A note on bounded hyperimaginaries

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The purpose of this note is to prove the following:

Theorem 1 *Every bounded hyperimaginary is a class of a bounded type–definable equivalence relation.*

Let $\mathbb{C}$ be a big saturated model. An equivalence relation E between tuples of elements of $\mathbb{C}$ is *type–definable* if it is defined by a set of formulas without parameters. The tuples may be infinite, but their length should be smaller than the size of $\mathbb{C}$. E is *bounded* if it has few classes compared with the size of $\mathbb{C}$. Bounded type–definable equivalence relations on a type–definable subclass of $\mathbb{C}$ are defined similarly.

A class of a type–definable equivalence relation is a *hyperimaginary*. A hyperimaginary is *bounded* if it has few conjugates under automorphisms of $\mathbb{C}$. Of course all classes of a bounded type–definable equivalence relation are bounded hyperimaginaries. By the proposed theorem all bounded hyperimaginaries occur in this way.

Let KP denote the finest bounded type–definable equivalence relation[1]. We will prove Theorem 1 in the following equivalent form:

Theorem 2 *Let $\mathbb{D}$ a type–definable subclass of $\mathbb{C}$. Then* $\mathrm{KP} \cap (\mathbb{D} \times \mathbb{D})$ *is the finest bounded type–definable equivalence relation on $\mathbb{D}$.*

To see that this implies Theorem 1 consider a bounded hyperimaginary Ea. The union $\mathbb{D}$ of all conjugates of Ea is type–definable and $E \cap (\mathbb{D} \times \mathbb{D})$ is a bounded type–definable equivalence relation on $\mathbb{D}$. Since KP refines E on $\mathbb{D}$ and since $\mathbb{D}$ is a union of KP–classes

$$F = \mathrm{KP} \cup (E \cap (\mathbb{D} \times \mathbb{D}))$$

defines an equivalence relation, which is bounded, type–definable and satisfies $Ea = Fa$.

[1] KP depends of course on the length of the tuples under consideration. To ease notation we will assume all tuples to be of length 1.

Definition *A reflexive and symmetric relation R on a set C is thick if there is no infinite R–anti-chain, i.e. an infinite set $A \subset C$ such that $\neg R(a,b)$ for all different $a, b \in A$.*

The following lemma is well–known and easy to prove.

Lemma 3 *Let $E(x,y)$ be the intersection of a set $\mathfrak{R}$ of definable reflexive and symmetric relations on $\mathbb{C}$ and $\mathbb{D}$ be the intersection of a set $\mathfrak{P}$ of definable subclasses of $\mathbb{C}$. We assume $\mathfrak{R}$ and $\mathfrak{P}$ to be closed under finite intersections.*

- *$E(x,y)$ is an equivalence relation iff for each $R \in \mathfrak{R}$ there is an $S \in \mathfrak{R}$ such that $S^2 \subset R$.*
- *Assume $E(x,y)$ is an equivalence relation. Then $E(x,y)$ is bounded on $\mathbb{D}$ iff each $R \in \mathfrak{R}$ is thick on some element P of $\mathfrak{P}$.*

To prove Theorem 2 we fix a type–definable class $\mathbb{D}$ and a bounded type–definable equivalence relation on $\mathbb{D}$. Extending by equality on $\mathbb{C} \setminus \mathbb{D}$ we see that this relation has the form $E \cap (\mathbb{D} \times \mathbb{D})$ for a type–definable equivalence relation E.

We write E as the intersection of a set $\mathfrak{R}$ of definable reflexive and symmetric relations and $\mathbb{D}$ as the intersection of definable classes from $\mathfrak{P}$. We can assume that both $\mathfrak{R}$ an $\mathfrak{P}$ are closed under finite intersections. Fix a relation $R \in \mathfrak{R}$. We will find a $P_0 \in \mathfrak{P}$ and a sequence $S_0, S_1, \ldots$ of definable symmetric and reflexive relations with the following properties:

(1) $S_n \cap (P_0 \times \mathbb{C}) \subset R$

(2) $S_{n+1}^2 \subset S_n$

(3) S_n^2 is thick.

The intersection of the S_n is then a bounded type–definable equivalence relation which refines R on $\mathbb{D}$. This shows $\mathrm{KP} \cap (\mathbb{D} \times \mathbb{D}) \subset R$ and proves Theorem 2.

Lemma *There is a $P \in \mathfrak{P}$ such that for all $x \in P$ there is a $y \in \mathbb{D}$ such that $R(x,y)$.*

Proof: Assume not. Then every $P \in \mathfrak{P}$ contains an x which is not R–related to any element of $\mathbb{D}$. By compactness there must be a subset P' of P which belongs to $\mathfrak{P}$ and does not contain elements R–related to x. We start with a set $P_0 \in \mathfrak{P}$ on which R is thick. Then we choose $x_0 \in P_0$ and a subset P_1 without elements R–related to x_0. In P_1 we choose x_1 and P_2 etc. The x_i form an infinite R–antichain, which is impossible.

By the last two lemmas we can find two sequences $R_0 \supset R_1 \supset \cdots$ and $P_0 \supset P_1 \supset \cdots$ of elements of $\mathfrak{R}$ and $\mathfrak{P}$ with the following properties:

(a) $R_0^3 \subset R$

(b) $R_{i+1}^4 \subset R_i$

(c) R_i is thick on P_i

(d) $\mathbb{C} \models \forall x \in P_i \; \exists y \in P_{i+1} \; R_{i+1}(x,y)$

Let x be an element of $\mathbb{C}$. We define recursively

$$\begin{aligned} X_0(x) &= \{z \in P_0 \mid R_0(x,z)\} \\ X_{i+1}(x) &= \{z \in P_{i+1} \mid R_{i+1}(y,z) \text{ for some } y \in X_i(x)\} \end{aligned}$$

and the relations S_n as

$$S_n(x,x') \iff X_n(x) \sim_{R_n} X_n(x'),$$

where $\sim_{R_n}$ is defined as follows:

Definition *Let S be a reflexive and symmetric relation on a set C. For subsets of X, X' of C define*

$$X \sim_S X' \iff \forall x \in X \; \exists x' \in X' \; S(x,x') \quad \wedge \quad \forall x' \in X' \; \exists x \in X \; S(x,x').$$

$\sim_S$ is a reflexive and symmetric relation on the power set of C.

Note that property (d) implies

$$X_i(x) \sim_{R_{i+1}} X_{i+1}(x).$$

Lemma 4 *Let S and S' be reflexive and symmetric relations on C.*

- *$\sim_{SS'}$ is the composition of $\sim_S$ and $\sim_{S'}$.*
- *If S is thick on C the relation $\sim_{S^2}$ is thick on the power set of C.*

Proof: The first claim is easy to see. Actually we will use only the trivial inclusion $\sim_S \sim_{S'} \subset \sim_{SS'}$. For the second claim assume that S is thick. Choose a maximal S–anti-chain A in C. Then every element of C is S–related to an element of A. It follows that for every subset X of C

$$X \sim_S X_A,$$

where $X_A = \{a \in A \mid S(x,a) \text{ for some } x \in X\}$. This implies that an $\sim_{S^2}$–anti-chain cannot have more elements than there are subsets of A.

Finally we show that the S_n have the properties (1)–(3).

(1) Assume $S_n(x,x')$. Let T denote the composition $R_1 R_2 \cdots R_n$. Then

$$X_0(x) \sim_T X_n(x) \sim_{R_n} X_n(x') \sim_T X_0(x'),$$

which implies $X_0(x) \sim_{TR_nT} X_0(x')$. By property (b) we have $TR_n \subset R_0$ (we use only $R_{i+1}^2 \subset R_i$), whence $X_0(x) \sim_{R_0^2} X_0(x')$. Now assume that in addition

$x \in P_0$. Then $x \in X_0(x)$ and we find an $y \in X_0(x')$ such that $R_0^2(x,y)$. Since $R_0(x',y)$ we can conclude that $R_0^3(x,x')$. Whence $R(x,x')$ by (a).

(2) If $S_{n+1}^2(x,x'')$ there is an element x' such that $S_{n+1}(x,x')$ and $S_{n+1}(x',x'')$. We have

$$X_n(x) \sim_{R_{n+1}} X_{n+1}(x) \sim_{R_{n+1}} X_{n+1}(x') \sim_{R_{n+1}} X_{n+1}(x'') \sim_{R_{n+1}} X_n(x'').$$

This implies $X_n(x) \sim_{R_n} X_n(x'')$ by property (b).

(3) By property (c) and the last lemma the relation $\sim_{R_n^2}$ is thick on the power set of P_n. This implies immediately the thickness of S_n.

Note added in June 2000

The theorem appeared already in Theorem 4.18 in *Hyperimaginairies and automorphism groups* by D.Lascar and A.Pillay. The proof given there is, if one circumvents the surrounding theory, the following: Let A be the set of all bounded hyperimaginaries a/E. Define the equivalence relation LP by

$$\mathrm{LP} \iff \mathrm{tp}(x/A) = \mathrm{tp}(y/A).$$

LP is bounded and type–definable. Whence it is refined by KP. Now let E be a type–definable equivalence relation on $\mathbb{D}$. The classes of E are bounded hyperimaginairies. Therefore elements of $\mathbb{D}$ which are LP–equivalent are also E equivalent. It follows that LP and therefore KP refines E on $\mathbb{D}$. In fact LP = KP.

That LP is type–definable is a basic observation: Let a/E be a hyperimaginary. Then x and y have the same type over a/E iff

$$\exists a' \; E(a,a') \wedge \mathrm{tp}(x,a) = \mathrm{tp}(y,a').$$

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