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ENRIQUE CASANOVAS AND MARTIN ZIEGLER

This short note presents a quick proof of the main result in [1], the existence of universal theories in many cardinals.

We consider (consistent) theories in the logic $L(Q_\omega^{\mathrm{cf}})$, the extension of first-order logic with the quantifier Q_ω^{cf} of cofinality ω. It is a compact logic (cf. [2]). In fact $L(Q_\omega^{\mathrm{cf}})$ is just the main example, but everything holds in any compact logic with finitary syntactical properties like those of first-order logic.

Definition. A theory T is a consistent set of $L(Q_\omega^{\mathrm{cf}})$-sentences, with similarity type (or language) $\tau(T)$. The cardinality $|T|$ of T is the cardinality of $\tau(T)$ if $\tau(T)$ is infinite and ω otherwise.

Definition. A theory T is *existentially closed* (an ec-theory, in short) if whenever a sentence $\sigma(\bar{R}, \bar{S})$ is consistent with T and $\bar{R}$ is a tuple of symbols of $\tau(T)$, and $\bar{R}, \bar{S}$ are all the non-logical symbols of σ, then T contains a sentence $\sigma(\bar{R}, \bar{S}')$ for some tuple $\bar{S}'$.

Clearly, we may assume that the symbols $\bar{S}$ do not occur in τ.

Remark 1 (Lemma 2.3 in [1])**.** *Every existentially closed theory T is an* amalgamation basis. *This means that whenever T_1 and T_2 are extensions of T with $\tau(T) = \tau(T_1) \cap \tau(T_2)$, then $T_1 \cup T_2$ is consistent.*

Note that an amalgamation basis is automatically complete.

Proof. Consider two finite conjunctions $\sigma_i(\bar{R}, \bar{S}_i)$ of sentences in T_i, with $\bar{R}$ in $\tau(T)$ an $\bar{S}_i$ in $\tau(T_i) \setminus \tau(T)$, for $i = 1, 2$. Then T contains for $i = 1, 2$ a sentence $\sigma_i(\bar{R}, \bar{S}_i')$. This shows that $\sigma_1(\bar{R}, \bar{S}_1) \wedge \sigma_2(\bar{R}, \bar{S}_2)$ is consistent. $\square$

Lemma 2. *Any theory T can be extended to an existentially closed theory T' with the same cardinality. If $T'' \supseteq T$ is ec, we may find such T' as a subset of T''.*

Proof. Let κ be the cardinality of T. We start with $T_0 = T$. Using the compactness of $L(Q_\omega^{\mathrm{cf}})$ it is easy to construct an extension T_1, of cardinality κ, such that for every sentence $\sigma(\bar{R}, \bar{S})$, with $\bar{R}$ in $\tau(T_0)$ and $\bar{S}$ disjoint from $\tau(T_1)$ and which is consistent with T_1, there is

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some $\sigma(\bar{R}, \bar{S}')$ in T_1. If we continue in this way, we obtain a sequence $T_0 \subseteq T_1 \subseteq T_2 \cdots$ whose union T' is ec and of cardinality κ.

If T is already contained in an ec theory T'', we can find a extension T' between T and T'' of cardinality κ such that for all $\sigma(\bar{R}, \bar{S})$, with $\bar{R}$ in $\tau(T')$, and $\bar{S}$ disjoint from $\tau(T'')$ the following holds

(1) if $\sigma(\bar{R}, \bar{S})$ is consistent with T'', then some $\sigma(\bar{R}, \bar{S}')$ is in T',
(2) if $\sigma(\bar{R}, \bar{S})$ is not consistent with T'', then $\sigma(\bar{R}, \bar{S})$ is not consistent with T'.

Obviously, T' is ec. □

Definition. Let T be a theory, let κ be a cardinal number. We say that T is κ-*existentially closed* (a κ-ec-theory) if whenever $\Sigma(\bar{R}, \bar{S})$ is a set of sentences consistent with T and $\bar{R}$ is a tuple (perhaps infinite) of symbols of $\tau(T)$ with $|\Sigma| \leq \kappa$ and $\bar{R}, \bar{S}$ are all the non-logical symbols of Σ) then there is a tuple $\bar{S}' \in \tau(T)$ such that $\Sigma(\bar{R}, \bar{S}') \subseteq T$. Note that $< \omega$-existentially closed means existentially closed.

This can be rephrased as follows. Let T_0 be contained in T and T' an extension of T_0. Assume $T' \cup T$ is consistent, and $|T'| \leq \kappa$. Then there is an *embedding* $T' \to T$ over T_0, that is, a mapping induced by a one-to-one mapping $\tau(T') \to \tau(T)$ preserving arities of symbols and fixing each symbol in $\tau(T_0)$.

Lemma 3. *If $|T| = 2^\lambda$, there is some λ-ec-theory $T' \supseteq T$ such that $|T'| = 2^\lambda$.*

Proof. Starting with $T_0 = T$ we construct a continuous ascending chain $(T_i \mid i < \lambda^+)$ of ec-theories T_i of cardinality 2^λ with the property that whenever $\Sigma(\bar{R}, \bar{S})$ is as in the definition above, with $|\Sigma| \leq \lambda$, $\bar{R}$ in $\tau(T_i)$ and $\bar{S}$ disjoint with $\tau(T_i)$, then for some $\bar{S}'$, $\Sigma(\bar{R}, \bar{S}') \subseteq T_{i+1}$. The number of sets Σ that we need to consider is 2^λ and we can assume that the common language of each two such sets is $\tau(T_i)$. Let $(\Sigma_j \mid j < 2^\lambda)$ enumerate these sets. Since T_i is an amalgamation basis, $T_i \cup \bigcup_{j<2^\lambda} \Sigma_j$ is consistent and can be extended to some ec-theory T_{i+1} of cardinality $\leq 2^\lambda$. Then $T' = \bigcup_{i<\lambda^+} T_i$ has the required properties. □

Proposition 4. *Let κ be uncountable and T be the union of an ascending sequence $(T_\lambda \mid \lambda < \kappa)$, where λ ranges over infinite cardinal numbers, and each T_λ is λ-ec. Let T^* be an amalgamation basis, contained in T_ω and of smaller cardinality than κ. Then every extension T' of T^*, of cardinality at most κ, is embeddable in T over T^*.*

Proof. By Lemma 2, we can assume that T' is ec and that $T' = \bigcup_{i<\kappa} T'_i$, where $T^* \subseteq T'_0$, each T'_i is ec, $|T'_i| < \kappa$ and $(T'_i \mid i < \kappa)$ is a continuous ascending chain. We inductively construct a continuous ascending chain $(f_i \mid i < \kappa)$ of embeddings $f_i : T'_i \to T$ over T^* such that $\mathrm{rng}(f_i) \subseteq T_{|T'_i|}$. Let $\mu_i = |T'_i|$. Since T_{μ_0} is μ_0-ec, it is easy to get $f_0 : T'_0 \to T_{\mu_0}$. In the limit case we take the union.

Consider now the case T_{i+1}. We have a surjective embedding $f_i : T'_i \to T^0 \subseteq T_{\mu_i}$ over T^*. Extend f_i to a bijective map f^1 between T'_{i+1} and an extension T^1 of T^0. We may assume that $\tau(T^1)$ and $\tau(T)$ intersect in $\tau(T^0)$. Since T^0 is an amalgamation basis, $T^1 \cup T_{\mu_{i+1}}$ is consistent. So there is an embedding $g : T^1 \to T_{\mu_{i+1}}$ over T^0. The composition $f_{i+1} = g \circ f_i$ is our extension of f_i. Clearly, $f = \bigcup_{i<\kappa} f_i$ is an embedding of T' on T over T^*. □

Definition. A theory T is universal over the subtheory T^* if every $T' \supseteq T^*$ of cardinality $\leq |\tau(T)|$ is embeddable in T over T^*.

Corollary 5 ([1] Theorem 2.7). *Assume $\kappa = 2^{<\kappa}$. Any amalgamation basis T^* of cardinality $< \kappa$ can be extended to a theory T of cardinality κ which is universal over T^*.*

Proof. By Lemma 3 and the choice of κ, we may obtain an ascending chain $(T_\lambda \mid \lambda < \kappa)$ (λ ranging over infinite cardinal numbers) such that $T^* \subseteq T_\omega$ and each T_λ is a λ-ec theory of cardinality $2^\lambda \leq \kappa$. Then, by Proposition 4, $T = \bigcup_{\lambda<\kappa} T_\lambda$ is universal. □

Let M be structure of cardinality κ with language $\tau(M) \leq \kappa$. We say that is *compactly expandable* if for every theory T with $\tau(M) \subset \tau(T)$ and $|T| \leq \kappa$ can be realised in an expansion of M, whenever every finite subset of T can be realised in an expansion of M. Note that then any restriction of M to a sublanguage of $\tau(M)$ is also compactly expandable.

We have now:

Remark 6 ([1]). *Let T be universal over T^*, and $|T| = \kappa$, and M a model of T of cardinality κ. Then the restriction of M to $\tau(T^*)$ is compactly expandable.*

References

[1] E. Casanovas and S. Shelah. Universal theories and compactly expandable models. *The Journal of Symbolic Logic*, 84:1215–1223, 2019.

[2] E. Casanovas and M. Ziegler. An exposition of the compactness of $L(Q^{cf})$, `https://arxiv.org/abs/1903.00579`, submitted, 2019.