Exercise sheet 7

02.07.2020

Exercise 7.1. Let $k = \bar{k}$ be an algebraically closed field.

1. Let $\varphi : X \to Y$ be a morphism of affine varieties over k. Show that if $\varphi^{\#} : \mathcal{O}(Y) \hookrightarrow \mathcal{O}(X)$ is an integral extension, then φ is a closed surjective map.
   Is the converse true? Is the converse true if one assumes further that the morphism φ has finite fibers?

2. Show (without elementary but annoying computations) that the map

$$\begin{aligned} \varphi : \mathbb{A}^1_k &\to \mathbb{A}^2_k \\ t &\mapsto (t^2 - 1, t(t^2 - 1)) \end{aligned}$$

   is surjective onto $Z(x^3 + x^2 - y^2)$.

Exercise 7.2. Let $k = \bar{k}$ be an algebraically closed field, and X, Y two affine varieties over k.

1. Show that $\mathrm{kdim}(X \times Y) = \mathrm{kdim}(X) + \mathrm{kdim}(Y)$.

2. Given a morphism $\varphi : X \to Y$, assuming X irreducible, show that there exists $V \subseteq \overline{\varphi(X)}$ open and dense such that the equality

$$\mathrm{kdim}(X) = \mathrm{kdim}(\overline{\varphi(X)}) + \mathrm{kdim}(\varphi^{-1}(y))$$

   holds for all $y \in V$.

Exercise 7.3. Let $K \subseteq L \subseteq M$ be fields.

1. Show that $\mathrm{trgr}(M/K) = \mathrm{trgr}(M/L) + \mathrm{trgr}(L/K)$.

2. Assuming L algebraic over K, show that if $x_1, ..., x_n \in M$ are algebraically independent over K, then they are also algebraically independent over L.