Interpreting in Graphs

Let L and S be two countable relational languages. An *L/S-notion D of definition* consists of an S-formula $\phi(x)$ and a family $(\phi_R)_{R\in L}$ of S-formulas $\phi_R(x_1,\dots x_n)$ where n is the arity of R. Such a D defines for every S-structure N a (possibly empty) L-structure $D(N)$ in the obvious way. Let L_G be the language of graphs, with one binary relation symbol.

We prove the following theorem:

Theorem 1 *For every countable relational language L there is an L/L_G-notion D of definition and a theory T of graphs such that every L-structure M is of the form $D(N)$ for a model N of T. If M is countable there are only countably many such N which are countable.*

Corollary 2 *Vaught's conjecture is true if it is true for complete theories of graphs.*

The proof has two steps: First we show how to define every structure in a unique *many sorted* graph. Then we prove the theorem for many sorted graphs. Here a *many sorted graph* is a directed graph where points and edges are *sorted* i.e. partitioned into countably many sorts where the sort of an edge determines the sorts of points edges of that sort can connect. We assume that no points of the same sort are connected. This notion can be subsumed under the usual notion of many sorted structure if one considers a graph as two-sorted structure consisting of points and edges.

Lemma 3 *For every countable relational language L there is a language S for many sorted graphs, an L/S-notion D of definition and a theory T of many sorted graphs such that every L-structure M is of the form $D(N)$ for a unique model N of T.*

Proof:
Let L be given. S will have the point sorts p and p_R for all relation symbols $R \in L$. The edge sorts are the $e_{R,i}$ where i is a natural number smaller than the arity of R for edges connecting points of sort p with points of sort p_R.

In a many sorted graph N of type S D defines the following L-structure $D(N)$:
The universe consists of the points of sort p. The relation R holds for the tuple $a_1 \dots a_n$ iff there is a b of sort p_R such for $i = 1 \dots n$ b is connected with a_i by an edge of sort $e_{R,i}$.

T says the following:

1. A point b of sort p_R is for every i smaller than the arity of R connected by an edge of sort $e_{R,i}$ to a unique point of type P. b belongs to no other edges.

2. A point of sort p_R is determined by the edges to which it belongs.

That D and T have the properties claimed in the lemma is clear. There results a small problem from the fact that there may be no edges of a given sort. We ignore here this problem. But see 5.

To make the proof more perspicuous we make first more precise what a manysorted graph is. The language L of a manysorted graph consists of a countable set P of sorts of points and a countable set E of sorts of edges. For every sort $e \in E$ two unary function symbols $tail_e$ and $head_e$ are given which map edges of sort e to points of sort t_e and h_e. We assume that always $t_e \neq h_e$. Note that this definition allows multiple edges but no loops.

Remark 4 *The manysorted graph constructed in the proof of 3 has* locally finite type *i.e. given two sorts $p_1, p_2 \in P$ there are only finitely many sorts $e \in E$ of edges which connect points of sort p_1 and p_2.*

Now we have to prove the theorem for many sorted graphs of locally finite type. Let L a language of manysorted graphs of locally finite type. For every $p \in P$ and every $e \in E$ we have to define L_G–formulas ϕ_p and ϕ_e which will define the sorts of a manysorted graph in suitable graphs. Furthermore we must find L_G–formulas $\chi_e(x, y)$ and $\tau_e(x, y)$ which will correspond to $head_e(x) = y$ and $tail_e(x) = y$. For this we fix enumerations (p_i) and (e_j) of P and E respectively.

Let G be a graph. We call a point of valency three an *E-point* and a point of at least valency four a *P-point*. If a is an E-point or a P-point we say that it is of type n if it has a *tail* of length n i.e. if it is the endpoint of a path $ab_0 \ldots b_n c$ where the b_i have valency two and c has valency one other a is of infinite type. Now if $p = p_i$ let $\phi_p(x)$ be the formula saying that x is a P-point of type i (we will also say that x has type p)and if $e = e_j$ let ϕ_e define the set of E-points of type j (where we also say that x has type p). The formulas $\chi_e(x, y)$ resp. $\tau_e(x, y)$ say that x is an E-point of type e, that y is a P-point of type h_e resp. t_e in y and that x and y are connected in G.

This gives already an L/L_G–notion D of definition. (This is not true as it stands: if G is graph then $D(G)$ is a many-sorted graph only iff the formulas χ_e and τ_e define functions on $\phi_e(G)$. But this problem will of course vanish if we compose D with notion of definition given in Lemma 3.)

If M is a manysorted graph we will define a graph G_M such that $M = D(G_M)$. Since G_M should also be a model of the theory T which we will define below the construction is a more complicated as expected.We construct G_M as the last step of an ascending sequence $G^1_M \subset G^2_M \subset G_M$. The points of G^1_M are the points and edges of M. We connect every edge of M with its two endpoints in M. To obtain G^2_M we add for every pair (a, b) of points of M a contable set E_{ab} of new points and connect each element of E_{ab} with a and b. For the final step we make use of the notion of an *infinite* tail. This is an infinite path $ab_0b_1 \ldots$ where all the b_i have valency two. In order to obtain G_M we will glue tails of finite or infinite length to the points of G^2_M. The points of M of sort p_i get a tail of length i, the edges of M of sort e_j get tails of length j and to the elements of E_{ab} we attach infinite tails. It is clear that $M = D(G_M)$.

Finally we have to describe the theory T. G_M should be a model of T. But T should be strong enough to force any other model G of T with $M = D(G)$ to be is very similar to G_M. There are three sets of axioms:

> There are no isolated points. Every point of valency one is connected to a point of valency two. Every point of valency at least three (i.e. an E- or a P-point) is connected to a unique point of valency two. No two points both of valency at least three or both of valency one can be connected by a path of points of valency two. There are no circles of points of valency two.

By these axioms a model G of T is build from its subgraph G' of points of valency at least three by adding to each point exactly one finite or infinite tail (call the result G") and by adding connected components to G'' which are one-sided or two-sided infinite paths. We call this the *ray components* and the *line components*.

> Every E-point a is connected to two P-points b and c. If a is finite type e b and c are of type h_e and t_e. A P-point is - besides with one point of valency two - only connected with E-points.

Now we know that if G is a model of T and if we remove the E- and P-points of infinite type the resulting subgraph G'' is determined by $D(G)$ and the number of vector and line components of G. The next set of axioms implies that each two P-points are jointly connected to infinitely many E-points of infinite type. Let p_{i_1} and p_{i_2} be two point sort. An assume that in L–structures points of that sort cannot be connected be edges of a sort e_j for all j bigger or equal to some j_0. We add then the axioms

Every two P-points a_1 and a_2 of type at least i_1 resp. i_2 (this includes infinite type) are jointly connected to infinitely many E-points of type at least j_0.

Now any model G of T is uniquely determined by $D(G)$, the number of P-points of infinite type, the number of vector components and the number of line components. This proves the theorem.

Remark 5 *Of course the proof remains true if some of the sorts of the manysorted graph M are empty.*

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