Exercise sheet 9

15.07.2020

In all the following exercises, k will always denote an algebraically closed field.

Exercise 9.1. A variety is said **normal** if $\mathcal{O}_{X,x}$ is an integrally closed domain for all $x \in X$.

1. Show that X is normal if and only if any affine open subvariety of X is normal.
2. Show that, if X is affine, this definition coincides with the one given in the previous exercise sheet.
3. Show that any normal variety is the disjoint union of its irreducible components.

Exercise 9.2. Define the **homogenization** map $h_0 : k[t_1, ..., t_n] \to k[x_0, ..., x_n]$ as $h_0(0) = 0$ and $h_0(p) := x_0^{\deg(p)} p(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0})$, if p is non-zero.
Consider then the embedding

$$\begin{aligned} i_0 : \mathbb{A}^n_k &\to \mathbb{P}^n_k \\ (t_1, ..., t_n) &\mapsto [1 : t_1 : ... : t_n] \end{aligned}$$

1. Show that if $I \subseteq k[t_1, ..., t_n]$ is a radical (resp. prime) ideal, then $\langle h_0(I)\rangle \subseteq k[x_0, ..., x_n]$ is a radical (resp. prime) homogeneous ideal.
   (**Hint:** it could be useful to realize that, for homogeneous ideals, one can check radicality (resp. primality) using just homogeneous elements...)
2. Show that, for any ideal $I \subseteq k[t_1, ..., t_n]$, it holds $\overline{i_0(Z(I))} = Z^*(h_0(I))$.
3. Show that in general, if $f_1, ..., f_n \in k[t_1, ..., t_n]$, it is not true that $\overline{i_0(Z(\{f_1, ..., f_n\}))} = Z^*(h_0(f_1), ..., h_0(f_n))$.
   (**Hint:** look, for example, at $Z(x, y - x^2) \subseteq \mathbb{A}^2_k$...)
4. Show that the projective closure of an affine variety strongly depends on the chosen embedding, i.e. find $X = Z(I) \subseteq \mathbb{A}^n_k$, $Y = Z(J) \subseteq \mathbb{A}^m_k$ such that $X \cong Y$ but $\overline{i_0(X)} \ncong \overline{i_0(Y)}$ (even assuming $n = m$).
   (**Hint:** look, for example, at $Z(y - x)$ and $Z(y - x^3) \subseteq \mathbb{A}^2_k$...and use the first point of the previous exercise.)

Exercise 9.3. Let $X \subseteq \mathbb{P}^n_k$, $Y \subseteq \mathbb{P}^m_k$ be two embedded projective varieties.

1. Show that if $\mathcal{O}^*(X)$ and $\mathcal{O}^*(Y)$ are isomorphic as graded rings, then X and Y are isomorphic.
2. Show that if X and Y are isomorphic, $\mathcal{O}^*(X)$ and $\mathcal{O}^*(Y)$ are not necessarily isomoprhic (even as ungraded rings).
   (**Hint:** look, for example, at $\mathbb{P}^1_k$ and $Z(x_0^2 - x_1x_2) \subseteq \mathbb{P}^2_k$...)

Exercise 9.4. Let $\operatorname{char} k = p > 0$ and let $X = (X, \mathcal{O})$ be a variety over k. Define its **Frobenius twist** $X^{[1]} := (X, \mathcal{O}^{[1]})$ via $\mathcal{O}^{[1]}(U) := \{f^p \mid f \in \mathcal{O}(U)\}$.

1. Show that $X^{[1]}$ is actually a variety, and that the identity on X defines a morphism of varieties $X \to X^{[1]}$ that in general is not an isomorphism.

2. Show that each morphism $X \to Y$ naturally induces a morphism $X^{[1]} \to Y^{[1]}$ and that, with this assignment, $X \mapsto X^{[1]}$ defines an auto-equivalence of the category of varieties over k.