Exercise sheet 5

16.06.2020

Exercise 5.1. Let R be a commutative ring, $S \subseteq R$ a subset, and $\mathrm{loc} : R \to S^{-1}R$ the localization map.

1. If $I \subseteq R$ is an ideal, show that $S^{-1}I = S^{-1}R$ if and only if $I \cap |S\rangle \neq \emptyset$.
2. During the lectures it has been explained that the map
$$\mathrm{loc}^{-1} : \{\text{ideals in } S^{-1}R\} \to \{\text{ideals in } R\}$$
is an injection, with the map $S^{-1} : \{\text{ideals in } R\} \to \{\text{ideals in } S^{-1}R\}$ as a left inverse.
Show with an example that this map is in general not also a right inverse of loc^{-1} (and so loc^{-1} is in fact not surjective), even if one restricts it to the set of ideals which do not intersect $|S\rangle$.
3. Show that an ideal $I \subseteq R$ lies in the image of loc^{-1} if and only if no element of S is a zero divisor in R/I.

Exercise 5.2. Let R be a commutative ring, $S \subseteq R$ a subset, and $\mathrm{loc} : R \to S^{-1}R$ the localization map.

1. Show that $\sqrt{S^{-1}I} = S^{-1}\sqrt{I}$, for any ideal $I \subseteq R$.
2. If R is a nilpotent-free ring, show that for any $f \in R$ the ideal $(fx - 1) \subseteq R[x]$ is a radical ideal.

   (**Remark:** This could be proven via elementary methods, but let's not allow them here. Rather, I want you to prove it showing that in general $R[x]/(fx-1)$ is isomorphic to a certain localization of R...and then using the previous point).

Exercise 5.3. Let X be an affine variety over $k = \bar{k}$ and $f \in \mathcal{O}(X)$. Show that the map

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{closed irreducible} \\ \text{subsets in } X_f\end{array}\right\} & \to & \left\{\begin{array}{c}\text{closed irreducible} \\ \text{subsets in X, not} \\ \text{contained in } Z(f)\end{array}\right\} \\ \\ Y & \mapsto & \bar{Y} \end{array}$$

is a bijection. What's the inverse map?

Exercise 5.4.

1. Give an example of a ring with only one minimal prime ideal without being an integral domain.
2. Show that a nilpotent-free ring with only one minimal prime ideal is necessarily an integral domain.