Fusion of structures of finite Morleyrank*

Martin Ziegler

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Abstract

Let T_1 and T_2 be two countable complete theories in disjoint languages, of finite Morley rank, the same Morley degree, with definable Morley rank and degree. Let N be a common multiple of the ranks of T_1 and T_2. We show that $T_1 \cup T_2$ has a nice complete expansion of Morley rank N.

1 Introduction

We call a countable complete L–theory T *good* if it has finite definable rank[1] $n > 0$ and definable degree[2]. A *conservative* expansion T' of T is a complete expansion of T, whose rank n' is a multiple of n, such that for all L–formulas $\phi(x,b)$.

$$\begin{aligned} \mathrm{MR}_{T'}\,\phi(x,b) &= \frac{n'}{n}\,\mathrm{MR}_T\,\phi(x,b) \\ \mathrm{MD}_{T'}\,\phi(x,b) &= \mathrm{MD}_T\,\phi(x,b). \end{aligned}$$

We call the quotient $\frac{n'}{n}$ the *index* of the expansion.

In this note we will prove the following theorem.

Theorem 1.1. *Let T_1 and T_2 be two good theories in disjoint languages of the same degree e and let N be a common multiple of their ranks. Then T_1 and T_2 have a common good conservative expansion T of rank N.*

Furthermore, if in T_i the predicates $P_i^1, \ldots, P_i^e$ define a partition of the universe into sets of degree 1, T can be chosen to imply $P_1^j = P_2^j$ for $j = 1, \ldots, e$.

If both, T_1 and T_2, have rank and degree 1, this is Hrushovski's fusion [5], except that we allow the language of T to be larger than the union of the languages of T_1 and T_2. The core of our proof is an adaption of the exposition of Hrushovski's fusion given in [3] and (in Section 2.2) of ideas from Poizat's [6].

As an immediate application we get an explanation of the title of Poizat's [6]:

Corollary 1.2 ([6],[1])**.** *In any characteristic there is an algebraically closed field K with a subset N such that (K,N) has rank 2.*

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[1]By "rank" we always mean "Morley rank", "degree" is "Morley degree".

[2]I.e. the DMP, the definable multiplicity property.

Proof. Apply 1.1 for T_1 the theory of algebraically closed fields of some fixed characteristic and for T_2 any good theory of rank 2 and degree 1, e.g. the "square of the identity". $\square$

For another account of 1.2 see [2].

Theorem 1.1 was motivated by the following surprising result of A. Hasson:

Corollary 1.3 ([4]). *Every good theory can be interpreted in a good strongly minimal set.*

Proof. Let T_1 be a good theory of rank n and degree e. Consider any good theory T_2 of rank n and degree e which can be interpreted in a strongly minimal set X defined in T_2. Use 1.1 to obtain a good theory T of rank n which conservatively expands T_1 and T_2. T_2 is then interpreted in X, which is still strongly minimal in T. $\square$

The simplest example of a theory T_2 as used in the above proof is the "disjoint union of e–copies of the n–th power of the identity": Let X be an infinite set, $Y_1,\ldots,Y_e$ be disjoint of copies of X^n and Δ the diagonal of Y_1. Then consider the structure

$$(M, Y_1, \ldots, Y_e, \Delta, f_1, \ldots, f_e)$$

where M is the disjoint union of the Y_j and f_j is the canonical bijection between Δ^n and Y_j.

The above proof shows that *every good theory of rank n and degree e with a partition $P_1 \cup \cdots \cup P_e$ into definable sets of degree* 1 *has a good conservative expansion of index* 1 *which contains a strongly minimal set X such that each P_j is in definable bijection with X^n*. This yields

Corollary 1.4. *Let T be a good theory and X and Y be two sets of maximal rank and the same degree. Then T has a good conservative expansion of index* 1 *with a definable bijection between X and Y.* $\square$

Let T be a good theory of rank N with a definable bijection between the universe and the N–th power of a strongly minimal set X. Then the rank of every good expansion of T is a multiple of N. This shows that in Theorem 1.1 one has to assume that N is a common multiple of the ranks of T_1 and T_2, even if one is not interested in the conservativeness of the expansions. A contrasting example is the case where the languages of the T_i have only unary predicates. Then the rank of a completion of $T_1 \cup T_2$ is bounded by $\mathrm{MR}(T_1) + \mathrm{MR}(T_2) - 1$. So, in 1.1, one has in general to increase the language to find an expansion whose rank is a common multiple of the ranks of T_1 and T_2.

I don't know if the last corollary remains true, if one assumes only that X and Y have the same rank (and degree). The following theorem can be used to prove a weaker result.

Theorem 1.5. *Let T be a two-sorted theory with sorts Σ_1 and Σ_2. Let $T_1 = T \restriction \Sigma_1$ be the theory of the full induced structure on Σ_1 and T_1^* a conservative expansion of T_1 of index* 1*. Assume that T and T_1^* have definable finite rank. Then $T^* = T_1^* \cup T$ is a conservative expansion of T of index* 1 *which has again definable rank.*

There are examples where T and T_1^* have the DMP, but T^* has not.

Corollary 1.6. *Let T be a good theory and X and Y be two sets of the same rank and the same degree. Then T has a conservative expansion of T^* of index 1 with a definable bijection between X and Y. T^* has definable rank.*

Proof. Let T' be the following (good) theory with sorts Σ_1 and Σ_2: Σ_2 is a model of T; Σ_1 is a disjoint union of two predicates X' and Y'; there are bijections between X and X' and between Y and Y'. In $T_1' = T' \restriction \Sigma_1$, X' and Y' have maximal rank and same degree. By 1.4 T_1' has a good conservative expansion $T_1'^*$ of index 1 with a definable bijection between X' and Y'. $T^* = (T' \cup T_1'^*) \restriction \Sigma_2$ has the required properties. □

In [4, Theorem 18] it is proved that for any m and n, any two good strongly minimal sets can be glued together to form a two–sorted structure, where both sets have rank one and there is a definable m-to-n function between them. By Remark 3 of [4] the proof "generalizes to finite-rank". A. Hasson has told me that the generalized proof shows that the union of two good theories of finite rank has a completion of finite rank. Since here the theories may have different degree, the expansions are in general not conservative.

2 Proof of Theorem 1.1

Theorem 1.1 follows from the next theorem, which we will prove in this section.

Theorem 2.1. *Let T_1 and T_2 be to good theories in disjoint languages L_1 and L_2 with ranks $N_1 \leq N_2$ and of degree e, and N be the least common multiple of N_1 and N_2. In T_i let the predicates $P_i^1, \ldots, P_i^e$ define a partition of the universe into sets of degree 1. Assume also that T_1 satisfies*

(*) *If N_1 divides $N_2 = N$, then each non-algebraic element is interalgebraic with infinitely many other elements. Otherwise, the universe is is a union of infinite $\emptyset$–definable $\mathbb{Q}$–vector spaces $V_0, \ldots, V_l$.*

Then $T_1 \cup T_2$ has a completion T of rank N which implies $P_1^j = P_2^j$ and is a good conservative expansion of T_1 and T_2.

Proof of 1.1. Denote the construction in 2.1 by $T_1 + T_2$. Let now T_1 and T_2 be as in 1.1. By adding constants we may assume that the predicates P_i^j are present. Let T_0 be the theory of the disjoint union of e infinite $\mathbb{Q}$–vector spaces. T_0 has rank 1 and degree e. Let N' be the least common multiple of the ranks of T_1 and T_2. Then

$$T' = (T_0 + T_1) + T_2$$

is a good conservative expansion of $T_1 \cup T_2$ of rank N'. Finally set $T = T' + T_3$ for any good theory T_3 of rank N and degree e. □

Actually we need the proposition only in the case that N_1 divides N_2. We have stated it in stronger form, since the proof can be given by a direct application of Hrushovski's fusion machinery to T_1 and T_2.

It is easy to see that, by naming parameters[3], we may assume the following.

(**) If $N_1 = N_2$, for each j, the theory T_2 has infinitely many 1–types over $\emptyset$ of rank $N_2 - 1$ which contain $P_2^j(x)$.

2.1 Hrushovki's machinery

In this section we will develop the theory without using the assumptions (*) and (**). This is a straightforward[4] generalization of sections 2–6 of [3]. We will omit most of the proofs.

2.1.1 Codes (see [3], Section 2)

Let T be a good theory of degree e with predicates $P^1, \ldots, P^e$ which define a partition of the universe in sets of degree 1. We call a formula $\chi(x, b)$ *simple*, if

- it has degree 1,
- the components of a generic realization are pairwise different and not algebraic over b.

A *code* c is a parameter-free formula

$$\phi_c(x, y),$$

where $|x| = n_c$ and y lies in some sort of T^{eq}, with the following properties.

(i) $\phi_c(x, b)$ is either empty[5] or simple. Furthermore there are indices $e_{c,i}$ such that $\phi_c(x, y)$ implies that the x_i are pairwise different and $P^{e_{c,1}}(x_1) \wedge \cdots \wedge P^{e_{c,n_c}}(x_{n_c})$.

(ii) All non-empty $\phi_c(x, b)$ have Morley rank k_c and Morley degree 1.

(iii) For each subset s of $\{1, \ldots, n_c\}$ there exists an integer $k_{c,s}$ such that for every realization a of $\phi_c(x, b)$

$$\mathrm{MR}(a/ba_s) \leq k_{c,s},$$

and equality holds for generic a.[6]

(iv) If both $\phi_c(x, b)$ and $\phi_c(x, b')$ are non-empty and $\phi_c(x, b) \sim^{k_c} \phi_c(x, b')$[7], then $b = b'$.

Lemma 2.2. *Let $\chi(x, d)$ be a simple formula. Then there is some code c and some $b_0 \in \mathrm{dcl}^{\mathrm{eq}}(d)$ such that $\chi(x, d) \sim^{k_c} \phi_c(x, b_0)$.*

We say that c *encodes* $\chi(x, d)$.

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[3] We can forget the new constants after the construction of T. So, the language is not increased.

[4] For the convenience of the reader many definition and statements are copied verbatim from [3].

[5] We assume that $\phi_c(x, b)$ is non-empty for some b.

[6] $a_s = \{a_i | i \in s\}$

[7] This means that the Morley rank of the symmetric difference of $\phi_c(x, b)$ and $\phi_c(x, b')$ is smaller than k_c.

Proof. As the proof of [3, 2.2]. Note is that, by definability of rank, the rank is *additive*

$$\mathrm{MR}(ab/B) = \mathrm{MR}(a/Bb) + \mathrm{MR}(b/B).$$

(see e.g. [7, 4.4]). □

Let c be a code, $\phi_c(x, b)$ non-empty and $p \in \mathrm{S}(b)$ the (stationary) type of rank k_c determined by $\phi_c(x, b)$. (iv) implies that b is the canonical base of p. Hence, b lies in the definable closure of a sufficiently large segment of a Morley sequence of p (which we call a *Morley sequence of* $\phi_c(x, b)$.) Let m_c be some upper bound for the length of such a segment. Note that one can always bound m_c by the rank of the sort of y in $\phi_c(x, y)$.

Lemma 2.3. *For every code c and every integer $\mu \geq m_c - 1$ there exists some formula $\Psi_c(x_0, \ldots, x_\mu, y)$ without parameters satisfying the following:*

(v) Given a Morley sequence $e_0, \ldots, e_\mu$ of $\phi_c(x, b)$, then $\models \Psi_c(e_0, \ldots, e_\mu, b)$.

(vi) For all $e_0, \ldots, e_\mu, b$ realizing Ψ_c the e_i's are pairwise disjoint realizations of $\phi_c(x, b)$.

(vii) Let $e_0, \ldots, e_\mu, b$ realize Ψ_c. Then b lies in the definable closure of any m_c many of the e_i's.

We say for $\Psi_c(x_0, \ldots, x_\mu, y)$ that "$x_0, \ldots, x_\mu$ is a *pseudo Morley sequence of c over y*".

Proof. As the proof of [3, 2.3]. □

We choose for every code (and every μ) a formula Ψ_c as above.

Let c be a code and σ some permutation of $\{1, \ldots, n_c\}$. Then c^σ defined by

$$\phi_{c^\sigma}(x^\sigma, y) = \phi_c(x, y)$$

is also a code. Similarly,

$$\Psi_{c^\sigma}(\bar{x}^\sigma, y) = \Psi_c(\bar{x}, y)$$

defines a pseudo Morley sequence of c^σ.

We call two codes c and c' *equivalent* if $n_c = n_{c'}$, $m_c = m_{c'}$ and

- for every b there is some b' such that $\phi_c(x, b) \equiv \phi_{c'}(x, b')$ and $\Psi_c(\bar{x}, b) \equiv \Psi_{c'}(\bar{x}, b')$ in T,
- similarly permuting c and c'.

Theorem 2.4. *There is a collection of codes C such that:*

(viii) Every simple formula can be encoded by exactly one $c \in C$.

(ix) For every $c \in C$ and every permutation σ, c^σ is equivalent to a code in C.

Proof. As the proof of [3, 2.4]. Note that we may have to change the Ψ_c. □

2.1.2 The δ–function (see [3], Section 3)

Let T_1 and T_2 be two good theories as in Theorem 1.1. We assume that the T_i has quantifier elimination in the relational language L_i. To deal with the predicates P_i^j in an effective way we replace both P_1^j and P_2^j by P^j. Then L_1 and L_2 intersect in $L_0 = \{P_1, \ldots, P_e\}$ and T_1 and T_2 intersect in the theory of a partition of the universe into e infinite sets.

Define $\mathcal{K}$ to be the class of all models of $T_{1,\forall} \cup T_{2,\forall}$. We allow also $\emptyset$ to be in $\mathcal{K}$.

Let N_i be rank of T_i, $N = \mathrm{lcm}(N_1, N_2)$ and $N = \nu_1 N_1 = \nu_2 N_2$. We define for finite $A \in \mathcal{K}$

(2.1) $\delta(A) = \nu_1 \,\mathrm{MR}_1(A) + \nu_2 \,\mathrm{MR}_2(A) - N \cdot |A|$.

By additivity of rank δ has the following properties.

(2.2)
$$\delta(\emptyset) = 0$$

(2.3)
$$\delta(\{a\}) \leq N \quad \text{for single elements } a$$

(2.4)
$$\delta(A \cup B) + \delta(A \cap B) \leq \delta(A) + \delta(B)$$

(2.3) is a special case of

(2.5) $\delta(a/B) \leq \nu_i \,\mathrm{MR}_i(a/B), \quad (i = 1, 2)$,

which holds for arbitrary tuples a.

If $A \setminus B$ is finite, we set

$$\delta(A/B) = \nu_1 \,\mathrm{MR}_1(A/B) + \nu_2 \,\mathrm{MR}_2(A/B) - N|A \setminus B|.$$

For finite B, it follows that $\delta(A/B) = \delta(A \cup B) - \delta(B)$.

B is *strong* in A if $B \subset A$ and $\delta(A'/B) \geq 0$ for all finite $A' \subset A$. We denote this by

$$B \leq A.$$

$B \lneqq A$ is *minimal* if $B \leq A' \leq A$ for no A' properly contained between B and A. a is *algebraic* over B, if a/B is algebraic in the sense of T_1 or T_2. A/B is non-algebraic if no $a \in A \setminus B$ is algebraic over B.

Lemma 2.5. *$B \leq A$ is minimal iff $\delta(A/A') < 0$ for all A' which lie properly between B and A.*

Proof. As the proof of [3, 3.1]. □

Lemma [3, 3.2] is not longer true, instead we have

Lemma 2.6. *Let $B \leq A$ be a minimal extension. There are three cases*

*(I) $\delta(A/B) = 0$, $A = B \cup \{a\}$ for an element $a \in A \setminus B$, which is algebraic over B. (*algebraic simple *extension)*

(II) $\delta(A/B) = 0$*, A/B is non-algebraic. (*prealgebraic *extension)*

(III) A/B is non-algebraic and $1 \leq \delta(A/B) \leq N$*, (*transcendental *extension). If $\delta(A/B) = N$, we have $A = B \cup \{a\}$ for an element a with* $\mathrm{MR}_i(a/B) = N_i$ *for $i = 1, 2$. (*transcendental simple *extension*[8]*)*

Proof. Assume first that A/B is algebraic. That means that some element $a \in A \setminus B$ is algebraic over B. This implies $\delta(a/B) = 0$ and $B \cup \{a\} \leq A$. So we are in case (I).

Now assume that A/B is transcendental and $\delta(A/B) \geq N$. Since $\delta(a/B) \leq N$ for all elements $a \in A \setminus B$, Lemma 2.5 implies $B \cup \{a\} = A$. □

Note that, unlike the situation in [3], there may be prealgebraic extensions A/B by single elements if N_1 and N_2 are not relatively prime. We do not call these extensions "simple".

Remark. If N_1 and N_2 are relatively prime, each strong extension by a single element is simple.

Proof. Let $A = B \cup \{a\}$ be a strong extension of B. If $\delta(A/B) > 0$, the extension is transcendental simple. Otherwise

$$\nu_1 \,\mathrm{MR}_1(a/A) + \nu_2 \,\mathrm{MR}_2(a/A) = N_2 \,\mathrm{MR}_1(a/A) + N_1 \,\mathrm{MR}_2(a/A) = N.$$

It follows that $\mathrm{MR}_1(a/A)$ is divisible by N_1 and $\mathrm{MR}_2(a/A)$ is divisible by N_2. Whence either $\mathrm{MR}_1(a/A)$ or $\mathrm{MR}_2(a/A)$ must be zero. So A/B is algebraic simple. □

We will work in the class

$$\mathcal{K}^0 = \{M \in \mathcal{K} | \emptyset \leq M\}.$$

Fix an element M of $\mathcal{K}^0$. We define for finite subsets of M.

$$\mathrm{d}(A) = \min_{A \subset A' \subset M} \delta(A').$$

d satisfies (2.2), (2.3), (2.4) and

(2.6)
$$\mathrm{d}(A) \geq 0$$

(2.7)
$$A \subset B \Rightarrow \mathrm{d}(A) \leq \mathrm{d}(B)$$
□

We define

$$\mathrm{d}(A/B) = \mathrm{d}(AB) - \mathrm{d}(B) = \delta(\mathrm{cl}(AB)/\,\mathrm{cl}(B)),$$

where $\mathrm{cl}(A)$, the *closure* of A, is the smallest strong subset of M which extends A. Note that the closure of a finite set is again finite (cf. [3, 3.4]).

[8] A transcendental simple extension is a transcendental extension by a single element. Note that simple extensions are *not* related to simple formulas.

2.1.3 Prealgebraic codes (see [3], Section 4)

For each T_i fix a set C_i of codes as in 2.4. We may assume that all ϕ_c and Ψ_c are quantifier free.

A *prealgebraic code* is a pair $c \in C_1 \times C_2$ such that

- $n_c = n_{c_1} = n_{c_2}$
- $e_{c_1,j} = e_{c_2,j}$ for all $j \in \{1, \ldots, n_c\}$.
- $\nu_1 k_{c_1} + \nu_2 k_{c_2} - N \cdot n_c = 0$
- $\nu_1 k_{c_1,s} + \nu_2 k_{c_2,s} - N(n_c - |s|) < 0$ for all non-empty proper subsets s of $\{1, \ldots, n_c\}$.

Set $m_c = \max(m_{c_1}, m_{c_2})$ and for each permutation σ $c^\sigma = (c_1^\sigma, c_2^\sigma)$. c^σ is again prealgebraic.

Some explanatory remarks: T_1^{eq} and T_2^{eq} share only their home sort. An element $b \in \mathrm{dcl}^{\mathrm{eq}}(B)$ is a pair $b = (b_1, b_2)$ with $b_i \in \mathrm{dcl}^{\mathrm{eq}}{}_i(B)$ for $i = 1, 2$. Likewise for $\mathrm{acl}^{\mathrm{eq}}(B)$. A *generic realization* of $\phi_c(x, b)$ (over B) is a generic realization of $\phi_{c_i}(x, b_i)$ (over B) in T_i for $i = 1, 2$. A *Morley sequence* of $\phi_c(x, b)$ is a Morley sequence both of $\phi_{c_1}(x, b_1)$ and $\phi_{c_2}(x, b_2)$. A *pseudo Morley sequence* of c over b is a realization of both $\Psi_{c_1}(\bar{x}, b_1)$ and $\Psi_{c_2}(\bar{x}, b_2)$. We say that M is *independent* from A over B if M is independent from A over B both in T_1 and T_2.

The following three lemmas are proved as Lemmas 4.1, 4.2 and 4.3 in [3].

Lemma 2.7. *Let $B \leq B \cup \{a_1, \ldots, a_n\}$ be a prealgebraic minimal extension and $a = (a_1, \ldots, a_n)$. Then there is some prealgebraic code c and $b \in \mathrm{acl}^{\mathrm{eq}}(B)$ such that a is a generic realization of $\phi_c(a, b)$.* □

Lemma 2.8. *Let $B \in \mathcal{K}$, c a prealgebraic code and $b \in \mathrm{acl}^{\mathrm{eq}}(B)$. Take a generic realization $a = (a_1, \ldots, a_{n_c})$ of $\phi_c(x, b)$ over B. Then $B \cup \{a_1, \ldots, a_{n_c}\}$ is a prealgebraic minimal extension of B.* □

Note that the isomorphism type of a over B is uniquely determined.

Lemma 2.9. *Let $B \subset A$ in $\mathcal{K}$, c a prealgebraic code, b in $\mathrm{acl}^{\mathrm{eq}}(B)$ and $a \in A$ a realization of $\phi_c(x, b)$ which does not lie completely in B. Then*

1. *$\delta(a/B) \leq 0$.*
2. *If $\delta(a/B) = 0$, then a is a generic realization of $\phi_c(x, b)$ over B.*

□

The next Lemma is the analogue of [3, 4.4].

Lemma 2.10. *Let $M \leq N$ an extension in $\mathcal{K}$ and $e_0, \ldots, e_\mu \in N$ a pseudo Morley sequence of c over b. Then one of the following holds:*

- *$b \in \mathrm{dcl}^{\mathrm{eq}}(M)$*
- *more than $\mu - m_c \cdot (N(n_c - 1) + 1)$ many of the e_i lie in $N \setminus M$.*

Proof. If b is not in $\mathrm{dcl}^{\mathrm{eq}}(M)$, less than m_c many of the e_i lie in M. Let r be the number of elements not in $N \setminus M$. We change the indexing so that $e_i \in N \setminus M$ implies $i \geq r$ and $e_i \in M$ implies $i < (m_c - 1)$. By Lemma 2.9 we have $\delta(e_i/Me_0, \dots, e_{i-1}) < 0$ for all $i \in [m_c, r)$. This implies, for $m = \min(m_c, r)$,

$$0 \leq \delta(e_0, \dots, e_{r-1}/M) \leq \delta(e_0, \dots, e_{m-1}/M) - (r - m_c).$$

On the other hand we have $\delta(e_0, \dots, e_{m-1}/M) \leq N \cdot m \cdot (n_c - 1)$, which implies

$$r \leq N \cdot m \cdot (n_c - 1) + m_c \leq N \cdot m_c \cdot (n_c - 1) + m_c.$$

□

2.1.4 The class $\mathcal{K}^\mu$ (see [3], Section 5)

Choose a function μ^* from prealgebraic codes to natural numbers similar to section 5 of [3]. μ^* must satisfy $\mu^*(c) \geq m_c - 1$ and be finite-to-one for every fixed n_c. Also we must have $\mu^*(c) = \mu^*(d)$, if c is equivalent to a permutation of d. Then set

$$\mu(c) = m_c \cdot (N(n_c - 1) + 1) + \mu^*(c).$$

From now on, a *pseudo Morley sequence* denotes a pseudo Morley sequence of length $\mu(c) + 1$ for a prealgebraic code c.

The class $\mathcal{K}^\mu$ consists of the all structures in $\mathcal{K}^0$ which do not contain any pseudo Morley sequence.

The following lemma and its corollary have the same proofs as their analogues [3, 5.1] and [3, 5.2].

Lemma 2.11. *Let B be a finite strong subset of $M \in \mathcal{K}^\mu$ and A/B a prealgebraic minimal extension. Then there are only finitely many B–isomorphic copies of A in M.* □

Corollary 2.12. *Let $B \leq M \in \mathcal{K}^\mu$, $B \subset A$ finite with $\delta(A/B) = 0$. Then there are only finitely many A' such that: $B \leq A' \subset M$ and A' is B-isomorphic to A.* □

Lemma [3, 5.4] may be wrong here. We have instead:

Lemma 2.13. *Let $M \in \mathcal{K}^\mu$ and N a simple extension of M. Then $N \in \mathcal{K}^\mu$.*

Proof. Let $(e_i) \in N$ a pseudo Morley sequence of c over b. At least $\mu(c)$ of the e_i lie in M. Since $\mu(c) \geq m_c$, we have $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$. Since M belongs to $\mathcal{K}^\mu$, one e_i does not lie in M. By 2.9 we conclude that e_i is disjoint from M and a generic realization of $\phi_c(x, b)$. So $n_c = 1$ and N/M is prealgebraic, i.e. not simple. □

Proposition 2.14. *$\mathcal{K}^\mu$ has the amalgamation property with respect to strong embeddings.*

Proof. The proof is the same as the proof of [3, 5.5], the main ingredient being Lemma 2.10. Only one point has to be checked: If A/B is strong and $a \in A$ is algebraic over b, say in the sense of T_1, then $\mathrm{tp}_2(a/B)$ is uniquely determined. This is the case, since $0 \leq \delta(a/B) = \nu_2 \,\mathrm{MR}_2(a/B) - N \leq \nu_2 N_2 - N = 0$ implies that $\mathrm{MR}_2(a/B) = N_2$. On the other hand, $tp_1(a/B)$ implies $a \in P^j$ for some j. So the T_2–type of a/B is uniquely determined since P^j has degree 1 in T_2. □

The proof has the following corollary.

Corollary 2.15. *Two strong extensions $B \leq M$ and $B \leq A$ in $\mathcal{K}^\mu$ can be amalgamated in $M, A \leq M' \in \mathcal{K}^\mu$ such that $\delta(M'/M) = \delta(A/B)$ and $\delta(M'/A) = \delta(M/B)$.* □

A structure $M \in \mathcal{K}^\mu$ is *rich* if for every finite $B \leq M$ and every finite $B \leq A \in \mathcal{K}^\mu$ there is some B-isomorphic copy of A in M. We will show in the next section that rich structures are models of $T_1 \cup T_2$.

Corollary 2.16. *There is a unique (up to isomorphism) countable rich structure K^μ. Any two rich structures are $(L_1 \cup L_2)_{\infty,\omega}$–equivalent.* □

2.1.5 The theory T^μ (see [3], Section 6)

Lemma 2.17. *Let $M \in \mathcal{K}^\mu$, $b \in \mathrm{acl}^{\mathrm{eq}}(M)$, $a \models \phi_c(x, b)$ generic over M and M' the prealgebraic minimal extension $M \cup \{a_1, \cdots a_{n_c}\}$. If M' is not in $\mathcal{K}^\mu$, then one of the following hold.*

(a) M' contains a pseudo Morley sequence of c over b, all whose elements but possibly one are contained in M.

(b) M' contains a pseudo Morley sequence for some code c' with more than $\mu^*(c')$ many elements in $M' \setminus M$.

Proof. As in the proof of [3, 6.1], this follows from 2.9 and 2.10. □

As in [3], Lemmas 2.7, 2.8 and 2.17 imply that we can describe all M with the following properties by an elementary theory T^μ.

Axioms of T^μ.

(a) $M \in \mathcal{K}^\mu$

(b) $T_1 \cup T_2$

(c) M has no prealgebraic minimal extension in $\mathcal{K}^\mu$.

To prove the analogue of Theorem [3, 6.3], which says that the rich structures are the ω–saturated models of T^μ we need the assumptions (*) and (**). Whithout this we can only show[9]

Lemma 2.18. *Rich structures are models of T^μ.*

Proof. Let K be rich. Consider an quantifier free L_1–formula $\chi(x)$ with parameters in K which is T_1–consistent. Let B be a finite strong subset of K which contains the parameters. If $\chi(x)$ is not realized in B, realize $\chi(x)$ by a new element a and define the structure $A = B \cup \{a\}$ in such a way that $\mathrm{MR}_2(a/B) = N_2$. Then $\delta(a/B) = \nu_1 \,\mathrm{MR}_1(a/B)$, so $B \leq A$ and A/B is simple. So by 2.13 B belong to $\mathcal{K}^\mu$. Since K is rich, it contains a copy of A/B. This proves that $\chi(x)$ is realized in K. This shows that K is model of T_1. The same proof shows that K is also a model of T_2.

Axiom (c) is proved like in the proof of [3, 6.3]. □

[9] It is conceivable that T^μ might be incomplete. We even do not know wether T^μ has an ω–stable completion. (This question was raised by the referee.)

2.2 Poizat's argument

We assume now conditions (*) and (**) of Theorem 2.1. We want to show that ω–saturated models of T^{μ} are rich. We start with two lemmas.

Lemma 2.19. *T_1 has the following property. Let $M_1 > 0$ and M_2 be two natural numbers, a an element of an $\emptyset$–definable $\mathbb{Q}$–vector space V_α. Let B be a set of parameters such that V_α contains elements which are of rank 1 over B. Then there are elements $c_1, \dots, c_{M_2}$ of V_α such that for all $s \subset \{1, \dots, M_2\}$*

(2.8)
$$\min(M_1, |s|) \leq \mathrm{MR}_1(c_s/Ba) \leq M_1$$

and, if $|s| > M_1$

(2.9)
$$\mathrm{MR}_1(c_s/B) = \mathrm{MR}_1(c_s/Ba) + \mathrm{MR}_1(a/B).$$

Proof. We start with a sequence $v_1, \dots, v_{M_2}$ of elements of $\mathbb{Q}^{M_1}$ such that

- any M_1 elements of the sequence are $\mathbb{Q}$–linearly independent,
- any $M_1 + 1$ elements of the sequence are linearly dependent, but affinely independent.

Then we choose any B–independent sequence $\bar{e} = (e_1, \dots, e_{M_1})$ of elements of V_α which have rank 1 over B, such that $\bar{e}$ is independent from a over B We consider $\bar{e}$ as a column vector and the v_i as a row vectors and define

$$c_i = v_i \cdot \bar{e} + a.$$

Since all c_i are algebraic over $Ba\bar{e}$, it is clear that

$$\mathrm{MR}_1(c_s/Ba) \leq \mathrm{MR}_1(\bar{e}/Ba) = M_1.$$

To show $\min(M_1, |s|) \leq \mathrm{MR}_1(c_s/Ba)$, we may assume that $|s| \leq M_1$. Since the v_i, $i \in s$ are linearly independent there is a subsequence $\bar{e}'$ of $\bar{e}$ of length $M_1 - |s|$ such that the elements of $\bar{e}'$ and $v_s \cdot \bar{e}$ span the same $\mathbb{Q}$–vector space as the elements of $\bar{e}$. So we have

$$M_1 = \mathrm{MR}_1(\bar{e}/Ba) = \mathrm{MR}_1(\bar{e}', v_s \cdot \bar{e}/Ba) \leq (M_1 - |s|) + \mathrm{MR}_1(v_s \cdot \bar{e}/Ba)$$

and hence
$$|s| \leq \mathrm{MR}_1(v_s \cdot \bar{e}/Ba) = \mathrm{MR}_1(c_s/Ba).$$

The last equation follows from the fact that each $M_1 + 1$ many of the e_i span an affine subspace which contains a. The reason for this is that the according v_i are linearly dependent, but affinely independent, and therefore span an affine space which contains 0. □

Lemma 2.20. *If $N_1 = N_2$, T_2 has the following property. Let B be any set of parameters, and p be the type over B of an M_2–tuple of independent elements of rank N_2 over B. Then p is the limit of types of tuples of independent elements of rank $N_2 - 1$ over B.*

Proof. We indicate the proof for $M_2 = 2$. Let $p = \mathrm{tp}(a_1a_2/B)$ and $\phi(x_1, x_2) \in p$. The formula $\phi_1(x_1) =$ "$\mathrm{MR}_{x_2}\,\phi(x_1, x_2) \geq N_2''$ has rank N_2. Therefore, by (**), there is a type q_1 over B which has rank $N_2 - 1$ and contains $\phi_1(x_1)$. Let b_1 be a realization of q_1. By the open mapping theorem, and (**) again, $\phi(b_1, x_2)$ contains a type q_2 over Bb_1, of rank $N_2 - 1$ which does not fork over B. Realize q_2 by b_2. The type of b_1b_2 over B contains ϕ, b_1 and b_2 are independent and of rank $N_2 - 1$ over B. □

Proposition 2.21. *The rich structures are exactly the ω–saturated models of T^μ.*

Proof. That rich structure are models of T^μ was proved in 2.18. As in the proof of [3, 6.3] one sees that it suffices to prove that ω–saturated models of T^μ are rich. So let K be an ω–saturated model, $B \leq K$ finite and $B \leq A$ a minimal extension which belongs to $\mathcal{K}^\mu$. We show that A/B can be strongly embedded in K by induction over $d = \delta(A/B)$.

If $d = 0$ the extension is algebraic or prealgebraic and the claim follows from 2.14, since K has no algebraic or prealgebraic extensions. So we assume $d > 0$. All we use from the minimality of A/B in this case is that $A \neq B$ and $\delta(X/B) > 0$ for all subsets of A, which are not contained in B.

We may assume that B is large enough to have, for each j, parameters for an L_2–formula in P^j which has rank $N_2 - 1$ in T_2. Choose two numbers M_1 and M_2 such that

$$\nu_1 M_1 - \nu_2 M_2 = -1.$$

The M_i are uniquely determined if we impose the condition $0 \leq M_1 < \nu_2$. We have then

$$M_1 = \frac{\nu_2 M_2 - 1}{\nu_1} < M_2,$$

since $\nu_2 \leq \nu_1$.

Let a be an arbitrary element of $A \setminus B$. Since $\delta(a/B) > 0$, a is not algebraic over B.

If N_1 divides N_2, i.e. if $\nu_2 = M_2 = 1$ and $M_1 = 0$, we choose an element $c_1 \notin A$, which is in the sense of T_1 interalgebraic with a and has rank N_2 over A in the sense of T_2. We set $C = A \cup \{c_1\}$. If N_1 does not divide N_2, we have $M_1 > 0$. We define then $C = A \cup \{c_1, \ldots, c_{M_2}\}$ where the c_i are given by Lemma 2.19 and are – in the sense of T_1 – independent from A over Ba. In the sense of T_2 they are chosen to be A–independent and of rank $N_2 - 1$ over A.

We compute

$$\delta(C/A) = \nu_1 M_1 + \nu_2 M_2(N_2 - 1) - N M_2 = \nu_1 M_1 - \nu_2 M_2 = -1.$$

Claim 1: $B \leq C$. Proof: Let X be a set between B and A and Y be a subset of $\{c_1, \ldots, c_{M_2}\}$ of size y. Note that $\delta(XY/B) \geq \delta(Y/A) + \delta(X/B)$ and by equation (2.8) we have

$$\delta(Y/A) \geq \nu_1 \min(M_1, y) + \nu_2 y(N_2 - 1) - Ny = \nu_1 \min(M_1, y) - \nu_2 y.$$

Case 1: $y \leq M_1$. Then $\delta(XY/B) \geq \delta(Y/A) \geq (\nu_1 - \nu_2)y \geq 0$.

Case 2: $M_1 < y$. Then we have $\delta(Y/A) = \nu_1 M_1 - \nu_2 y \geq \nu_1 M_1 - \nu_2 M_2 = -1$ and distinguish two cases: If $X = B$, then, by (2.9), $\mathrm{MR}_1(Y/B) > \mathrm{MR}_1(Y/A)$

and therefore $\delta(XY/B) = \delta(Y/B) > \delta(Y/A) \geq -1$. If X is different from B we have $\delta(XY/B) \geq -1 + \delta(X/B) \geq 0$. This proves the claim.

Claim 2: The closure of A in C equals C. Proof: Let Y be a proper subset of $\{c_1, \ldots, c_{M_2}\}$ of size y. We have to show that $\delta(Y/A) > -1$. By the above this is clear if $y \leq M_1$. Otherwise we have

$$\delta(Y/A) = \nu_1 M_1 - \nu_2 y > \nu_1 M_1 - \nu_2 M_2 = -1.$$

This proves the claim.

It follows (if N_1 does not divide N_2, from the proof of Lemma 2.19) that one can produce a sequence of extensions $A \subset C_i$ like above such that the types $\mathrm{tp}_1(C_i/A)$ converge against a type $\mathrm{tp}_1(D/A)$ where the elements $d_0, \ldots, d_{M_2}$ are of rank ≥ 1 and algebraically independent[10] over A in the sense of T_1. If $N_1 < N_2$ we simply choose the types $\mathrm{tp}_2(C_i/A)$ and $\mathrm{tp}_2(D/A)$ to be all the same and with components of rank $N_2 - 1$ independent over A in the sense of T_2. If $N_1 = N_2$, it follows from Lemma 2.20 that we may assume that the types $\mathrm{tp}_2(C_i/A)$ converge to $\mathrm{tp}_2(D/A)$ and that the d_i have rank N_2 over A and are independent over A in the sense of T_2.

If $N_1 < N_2$, we have

$$\delta(d_i/Ad_0 \ldots d_{i-1}) \geq \nu_1 \cdot 1 + \nu_2(N_2 - 1) - N = \nu_1 - \nu_2 > 0.$$

If $N_1 = N_2$ we have for every i

$$\delta(d_i/Ad_0 \ldots d_{i-1}) \geq \nu_1 \cdot 1 + \nu_2 N_2 - N = \nu_1 > 0.$$

So D is a strong extension of A which splits into a sequence of transcendental simple extensions. So, by Lemma 2.13, D belongs to $\mathcal{K}^\mu$.

Claim: For large enough i we have $C_i \in \mathcal{K}^\mu$. Proof: Since the C_i have all the same size, if C_i does not belong to $\mathcal{K}^\mu$ and μ is finite-to-1 for fixed n_c, there is a certain finite set of prealgebraic codes which can be responsible for this. Since $D \in \mathcal{K}^\mu$, almost all C_i belong to $\mathcal{K}^\mu$.

Now by induction for large enough i, C_i can be strongly embedded over B into K. Since K is ω–saturated this implies that D can be strongly embedded into K. Such an embedding also strongly embeds A, since $A \leq D$. □

Corollary 2.22. *T^μ is complete. In models of T^μ two tuples have the same type iff they have isomorphic closures.*

Proof. Same as the proof of [3, 7.1]. □

2.3 Rank computation

Proposition 2.23. *In T^μ we have for tuples a*

$$\mathrm{MR}(a/B) = \mathrm{d}(a/B).$$

[10] It suffices that d_i is not in $\mathrm{acl}_1(Ad_0 \ldots d_{i-1})$.

Proof. We prove first $\mathrm{MR}(a/B) \leq \mathrm{d}(a/B)$. Since the closure is algebraic we may assume that B and $A = B \cup \{a\}$ are closed. Then $\mathrm{d}(a/B) = \delta(a/B)$, so it suffices to show that $\mathrm{MR}(a/B) \leq \delta(a/B)$ for all closed B and arbitrary a. We do this by induction on $d = \delta(a/B)$.

Let M be an ω–saturated model, which contains B such that the (a priori infinite) rank of a over M is the same as the rank of a over B. Then $\delta(a/M) \leq \delta(a/B)$ and by induction we may assume that $\delta(a/M) = d$. Also we may assume that a is disjoint from M. Write $a = (a_1, \dots, a_n)$.

Choose for $i = 1, 2$ an $L_i(M)$-formula $\phi_i(x) \in \mathrm{tp}_i(a/M)$ with the following properties.

(i) ϕ_i has degree 1

If a' is any realization of $\phi(x)$, then

(ii) the components of a' are pairwise different

(iii) $\mathrm{MR}_i(a'/Ma'_s) \leq k_{i,s}$, where s is any subset of $\{1, \dots, n\}$ and $k_{i,s} = \mathrm{MR}_i(a/Ma_s)$.

It follows that $\mathrm{MR}_i\, \phi_i = k_{i,\emptyset} = \mathrm{MR}_i(a/M)$.

Let a' be any realization of $\phi(x, b) = \phi_1(x, b) \wedge \phi_2(x, b)$. The inequality $\mathrm{MR}(a/M) \leq d$ follows the from ω–saturation of M and the next claim.

Claim: Either $\mathrm{MR}(a'/M) < d$ or $\mathrm{tp}(a'/M) = \mathrm{tp}(a/M)$.

Proof:
Case 1. $\delta(a'/M) < d$. Then $\mathrm{MR}(a'/M) < d$ by induction.

Case 2. $\delta(a'/M) \geq d$. Set $s = \{i \mid a'_i \in M\}$ consider the inequality

$$\begin{aligned}\delta(a'/M) &= \nu_1 \cdot \mathrm{MR}_1(a'/Ma'_s) + \nu_2\, \mathrm{MR}_2(a'/Ma'_s) - N \cdot (n - |s|)\\ &\leq \nu_1 \cdot k_{1,s} + \nu_2 k_{2,s} - N \cdot (n - |s|)\\ &= \delta(a/Ma_s) \leq \delta(a/M).\end{aligned}$$

Our assumption implies $\mathrm{MR}_i(a'/Ma'_s) = k_{i,s}$ and $\delta(a/Ma_s) = \delta(a/M)$. The latter implies $\delta(a_s/M) = 0$, so a_s/M is algebraic in the sense of T^μ (2.12), which is only possible if s is empty. So we have $\mathrm{MR}_i(a'/M) = \mathrm{MR}_i(a/M)$, which implies that a' and a are isomorphic over M, and $\delta(a'/M) = d$.

Case 2.1 $M \cup \{a'\}$ is not closed. Then a' has an extension a'' with $\delta(a''/M) < d$. It follows $\mathrm{MR}(a'/M) \leq \mathrm{MR}(a''/M) < d$ by induction.

Case 2.2 $M \cup \{a'\}$ is closed. Then $\mathrm{tp}(a'/M) = \mathrm{tp}(a/M)$.

Now we prove $\mathrm{d}(a/B) \leq \mathrm{MR}(a/B)$ by induction on $d = \mathrm{d}(a/B)$. We may we may assume that B is finite, that B and $B \cup \{a\}$ are closed and (using 2.15) that B has, for each j, parameters for an L_2–formula in P^j which has rank $N_2 - 1$ in T_2. If $d = 0$, there is nothing to show. If $d > 0$, we decompose A/B into $B \leq B' \leq A$, where B' is maximal with $\delta(B'/B) = 0$.

Now we can use the construction in proof of 2.21 to obtain a sequence of extensions $A \subset C_i$ and $A \leq D$, such that $B' \leq C_i$, $\delta(C_i/A) = d - 1$, all in

$\mathcal{K}^\mu$, such that C_i is the closure of A and the qf-types of the C_i over A converge against the qf-type of D over A. We may assume that D is closed (in the monster model). We also choose a copies C_i' of C_i over B' which are closed. Let A_i' be the corresponding copy of A in C_i'. Since the types of the $\operatorname{tp}(C_i'/B)$ converge against $\operatorname{tp}(D/B)$, the types $\operatorname{tp}(A_i'/B)$ converge against $\operatorname{tp}(A/B)$. Now $\mathrm{d}(A_i'/B) = \delta(C_i'/B) = d-1$, so by induction $d-1 \leq \mathrm{MR}(C_i'/B)$, which implies $d \leq \mathrm{MR}(A/B)$. □

The referee has pointed out that our proof of $\mathrm{MR}(a/B) \leq \mathrm{d}(a/B)$ can be rephrased as follows: It it easy to see that d–independence defines a notion of independence. The claim in the proof of 2.23 shows that types over ω–saturated models are isolated among the types of at least the same rank. This implies the above inequality.

Lemma 2.24. *Let $\phi(x)$ be an L_i–formula (with parameters). Then*

$$\mathrm{MR}\,\phi = \nu_i \,\mathrm{MR}_i\,\phi.$$

Proof. Consider $i=1$, the case $i=2$ works the same. Let $\phi(x)$ be defined over the closed set B. If a is any realization of ϕ, we have by (2.5)

$$\mathrm{MR}(a/B) \leq \delta(a/B) \leq \nu_1\,\mathrm{MR}_1(a/B) \leq \nu_1\,\mathrm{MR}_1\,\phi.$$

So $\mathrm{MR}\,\phi \leq \nu_1\,\mathrm{MR}_1\,\phi$. For the converse choose a generic realization $a = (a_1, \ldots, a_n)$ of ϕ. Choose $\operatorname{tp}_2(a/B)$ of maximal possible rank[11]. Then clearly $\delta(a/B) = \nu_1\,\mathrm{MR}_1(a/B) = \nu_1\,\mathrm{MR}_1\,\phi$. Also, for every i, $B \cup \{a_1, \ldots, a_i\}$ is equal to, or a simple extension of, $B \cup \{a_1, \ldots, a_{i-1}\}$. So, by 2.13, $B \cup \{a\}$ belongs to $\mathcal{K}^\mu$. We can therefore find $B \cup \{a\}$ as a closed subset of a model of T^μ. This implies $\mathrm{MR}(a/B) = \delta(a/B) = \nu_1\,\mathrm{MR}_1\,\phi$. □

Lemma 2.25. *Let $\phi(x)$ be an L_i–formula (with parameters). Then*

$$\mathrm{MD}\,\phi = \mathrm{MD}_i\,\phi.$$

Proof. Consider $i=1$. Let $\phi(x)$ be defined over the closed set B. We may assume that ϕ is simple in the sense of T_1. Let a be a realization of $\phi(x)$ with $\mathrm{MR}(a/B) = \mathrm{MR}\,\phi$. Then $\mathrm{MR}_1(a/B) = \mathrm{MR}_1\,\phi$, which determines $\operatorname{tp}_1(a/B)$ uniquely, since $\mathrm{MD}_1\,\phi = 1$. In the sense of T_2 the a_i are B–independent generic elements of certain P^j's, so the type $\operatorname{tp}_2(a/B)$ is uniquely determined. Finally $B \cup \{a\}$ must be closed. This implies that $\operatorname{tp}(a/B)$ is uniquely determined and $\mathrm{MD}\,\phi = 1$. □

2.4 Definable rank and degree

It remains to show that T^μ has definable rank and degree. If N_1 does not divide N_2 the definability of rank follows from the fact that the universe of T^μ is covered by a finite set of definable groups. We give a proof which works also for the case $N_1 | N_2$.

We use the following observation, due to M. Hils. Call a formula $\phi(x,b)$ of rank n and degree 1 *normal* if b satisfies a formula $\theta(y)$ such that $\phi(x,b')$ has rank n and degree 1 for all realizations b' of θ. A type is *normal* if it contains a normal formula of the same rank. We have then

[11] This is N_2 times the number of different a_i's

Lemma 2.26. *Let T be a complete theory of finite rank. Then*

1. *T has definable rank and degree iff every type over a model M is normal.*

2. *If $\mathrm{tp}(a, a'/M)$ is normal, and a' is algebraic over Ma, then also $\mathrm{tp}(a/M)$ is normal.* $\square$

In 1. it suffices to consider ω–saturated models M. Also, if M is ω–saturated and $b \in M$, then $\phi(x, b)$ is normal iff there is a $\theta(y)$ defined over M such that $\phi(x, b')$ has rank n and degree 1 for all b' in $\theta(M)$.

Consider an ω–saturated model M of T^μ and a type $p = \mathrm{tp}(a/M)$ of rank $d = \mathrm{d}(a/M)$. We want to show that p is normal. By 2.26.2 we may assume that $M \cup \{a\}$ is closed, i.e. $d = \delta(a/M)$. We may also assume that a is disjoint from M and that all components of a are different. Choose for each $i = 1, 2$ formulas $\phi_i(x, m) \in \mathrm{tp}_i(a/M)$ with properties (i), (ii), (iii) as in the first part of the proof of proposition 2.23. Choose a formula $\theta(x)$ over M, which is satisfied by m, such that for all $m' \in \theta(M)$ the formulas $\phi(x, m')$ satisfy (i), (ii), and (iii) and $\mathrm{MR}_i\, \phi_i(x, m') = k_{i,\emptyset}$ for $i = 1, 2$. Let a' be a generic realization of $\phi(x, m')$, which has a unique qf-type over M. Then $\delta(a'_s/M) = \delta(a_s/M)$ for all $s \subset \{1, \ldots, n\}$, especially $\delta(a'/M) = d$. This implies that $M_{m'} = M \cup \{a'\}$ is a strong extension of M. One sees easily, like in [3, 6.2], that we can strengthen θ to ensure that $M_{m'} \in \mathcal{K}^\mu$[12]. So we can find a' with $M_{m'}$ closed in the universe. This implies $\mathrm{MR}(a'/M) = d$.

The proof of 2.23 shows that for all realizations a'' of $\phi(x, m')$ either $\mathrm{MR}(a''/M) < d$ or $\mathrm{tp}(a''/M) = \mathrm{tp}(a'/M)$. This shows that $\phi(x, m')$ has rank d degree 1 and that $\phi(x, m)$ is normal.

This completes the proof of Theorem 2.1.

3 Proof of Theorem 1.5

We start with an easy lemma.

Lemma 3.1. *Let T be a complete two–sorted theory with sorts Σ_1 and Σ_2. Then the following are equivalent.*

a) Σ_1 is stably embedded.

b) Let T_1^* be a one–sorted complete expansion of $T_1 = T \upharpoonright \Sigma_1$. Then $T^* = T_1^* \cup T$ is complete.

Proof. a)→b): Consider $S = (S_1^*, S_2)$ two saturated models $S' = (S_1'^*, S_2')$ of T^* of the same cardinality. Since T and T_1^* are complete, there are isomorphisms $f : (S_1, S_2) \to (S_1', S_2')$ and $g : S_1^* \to S_1'^*$. $f^{-1}g \upharpoonright S_1$ is an automorphism of the structure induced on S_1. Since S_1 is stably embedded, there is an extension of

[12] The argument is as follows. Decompose the extension $M \leq M \cup \{a\}$ into a sequence of minimal extensions, where the prealgebraic extensions are given by codes $c_1, \ldots, c_k$. Strengthen θ so that the extensions $M \leq M \cup \{a'\}$ are also composed of prealgebraic extension coming from $c_1, \cdots, c_k$. The argument of [3, 6.2] shows now that "$M \cup \{a'\} \in \mathcal{K}^\mu$" is an elementary property of m'.

$f^{-1}g \restriction S_1$ to an automorphism h of (S_1, S_2). Then fh is an isomorphism $S \to S'$.

b)$\to$a): This is not used in this article and left to the reader. $\square$

We fix for the rest of the section T, T_1, T_1^* and T^* be as in 1.5. Let L, L_1, L_1^* and $L^* = L_1^* \cup L$ be the respective languages. We may assume that T_1 has elimination of imaginaries.[13]

The following lemma is due to Anand Pillay. We need only that Σ_1 is stably embedded.

Corollary 3.2. *In T^* every L^*–formula $\Phi(x)$ is equivalent to a formula of the form*

$$\psi^*(t(x)),$$

where $\psi^*(y)$ is an L_1^*–formula and t is a T–definable function with values in some power of Σ_1.

Proof. Let $S = (S_1, S_2)$ be a model of T, where S_1 is a model of T_1 and S^* be an expansion to a model of T^*. Let a be a tuple from S. Since S_1 is stably embedded and has elimination of imaginaries, every a–definable relation on S_1 has a canonical parameter in S_1. $B = \mathrm{dcl}(a) \cap S_1$ is the set of these parameters and $(S_1, b)_{b\in B}$ is the structure induced by (S, a) on S_1.

By 3.1

$$\mathrm{Th}(S^*, a) = \mathrm{Th}(S_1, b)_{b\in B} \cup \mathrm{Th}(S, a).$$

This means that $\mathrm{tp}^*(a)$ is axiomatized by $\mathrm{tp}^1(B) \cup \mathrm{tp}(a)$, which implies the lemma. $\square$

Corollary 3.3. *S_1^* is the structure induced by S^* on S_1.* $\square$

Proof of Theorem 1.5: We prove the following claim by induction on k.

1) For every L–definable X with $\mathrm{MR}\, X \le k$ we have $\mathrm{MRD}^*\, X = \mathrm{MRD}\, X$.

2) For all L^*–formulas $\Phi(x, y)$ is "$\mathrm{MR}^*\, \Phi(x, b) = k$" an L^*–elementary property of b.

Case $k = 0$: Let $\Phi(x, b)$ be of the form $\psi^*(t(x))$, where ψ^* and t are defined from b. Consider t as a map $S \to S_1$. Then $\psi^*(t(x))$ is finite iff the L_1^*–formula $\exists x\, (y \doteq t(x) \wedge \psi^*(y))$ and all the fibers $t(x) = a$ for $\models \psi^*(a)$ are finite. This can be elementarily expressed since finiteness can be expressed in T_1^* and T. This proves 2). 1) is clear.

Case $k + 1$:

1): Assume $\mathrm{MR}\, X \le k+1$. If all L^*–definable subsets of X are L–definable, it is clear that $\mathrm{MRD}^*\, X = \mathrm{MRD}\, X$. So assume that there is an L^*–definable $A \subset X$ which is not L–definable. By Corollary 3.2 there is an L–definable surjection $t : X \to Y \subset S_1^n$ and an L^*–definable $B \subset Y$ such that $A = t^{-1}B$. Since MR is definable in T we can partition Y into finitely many L-definable sets

[13] For this we replace T_1 by T_1^{eq}. Actually the sort Σ_1 may be itself a many-sorted structure.

on each of which the ranks of the fibers $t^{-1}y$ have constant rank. The inverse image of this partition is an L–definable partition of X. Since it is enough to prove 1) for each of the sets of the partition, we may assume that all fibers of t have the same rank f. Since A is not L–definable, Y must be infinite. So we have $f = \mathrm{MR}\, X - \mathrm{MR}\, Y \leq k$. By induction all fibers have T^*–rank f. Since, again by induction, all T^*–ranks $\leq k$ are definable, it follows[14] that $\mathrm{MR}^*\, X = f + \mathrm{MR}^*(Y) = f + \mathrm{MR}\, Y = \mathrm{MR}\, X$.

To prove that $\mathrm{MD}^*\, X = \mathrm{MD}\, X$, we may assume that $\mathrm{MD}\, X = 1$. We have to show that $\mathrm{MR}^*(X \setminus A) < \mathrm{MR}\, X$ for every L^*–definable $A \subset X$ of T^*–rank $\mathrm{MR}\, X$. This is clear if A is L–definable. If not, we choose Y, t and B as above. Again we may assume that all fibers have rank f. We have then $\mathrm{MD}^*\, Y = \mathrm{MD}\, Y = 1$. Since $f \leq k$, we have again by induction that $\mathrm{MR}^*\, B = \mathrm{MR}^*\, A - f = \mathrm{MR}\, X - f = \mathrm{MR}\, Y = \mathrm{MR}^*\, Y$. So $\mathrm{MR}^*(X \setminus A) = f + \mathrm{MR}^*(Y \setminus B) < f + \mathrm{MR}^*(Y) = f + \mathrm{MR}\, Y = \mathrm{MR}(X)$.

2): Consider L^*–definable sets $A \subset S^m$. Let N be the T–rank of S^m. $\mathrm{MR}^*\, X \geq k+1$ is $\bigwedge$–definable and $\bigvee$–definable , since this is equivalent to *"for all/some L–definable $t : S^m \to S_1^n$ with $A = t^{-1}B$ for $B = t(A)$ there is a number $f \leq N$ such that the T^*–rank of $C_f = \{b \in B \mid \mathrm{MR}(t^{-1}b) = f\}$ is $\geq k+1-f$"*. Indeed, if there is such a t and f, we have

$$\mathrm{MR}^*\, A \geq f + \mathrm{MR}^*\, C_f \geq k+1.$$

If conversely $\mathrm{MR}^*\, A \geq k+1$ and t is such that $A = t^{-1}B$ for $B = t(A)$, there is a C_f such that $\mathrm{MR}^*\, t^{-1}C_f \geq k+1$. If $\mathrm{MR}^*\, C_f \leq k-f$ we would have $f \leq k$ and by definability of T^*–ranks $\leq k$ we have $\mathrm{MR}^*\, t^{-1}C_f = f + \mathrm{MR}^*\, C_f \leq k$. So $\mathrm{MR}^*\, C_f \geq k+1-f$. □

Finally let us state an open problem: *Let T be a good theory with two sorts Σ_1 and Σ_2 and T' be a conservative expansion of $T \restriction \Sigma_1$. Does $T' \cup T$ have finite Morley rank?*

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[14] The reader may consult Lemma 3.11 and (the proof of) Folgerung 4.4 in [7].

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