^2$, which is not an isomorphism of affine varieties.

2. Assume $\mathrm{char}(k) = p > 0$ and consider the map
$$\begin{aligned} k &\to k \\ x &\mapsto x^p \end{aligned}$$
Show that this map induces a bijective morphism between k and itself, which is not an isomorphism of affine varieties.

Exercise 3.2. Assume $\mathrm{char}(k) = p > 0$. Given an affine k-variety $(X, \mathcal{O}(X))$ define its *Frobenius twist* $(X^{[1]}, \mathcal{O}(X^{[1]}))$ by $X^{[1]} := X$, $\mathcal{O}(X^{[1]}) := \{f^p \mid f \in \mathcal{O}(X)\}$.

1. Show that $\mathcal{O}(X^{[1]})$ is actually a k-algebra and that $(X^{[1]}, \mathcal{O}(X^{[1]}))$ is actually a k-variety.
2. Show that the identity map on X induces a morphism $X \to X^{[1]}$ which is in general not an isomorphism.
3. Show that each morphism $X \to Y$ induces a morphism $X^{[1]} \to Y^{[1]}$ in a natural way, and that the assignment $X \mapsto X^{[1]}$ defines an autoequivalence of the category Varaff_k.

Exercise 3.3. Let $X \subseteq k^n$ and $Y \subseteq k^m$ be closed subsets.

1. Show that if two finitely generated commutative k-algebras A and B are nilpotent free, then also $A \otimes_k B$ is nilpotent free.
(**Hint:** suppose $\alpha \in A \otimes_k B$ is a nilpotent element; after having written it in a convenient way, map it to $A/\mathfrak{m} \otimes_k B$ for each $\mathfrak{m} \subseteq A$ maximal ideal, using the körpertheoretische Nullstellensatz to describe $A/\mathfrak{m}$...).
2. Show that the set $X \times Y \subseteq k^{n+m}$ is closed and that, together with the obvious projections, it gives the product of X and Y in the category $\{\text{algebraic sets in some } k^n\}$.
Deduce at once, using Satz 2.2.4 together with the previous point, that $\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$.

3. Show that the set $X \coprod Y \subseteq k^{\max\{n,m\}+1}$ (defined through the inclusions $k^n \to k^{\max\{n,m\}+1}$, $(x_1,...,x_n) \mapsto (x_1,...,x_n,0,...,0)$ and $k^m \to k^{\max\{n,m\}+1}$, $(y_1,...,y_m) \mapsto (y_1,...,y_m,1,...,1)$) is closed and that, together with the obvious inclusions, it gives the coproduct of X and Y in the category $\{$algebraic sets in some $k^n\}$.
   Deduce at once, using Satz 2.2.4, that $\mathcal{O}(X \coprod Y) \cong \mathcal{O}(X) \times \mathcal{O}(Y)$.

4. Show that the connected components of an algebraic set Z are in bijection with the blocks of $\mathcal{O}(Z)$.
   (a block of a commutative ring R is an indecomposable direct summand of R, regarded as module over itself; for more about blocks see Section 1.2.10).

Exercise 3.4. Let $\varphi : X \to Y$ a morphism of affine varieties.

1. Show that $\varphi^{\#}$ is injective if and only if φ has dense image in Y.

2. Show that $\varphi^{\#}$ is surjective if and only if φ is a closed embedding (i.e. it has closed image and it induces an isomorphism $X \tilde{\to} \varphi(X)$, where $\varphi(X)$ has the structure of variety induced by Y).

Exercise 3.5. Let $\mathcal{C}$ be a category, and let $\varphi : X \to Y$ be a morphism in $\mathcal{C}$:

- φ is called a *monomorphism* if for any $W \in \mathrm{Ob}\,\mathcal{C}$ and any pair of morphisms $f, g \in \mathcal{C}(W, X)$, if $\varphi \circ f = \varphi \circ g$ then $f = g$.

- φ is called an *epimorphism* if for any $W \in \mathrm{Ob}\,\mathcal{C}$ and any pair of morphisms $f, g \in \mathcal{C}(Y, W)$, if $f \circ \varphi = g \circ \varphi$ then $f = g$.

1. Show that if $\mathcal{C} = \mathrm{Ens}$ is the category of sets, a morphism is a monomorphism if and only if it is injective and it is an epimorphism if and only if it is surjective.

2. Show that if $\mathcal{C} = k$-Alg is the category of k-algebras, a morphism is a monomorphism if and only if it is injective, but it could be an epimorphism also without being surjective.

3. Show that an isomorphism is always both a monomorphism and an epimorphism, but give an explicit example of a morphism that is both a monomorphism and an epimorphism, without being an isomorphism.
   (**Hint:** work again in $\mathcal{C} = k$-Alg).

4. Show that under a controvariant equivalence of categories $F : \mathcal{C} \overset{\approx}{\to} \mathcal{D}^{opp}$, monomorphisms and epimorphisms get swapped, i.e. $\varphi \in \mathcal{C}(X, Y)$ is a monomorphism (resp. an epimorphism) if and only if $F(\varphi) \in \mathcal{D}(F(X), F(Y))$ is an epimorphism (resp. a monomorphism).

5. Show that if $\mathcal{C} = \mathrm{Varaff}_k$ is the category of affine varieties over k, a morphism is a monomorphism if and only if it is injective, but it could be an epimorphism also without being surjective.
   More precisely, which geometric property characterizes the epimorphisms in this category?