Exercise sheet 6

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In all the following exercises, R will always denote a commutative ring with 1, and $S \subseteq R$ a subset in R.

Exercise 6.1. Let M be a R-module.

1. Show that if M is finitely generated, then $S^{-1}M = \{0\}$ if and only if there exists $s \in |S\rangle$ such that $sM = \{0\}$. What if M is not finitely generated?

2. Let $L \to M \to N$ be a sequence of R-modules. Show that it is exact if and only if the corresponding sequence $L_{\mathfrak{m}} \to M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is exact for every maximal ideal $\mathfrak{m} \subseteq R$.

3. Show that localization commutes with finite intersections, i.e. if $M_i \subseteq M$ are submodules, $i = 1, ..., n$, then inside $S^{-1}M$ one gets the equality $S^{-1}(\bigcap_{i=1}^n M_i) = \bigcap_{i=1}^n S^{-1}M_i$. What about infinite intersections?

Exercise 6.2.

1. Let R be and integral domain and suppose $0 \notin S$, $S \nsubseteq R^\times$. Show that $S^{-1}R$ is not a finitely generated R-module.

2. Let M be a finitely generated R-module (R not necessarily integral anymore). Show that a surjective R-module endomorphism $\phi : M \to M$ is necessarily an isomorphism.
Deduce that in a free R-module of rank n, a set of n generators is necessarily a basis.
(**Hint:** Look at M as an $R[x]$ module, defining $x \cdot m := \phi(m)$; the hypothesis now translates to $M = IM$, where $I = (x)$...)

Exercise 6.3. An integral domain R is called *integrally closed* if R coincides with its own integral closure inside its quotient field.

1. Show that if R is an UFD, then it is integrally closed.

2. Show that an integral domain R is integrally closed $\Leftrightarrow$ $R_{\mathfrak{p}}$ is integrally closed for all $\mathfrak{p} \subseteq R$ prime ideals $\Leftrightarrow$ $R_{\mathfrak{m}}$ is integrally closed for all $\mathfrak{m} \subseteq R$ maximal ideals.

3. Describe the integral closures of the rings of regular functions on the Neil'sche parabola and on the nodal cubic.

4. Show that "being an integral extension is something about maps, not about spaces", i.e. find three affine varieties X, Y_1, Y_2 with $Y_1 \cong Y_2$, and dense morphisms $\varphi_i : Y_i \to X$ such that $\mathcal{O}(X) \subseteq \mathcal{O}(Y_1)$ is an integral extension, while $\mathcal{O}(X) \subseteq \mathcal{O}(Y_2)$ is not.