An exposition of the compactness of $L(Q^{\mathrm{cf}})$

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Abstract

We give an exposition of the compactness of $L(Q^{\mathrm{cf}})$, for any set C of regular cardinals.

1 Introduction

We present here a new and short exposition of the proof of the compactness of the logic $L(Q^{\mathrm{cf}}_C)$, first-order logic extended by the cofinality quantifier Q^{cf}_C, where C is a class of regular cardinals. The logic and the proof of compactness are due to S. Shelah. The Compactness Theorem was stated and proved in [7], but this article is not self-contained and some fundamental steps of the proof must be found in the earlier article [6]. The interested reader consulting these two articles will soon realise that the structure of the proof is not completely transparent and that to fully understand the details requires a lot of work.

The most popular case of the cofinality quantifier is the logic $L(Q^{\mathrm{cf}}_\omega)$ of the quantifier of cofinality ω, that is, $C = \{\omega\}$. Our motivation comes from the application of $L(Q^{\mathrm{cf}}_\omega)$ in [1] to an old problem on expandability of models. An anonymous referee of a preliminary version of [1] did not accept the validity (in ZFC) of the compactness proof presented in [7], apparently confused by the assumption of the existence of a weakly compact cardinal made at the beginning of the article. The assumption only applies to a previous result on a logic stronger than first-order logic even for countable models.

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Our proof of compactness of $L(Q^{\mathrm{cf}}_C)$ uses some ideas of [7], but it is more in the spirit of Keisler's proof in [4] of countable compactness of the logic $L(Q_1)$ with the quantifier of uncountable cardinality. However we use a simpler notion of weak model. J. Väänänen in the last chapter of [8] offers also a proof of compactness of $L(Q^{\mathrm{cf}}_\omega)$ in Keisler's style, but it is incomplete and only gives countable compactness (see I. Hodkinson's review in [3]).

There are some other proofs in the literature, but also unsatisfactory. The proof by H-D. Ebbinghaus in [2], based on a set-theoretical translation, is just an sketch and the proof of J.A. Makowsky and S. Shelah in [5] only replaces part of Shelah's argument in [7] by a different reasoning and does not include all details.

2 Connections

For a linear ordering $(X,<)$ we use the expressions

$$\exists^{\mathrm{cf}} x\, A(x), \text{ and } \forall^{\mathrm{cf}} x\, A(x)$$

for $\forall x'\, \exists x\ (x' \leq x \wedge A(x))$, and $\exists x'\, \forall x\ (x' \leq x \rightarrow A(x))$, respectively.

Definition. Let X and Y be two linear orderings. A connection between X and Y is a relation $G \subset X \times Y$ with satisfies

$$\exists^{\mathrm{cf}} x\, \forall^{\mathrm{cf}} y\, G(x,y) \text{ and } \tag{1}$$

$$\exists^{\mathrm{cf}} y\, \forall^{\mathrm{cf}} x\, \neg G(x,y). \tag{2}$$

Note that X and Y cannot be connected if X or Y has a last element.

Remark 2.1. *1. If X has no last element, the relation $x \leq y$ connects X with itself.*

2. If G connects X and Y, then $\neg G^{-1} = \{(y,x) \mid \neg G(x,y)\}$ connects Y and X.

3. If G connects X and Y, and H connects Y and Z, then

$$K = \Big\{(x,z)\ \Big|\ \exists y'\ \big(\forall y\ (y' \leq y \rightarrow G(x,y))\ \wedge\ H(y',z)\big)\Big\}$$

connects X and Z.

Proof. We will not make use of this remark, but we give a proof of 3, nevertheless.

$\exists^{\mathrm{cf}} x \forall^{\mathrm{cf}} z\ K(x,z)$: If x' is given, there are x and y' that $x' \leq x$ and $y' \leq y \to G(x,y)$ for all y. If we choose y' large enough, there is also a z' such that $z' \leq z \to H(y',z)$ for all z. This shows that $z' \leq z \to K(x,z)$ for all z.

$\exists^{\mathrm{cf}} z \forall^{\mathrm{cf}} x\ \neg K(x,z)$: If z' is given, we find z, y' and x' such that $z' \leq z$ and and for all x and y we have $y' \leq y \to \neg H(y,z)$ and $x' \leq x \to \neg G(x,y')$. Now this implies that $x' \leq x \to \neg K(x,z)$ for all x. To see this assume $x' \leq x$. We will show that $\forall y\ (y'' \leq y \to G(x,y)) \ \wedge\ H(y'',z)\big)$ is wrong for all y''. Indeed, if $y'' \leq y'$, this follows from $\neg G(x,y')$. And if $y' \leq y''$, we have $\neg H(y'',z)$. ☐

Remark 2.2. *If X and Y are connected by G, then also by*

$$G' = \Big\{(x,y) \ \Big|\ \exists x' \ \big(x \leq x' \wedge \forall y' \ (y \leq y' \to G(x',y'))\big)\Big\}.$$

G' is antitone in x and monotone in y.

Proof. It is easy to see that $G^{\mathrm{anti}} = \big\{(x,y) \ \big|\ \exists x' \ (x \leq x' \wedge G(x',y))\big\}$ connects X and Y and is antitone in x. Now set

$$G' = (\neg((\neg G^{-1})^{\mathrm{anti}})^{-1})^{\mathrm{anti}}.$$

☐

Lemma 2.3. *Two linear orders without last element are connected if and only if they have the same cofinality.*

Proof. If $\mathrm{cf}(X) = \mathrm{cf}(Y) = \kappa$, choose two increasing cofinal sequences $(x_\alpha \mid \alpha < \kappa)$ and $(y_\alpha \mid \alpha < \kappa)$ in X and Y. Then

$$G = \{(x,y) \mid \exists \alpha \ (x \leq x_\alpha \wedge y_\alpha \leq y)\}$$

connects X and Y.[1]

For the converse assume that $\mathrm{cf}(X) = \kappa$, and that G connects X and Y. Choose a cofinal sequence $(x_\alpha \mid \alpha < \kappa)$ in X and elements y_α in Y such that $y_\alpha \leq y \to G(x_\alpha, y)$ for all y. Then the y_α are cofinal in Y. To see this let y be an element of Y. Since the x_α are cofinal, we have $\neg G(x_\alpha, y)$ for some α. It follows that $y < y_\alpha$. ☐

[1]It suffices to assume that the y_α are increasing. Also one can use $G = \{(x_\alpha, y) \mid y_\alpha \leq y)\}$.

Lemma 2.4. *Assume that $G \subset X \times Y$ satisfies*

$$\exists^{\mathrm{cf}} x \, \exists y \, G(x,y) \tag{3}$$
$$\forall y' \, \exists x' \, \forall xy \, (x' \leq x \wedge y \leq y') \to \neg G(x,y). \tag{4}$$

Then $G' = \{(x,y) \mid \exists y' \, (y' \leq y \wedge G(x,y'))\}$ connects X and Y.

Note that a connecting G which is monotone in y satisfies (3) and (4).

Proof. This is a straightforward verification. □

3 The Main Lemma

Consider a L-structure M with two (parametrically) definable linear orderings, $<_\varphi$ and $<_\psi$ of its universe, both without last element. We say that φ and ψ are *definably connected* if there is a definable connection between $(M, <_\varphi)$ and $(M, <_\psi)$.

Lemma 3.1. *If φ and ψ are not definably connected, and c is a new constant, the theory*

$$T' = \mathrm{Th}(M,m)_{m \in M} \cup \{m <_\varphi c \mid m \in M\}$$

does not isolate the partial type $\Sigma(y) = \{n <_\psi y \mid n \in M\}$.

Proof. Assume that $\gamma(c,y)$, for some $L(M)$-formula $\gamma(x,y)$, isolates $\Sigma(y)$ in T'. This means that

1. $T' \cup \{\gamma(c,y)\}$ is consistent.
2. $T' \vdash \gamma(c,y) \to n <_\psi y$ for all $n \in M$.

We show that the relation G defined by $\gamma(x,y)$ has properties (3) and (4) of Lemma 2.4, where $X = (M, <_\varphi)$ and $Y = (M, <_\psi)$. This will contradict the hypothesis of our Lemma.

That $T' \cup \{\gamma(c,y)\}$ is consistent means that for all $m \in M$ the theory $\mathrm{Th}(M,m)_{m \in M}$ does not prove $m \leq_\varphi c \to \neg\exists y \; \gamma(c,y)$, which means that $M \models \exists x (m \leq_\psi x \wedge \exists y \; \gamma(x,y))$. This is exactly condition (3) of 2.4.

That $T' \vdash \gamma(c,y) \to n <_\psi y$ means that there is an $m \in M$ such that $\mathrm{Th}(M,m)_{m \in M}$ proves $(m \leq_\varphi c \wedge \gamma(c,y)) \to n <_\psi y$, which means $M \models \forall xy \; (m \leq_\varphi x \wedge y \leq_\psi n \to \neg\gamma(x,y)$. The existence of such m for all n is exactly condition (4) of 2.4. □

Corollary 3.2. *Assume κ is regular, $|M|, |L| \leq \kappa$, and $<_\varphi$ is a definable linear ordering of M without last element. Then there is an elementary extension N of M such that:*

1. *M is not $<_\varphi$-cofinal in N.*

2. *If $<_\psi$ is a definable linear ordering of M of cofinality κ, and ψ and φ are not definably connected, then M is $<_\psi$-cofinal in N.*

Proof. Let c be a new constant and let $T' = \mathrm{Th}(M, m)_{m \in M} \cup \{m <_\varphi c \mid m \in M\}$. By Lemma 3.1, T' does not isolate any of the types $\Sigma_\psi(y) = \{n <_\psi y \mid n \in M\}$. By the form of the types and regularity of κ, for any $<_\psi$ of cofinality κ the type $\Sigma_\psi(y)$ cannot be isolated neither by means of a set of $< \kappa$ formulas. By the κ-Omitting Types Theorem, there is a model of T' omitting all types $\Sigma_\psi(y)$ for any $<_\psi$ of cofinality κ. This gives the elementary extension N. □

This corollary applies in particular to the case $\kappa = \omega$. Here the assumption on the cofinality of $<_\psi$ is not needed since it is the only possible cofinality in a countable model, and the Omitting Types Theorem used in the proof is the ordinary one for countable languages and countably many non-isolated types.

4 Completeness

For a language L let $L(Q^{\mathrm{cf}})$ be the set of formulas which are built like first-order formulas but using an additional two-place quantifier $Q^{\mathrm{cf}}xy\ \varphi$, for different variables x and y. Let C be class a of regular cardinals and M an L-structure. For a binary relation R on M, we write "$\mathrm{cf}\, R \in C$" for "R is a linear ordering of M, without last element and cofinality in C ".

The satisfaction relation $\models_C$ for L-structures M, $L(Q^{\mathrm{cf}})$-formulas $\psi(\bar{z})$, and tuples $\bar{c}$ of elements of M is defined inductively, where the Q^{cf}-step is

$$M \models_C Q^{\mathrm{cf}}xy\ \varphi(x, y, \bar{c}) \iff \mathrm{cf}\, \{(a, b) \mid M \models_C \varphi(a, b, \bar{c})\} \in C.$$

We say that M is a *C-model of* T, a set of $L(Q^{\mathrm{cf}})$-sentences, if $M \models_C \psi$ for all $\psi \in T$.

A *weak* structure $M^* = (M, \dots)$ is an L^*-structure, where L^* is an extension of L by an n-ary relation R_φ for every $L(Q^{\mathrm{cf}})$-formula $\varphi(x, y, z_1, \dots, z_n)$. Satisfaction is defined using the rule

$$M^* \models Q^{\mathrm{cf}}xy\ \varphi(x,y,\bar{c}) \ \Leftrightarrow\ M^* \models R_\varphi(\bar{c}).$$

In weak structures every $L(Q^{\mathrm{cf}})$-formula is equivalent to a first-order L^*-formula, and conversely. So the $L(Q^{\mathrm{cf}})$-model theory of weak structures is the same as their first-order model theory.

Note that the C-semantics of M is given by the semantics of the weak structure M^* if one sets

$$M^* \models R_\varphi(\bar{c}) \ \Leftrightarrow\ M \models_C Q^{\mathrm{cf}}xy\ \varphi(x,y,\bar{c}).$$

The following lemma is clear:

Lemma 4.1. *A weak structure M^* describes the C-semantics of M if and only if*

$$M^* \models Q^{\mathrm{cf}}xy\ \varphi(x,y,\bar{c}) \ \Leftrightarrow\ \mathrm{cf}\,\{(a,b) \mid M^* \models \varphi(a,b,\bar{c})\} \in C$$

for all φ and $\bar{c}$.

The following property of weak structures M^* can be expressed by a set SA of $L(Q^{\mathrm{cf}})$ sentences (the Shelah Axioms):

> *If the $L(Q^{\mathrm{cf}})(M)$-formula $\varphi(x,y)$ satisfies $M^* \models Q^{\mathrm{cf}}xy\ \varphi(x,y)$ then φ defines a linear ordering $<_\varphi$ without last element. Furthermore, if $\psi(x,y)$ defines a linear ordering $<_\psi$ and $M^* \models \neg Q^{\mathrm{cf}}xy\ \psi(x,y)$, there is no definable connection between $(M,<_\varphi)$ and $(M,<_\psi)$.*

Lemma 4.2. *L-structures with the C-semantics are models of* SA.

Proof. This follows from Lemma 2.3. □

Theorem 4.3. *Let C be a non-empty class of regular cardinals, different from the class of all regular cardinals. An $L(Q^{\mathrm{cf}})$-theory T has a C-model if and only if $T \cup \mathrm{SA}$ has a weak model.*

Proof. One direction follows from Lemma 4.2. For the other direction assume that $T \cup \mathrm{SA}$ has a weak model.

Claim 1: If L is countable, T has a $\{\omega\}$-model of cardinality ω_1.

Proof. Let M_0^* be countable weak model of $T \cup$SA. Consider a linear ordering $<_\varphi$ without last element and $M_0^* \models \neg\, Q^{\mathrm{cf}} xy\, \varphi$. Then by Corollary 3.2 for $\kappa = \omega$ and the axioms SA, there is an elementary extension M_1^* such that M_0 is not $<_\varphi$-cofinal in M_1, but $<_\psi$-cofinal in M_1 for every ψ with $M_0^* \models Q^{\mathrm{cf}} xy\, \psi$. We may assume that M_1^* is countable. Continuing in this manner, taking unions at limit stages, one constructs an elementary chain of countable weak models $M_0^* \prec M_1^* \cdots$ of length ω_1 with union M^*, such that

1. If $<_\varphi$ is a linear ordering of M^* without last element and $M^* \models \neg\, Q^{\mathrm{cf}} xy\, \varphi$, and if the parameters of φ are in M_α, then for uncountably many $\beta \geq \alpha$, M_β is not $<_\varphi$-cofinal in $M_{\beta+1}$.
2. If $M^* \models Q^{\mathrm{cf}} xy\, \psi$, and the parameters of φ are in M_α, then M_α is $<_\psi$-cofinal in M.

It follows that, if $M^* \models \neg\, Q^{\mathrm{cf}} xy\, \varphi$, then either φ does not define a linear ordering without last element, or $<_\varphi$ has cofinality ω_1. And, if $M^* \models Q^{\mathrm{cf}} xy\, \psi$, then $<_\psi$ has cofinality ω. By Lemma 4.1 M is an $\{\omega\}$-model of the $L(Q^{\mathrm{cf}})$-theory of M^*, and whence an $\{\omega\}$-model of T. This proves Claim 1.

Let L' be the extension of L which has for every $L(Q^{\mathrm{cf}})$-formula $\varphi(x, y, \bar{z})$ a new relation symbol V_φ of arity $2 + 2 \cdot |\bar{z}|$. Let SK be the set of axioms which state that if $\varphi(x, y, \bar{c}_1)$ and $\varphi(x, y, \bar{c}_2)$ define linear orderings without last elements, and

$$Q^{\mathrm{cf}} xy\, \varphi(x, y, \bar{c}_1) \;\leftrightarrow\; Q^{\mathrm{cf}} xy\, \varphi(x, y, \bar{c}_2),$$

then $V_\varphi(x, y, \bar{c}_1, \bar{c}_2)$ defines a connection between the two orderings.

Claim 2: $T \cup \mathrm{SA} \cup \mathrm{SK}$ has a weak model.

Proof: By compactness we may assume that L is countable. Then T has an $\{\omega\}$-model M of cardinality ω_1, by Claim 1. If $\varphi(x, y, \bar{c}_1)$ and $\varphi(x, y, \bar{c}_2)$ define linear orderings without last element, and $M \models_C Q^{\mathrm{cf}} xy\, \varphi(x, y, \bar{c}_1) \leftrightarrow Q^{\mathrm{cf}} xy\, \varphi(x, y, \bar{c}_2)$, then the two orderings have the same cofinality, namely ω or ω_1, and there is a connection between them by Lemma 2.3. This proves Claim 2.

To prove the theorem, we choose two regular cardinals λ, κ such that $|L| \leq \kappa$ and either $\lambda \notin C$ and $\kappa \in C$ or conversely. Let M_0^* be a weak model of $T \cup \mathrm{SA} \cup \mathrm{SK}$. It M_0^* is finite, it is a C-model of T for trivial reasons[2]. Otherwise we may assume that M_0^* has cardinality κ and all $L(Q^{\mathrm{cf}})$-definable linear orderings without last element have cofinality κ. Let us first assume that $\lambda \notin C$ and $\kappa \in C$.

Consider an $L(Q^{\mathrm{cf}})$-definable linear ordering $<_\varphi$ without last element and $M_0^* \models \neg\, Q^{\mathrm{cf}} xy\, \varphi$. Then by Corollary 3.2 and the axioms SA, there is an elementary extension M_1^* such that M_0 is not $<_\varphi$-cofinal in M_1, but $<_\psi$-cofinal in M_1 for every $L(Q^{\mathrm{cf}})$-formula ψ with $M_0^* \models Q^{\mathrm{cf}} xy\, \psi$. We may assume that M_1^* has cardinal κ. The axioms SK imply that in M_1^* every $L(Q^{\mathrm{cf}})$-definable linear ordering without last element is connected to a linear ordering defined in M_0, and so has also cofinality κ.

Continuing in this manner, taking unions at limit stages, one constructs an elementary chain of weak models $M_0^* \prec M_1^* \cdots$ of length λ with union M^*, such that

1. If $<_\varphi$ is an $L(Q^{\mathrm{cf}})$-definable linear ordering $<_\varphi$ of M^* without last element and $M^* \models \neg\, Q^{\mathrm{cf}} xy\, \varphi$, and if the parameters of φ are in M_α, then for λ-many $\beta \geq \alpha$, M_β is not $<_\varphi$-cofinal in $M_{\beta+1}$.

2. If $M^* \models Q^{\mathrm{cf}} xy\, \psi$, and the parameters of φ are in M_α, then M_α is $<_\psi$-cofinal in M.

It follows that, if $M^* \models \neg\, Q^{\mathrm{cf}} xy\, \varphi$, then either φ does not define a linear ordering without last element, or $<_\varphi$ has cofinality λ. And, if $M^* \models Q^{\mathrm{cf}} xy\, \psi$, then $<_\psi$ has cofinality κ. By Lemma 4.1 $M \restriction L$ is an C-model of the $L(Q^{\mathrm{cf}})$-theory of M^*, and whence a C-model of T.

The proof in the case $\lambda \in C$ and $\kappa \notin C$ is, mutatis mutandis, the same. $\square$

Corollary 4.4. *For every class C of regular cardinals, the logic $L(Q^{\mathrm{cf}}_C)$ is compact.*

We have always assumed that whenever $Q^{\mathrm{cf}} xy \varphi(x, y, \bar{c})$, the definable ordering $<_\varphi$ linearly orders the universe. This is not exactly the assumption of Shelah in [7]: with his definition $<_\varphi$ linearly orders $\{x \mid \exists y\, \varphi(x, y, \bar{c})\}$, the domain of φ. The results presented here, in particular completeness and compactness, also apply to this modification of the semantics, it suffices to add, for each such φ, new relation symbols R_φ and H_φ, and declare

[2] SA is used here.

that for every $\bar{c}$, $R_\varphi(x, y, \bar{c})$ defines a linear ordering $<'_\varphi$ on the universe and $H_\varphi(x, y, \bar{c})$ connects $<_\varphi$ and $<'_\varphi$. This gives compactness. For the formulation of completeness (Theorem 4.3) one must adapt the axioms SA to the new situation.

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