Exercise sheet 8

10.07.2020

In all the following exercises, R will always denote a commutative ring and k will always denote an algebraically closed field.

Exercise 8.1. Let R be a kettenlängenring.

1. Show that any localization of R and any quotient of R by a prime ideal are kettenlängenrings as well.
2. Assuming $R \neq \{0\}$ Noetherian, and given $f \in R$ a non-zero-divisor, show that $R/(f)$ is a kettenlängenring as well. Furthermore, if f is not a unit, show that it holds the equality

$$\text{kdim } R/(f) = \text{kdim } R - 1.$$

Exercise 8.2.

1. Assuming that R has only finitely many minimal prime ideals $P_1, ..., P_n$, show that R is normal if and only if the obvious map

$$R \to R/P_1 \times \cdots \times R/P_n$$

is an isomorphism and each R/P_i is integrally closed.

2. An affine variety X is said **normal** if $\mathcal{O}(X)$ is a normal ring.
Show that an affine variety X over k is normal if and only if its irreducible components are all disjoint and each of them has an integrally closed domain as ring of functions.

Exercise 8.3. Let X be an affine variety over k, with $\mathcal{O}(X)$ being an UFD. Let $Y \subseteq X$ be a hypersurface and $Z \subseteq X$ be an irreducible closed subset. Show that if Z does not contain Y, then $Y \cap Z$ is an hypersurface in Z.

Exercise 8.4. Show that if $(U_i)_{i \in I}$ is an open cover of a k-ringed space X, then X carries the final structure with respect to the embeddings $U_i \hookrightarrow X$.
Deduce that, if Y is another k-ringed space, a map $X \to Y$ is a morphism if and only if its restriction to U_i is a morphisms $\forall i \in I$.

Exercise 8.5. Show that for any open subset $U \subseteq \mathbb{A}^n_k$, the ring $\mathcal{O}_{\mathbb{A}^n_k}(U)$ is a finitely generated k-algebra.