Exercise sheet 4

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In all the following exercises, k will always denote an algebraically closed field.

Exercise 4.1.

1. Show that for a topological space X the following are equivalent:
   - X is irreducible
   - X is non-empty and each non-empty open subset in X is dense
2. Show that a subspace $Y \subseteq X$ is irreducible if and only if its closure $\bar{Y} \subseteq X$ is irreducible.
3. Show that if $f : X \to Y$ is a continuous map between topological spaces and X is irreducible, then also $f(X)$ is irreducible.
4. Show that for any subspace $Y \subseteq X$ it holds $\mathrm{kdim}(Y) \leq \mathrm{kdim}(X)$; furthermore if X is irreducible and finite dimensional, and Y is closed with $\mathrm{kdim}(Y) = \mathrm{kdim}(X)$, then necessarily $Y = X$.
   Disprove the last statement when X is not irreducible or Y is not closed.

Exercise 4.2.

1. Show that $Z(y - x^2)$ and $Z(xy - 1)$ are irreducible subspaces of $\mathbb{A}^2_k$.
   (For your info: these are all the irreducible conics in the plane up to isomorphism).
2. Show that $Z(x^3 - y^2),\ Z(x^3 + x^2 - y^2) \subseteq \mathbb{A}^2_k$ are irreducible.
3. Show that $Z(x^2 + y^2 - z^2 - 1),\ Z(x^2 - y, x^3 - z) \subseteq \mathbb{A}^3_k$ are irreducible.
4. Describe the irreducible components of $Z(x^2 - yz, xz - x) \subseteq \mathbb{A}^3_k$.

(**Hint:** to test irreducibility one can use various strategies, I'll recall here the *Eisenstein criterion*, which is often useful:
Let R be an UFD and $p(x) = a_n x^n + ... + a_0 \in R[x]$ a primitive (i.e. the gcd of its coefficients is 1) polynomial with coefficients in R.
If there exists a prime ideal $P \subseteq R$ such that

- $a_n \notin P$
- $a_i \in P$, $\forall i = 0, ..., n-1$
- $a_0^2 \notin P$

then $p(x)$ is irreducible in $R[x]$.)

Exercise 4.3.

1. Show that the determinant $det \in k[x_{ij}]_{1\le i,j\le n}$ is an irreducible polynomial.

   (**Hint:** show first that $Z(det) \subseteq \mathbb{A}_k^{n^2}$ is irreducible, since it is the image of the map

   $$\begin{aligned} \mathrm{Mat}_n(k) \times \mathrm{Mat}_n(k) &\longrightarrow \mathrm{Mat}_n(k) \\ (A, B) \qquad &\longmapsto A\mathrm{diag}(1, ..., 1, 0)B \end{aligned}$$

   Now use the Nullstellensatz and some other argument to conclude...)

2. Show that $\mathrm{SL}_n(k) \subseteq \mathbb{A}_k^{n^2}$ is irreducible.

Exercise 4.4. Let X be an affine variety over k, and $Y, Z_1, ..., Z_n \subseteq X$ closed subspaces. Suppose that Z_i is irreducible and that $Z_i \nsubseteq Y$ for all i.
Show then that there exists a regular function $f \in \mathcal{O}(X)$ vanishing identically on Y and not vanishing identically on each Z_i.
(**Hint:** Translate the problem in algebraic terms...).

Exercise 4.5. Let R be a commutative ring, $S \subseteq R$ a subset. Show that:

1. loc: $R \to S^{-1}R$ is injective if and only if S does not contain zero divisors.
2. loc: $R \to S^{-1}R$ is an isomorphism if and only if $S \subseteq R^\times$, i.e. all elements in S are invertible in R.
3. $S^{-1}R = \{0\}$ if and only if $0 \in |S\rangle$.
4. if $I \subseteq R$ is an ideal and $\bar{S} \subseteq R/I$ denotes the image of S under the canonical projection to the quotient, then there is a canonical isomorphism

   $$\bar{S}^{-1}(R/I) \tilde{\to} (S^{-1}R)/(S^{-1}I)$$

5. if R is an integral domain, then inside $\mathrm{Quot}(R)$ holds the equality

   $$R = \bigcap_{\mathfrak{m} \in \mathrm{Max} R} R_{\mathfrak{m}}$$