Introduction to the Lascar Group

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1 Introduction

The aim of this article is to give a short introduction to the Lascar Galois group $\mathrm{Gal}_{\mathrm{L}}(T)$ of a complete first order theory T. We prove that $\mathrm{Gal}_{\mathrm{L}}(T)$ is a quasicompact topological group in section 5. $\mathrm{Gal}_{\mathrm{L}}(T)$ has two canonical normal closed subgroups: $\Gamma_1(T)$, the topological closure of the identity, and $\mathrm{Gal}^0_{\mathrm{L}}(T)$, the connected component. In section 6 we characterize these two groups by the way they act on bounded hyperimaginaries. In the last section we give examples which show that every compact group occurs as a Lascar Galois group and an example in which $\Gamma_1(T)$ is non–trivial.

None of the results, except possibly Corollary 26, are new, but some technical lemmas and proofs are. In particular, the treatment of the topology of $\mathrm{Gal}_{\mathrm{L}}(T)$ in sections 4 and 5 avoids ultraproducts, by which the topology was originally defined in [6]. Most of the theory expounded here was taken from that article, and the more recent [7], [4] and [2].

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2 The group

We fix a complete theory T. Let $\mathbb{C}$ be a saturated[1] model of T, of cardinality larger than as $2^{|T|}$, and let $\mathrm{Aut}(\mathbb{C})$ its automorphism group. The subgroup $\mathrm{Autf}_{\mathrm{L}}(\mathbb{C})$ generated by all point–wise stabilizers $\mathrm{Aut}_M(\mathbb{C})$ of elementary[2] submodels M is called the group of *Lascar strong* automorphisms. $\mathrm{Autf}_{\mathrm{L}}(\mathbb{C})$ is a

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[1] T may not have saturated models. In this case we take for $\mathbb{C}$ a *special* model (see [3] Chapter 10.4) of T and use the $\mathrm{cf}|\mathbb{C}|$ instead of $|\mathbb{C}|$. Especially we assume that $\mathrm{cf}|\mathbb{C}| > 2^{|T|}$.

[2] In the sequel *submodel* will always mean *elementary submodel.*

normal subgroup of $\mathrm{Aut}(\mathbb{C})$. The quotient is the *Lascar (Galois) group* of $\mathbb{C}$:

$$\mathrm{Gal_L}(\mathbb{C}) = \mathrm{Aut}(\mathbb{C})/\mathrm{Autf_L}(\mathbb{C}).$$

We will show that $\mathrm{Gal_L}(\mathbb{C})$ does not depend on the choice of $\mathbb{C}$.

Lemma 1 *Let M and N be two small* [3] *submodels of $\mathbb{C}$ and f an automorphism. Then the class of f in $\mathrm{Gal_L}(\mathbb{C})$ is determined by the type of $f(M)$ over N.*

PROOF: Let $(m_i)_{i\in I}$ be an enumeration of M. By the type of $f(M)$ over N we mean the type of the infinite tuple $(f(m_i))_{i\in I}$ over N. This is a type in variables $(x_i)_{i\in I}$. We denote by $\mathrm{S}_I(N)$ the set of all such types over N.

Let $g(M)$ have the same type over N as $f(M)$. Choose an automorphism s which fixes N and maps $f(M)$ to $g(M)$. Then s is a Lascar strong automorphism, as is $t = (sf)^{-1}g$, which fixes M. Now we see that $g = sft$ and f have the same class in $\mathrm{Gal_L}(\mathbb{C})$. □

Two possibly infinite tuples a and b from $\mathbb{C}$ are said to have the same *Lascar strong type* iff $f(a) = b$ for a Lascar strong automorphism f.

Lemma 2 *a and b have the same Lascar strong type iff there is a sequence of tuples $a = a_0, \dots, a_n = b$ and a sequence of small submodels $N_1, \dots, N_n$ such that, for each i, a_{i-1} and a_i have the same type over N_i.*

PROOF: Clear □

Corollary 3 *a and b have the same Lascar strong type in $\mathbb{C}$ if they have the same Lascar strong type in an elementary extension of $\mathbb{C}$.*

PROOF: If $a_0, \dots, N_n$ exist in an elementary extension of $\mathbb{C}$, we find by saturation in $\mathbb{C}$ a sequence $a'_0, \dots, N'_n$ which has the same type over ab as $a_0, \dots, N_n$. This sequence shows that a and b have the same Lascar strong type in $\mathbb{C}$. □

Theorem 4 ([6]) $\mathrm{Gal_L}(\mathbb{C})$ *depends only on T and not on the choice of $\mathbb{C}$.*

PROOF: If $\mathbb{C}'$ is another big saturated model of T we can assume that $\mathbb{C}'$ is an elementary extension of $\mathbb{C}$ and of larger cardinality. We can extend every automorphism f of $\mathbb{C}$ to an automorphism f' of $\mathbb{C}'$. Since all such f' differ only by elements of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}')$, this defines a homomorphism $\mathrm{Aut}(\mathbb{C}) \to \mathrm{Gal_L}(\mathbb{C}')$. If f Lascar strong, f' is Lascar strong as well. Whence we have a well defined natural map

$$\mathrm{Gal_L}(\mathbb{C}) \to \mathrm{Gal_L}(\mathbb{C}'),$$

[3] of smaller cardinality than $\mathbb{C}$

which will turn out to be an isomorphism.

To prove surjectivity, fix an automorphism g of $\mathbb{C}'$. Choose two small submodels M and N of $\mathbb{C}$. By saturation we find a submodel M' of $\mathbb{C}$ which has the same type over N as $g(M)$. There is an automorphism f of $\mathbb{C}$ which maps M to M'. Extend f to an automorphism f' of $\mathbb{C}'$. Then $f'(M)$ and $g(M)$ have the same type over N. Whence, by the last lemma f' and g represent the same element of $\mathrm{Gal}_{\mathrm{L}}(\mathbb{C}')$.

Now assume that $f \in \mathrm{Aut}(\mathbb{C})$ extends to a Lascar strong automorphism f' of $\mathbb{C}'$. Fix a small submodel M of $\mathbb{C}$. Then M and $f(M)$ have the same Lascar strong type in $\mathbb{C}'$, whence also in $\mathbb{C}$ by Corollary 3. So M can be mapped to $f(M)$ by a Lascar strong automorphism of $\mathbb{C}$. Such an automorphism agrees with f on M, whence f is also strong. This shows that $\mathrm{Gal}_{\mathrm{L}}(\mathbb{C}) \to \mathrm{Gal}_{\mathrm{L}}(\mathbb{C}')$ is injective. $\square$

Definition *The Lascar group of T is the quotient*

$$\mathrm{Gal}_{\mathrm{L}}(T) = \mathrm{Aut}(\mathbb{C})/\mathrm{Autf}_{\mathrm{L}}(\mathbb{C}),$$

where $\mathbb{C}$ is any big saturated model of T.

Corollary 5 *The cardinality of $\mathrm{Gal}_{\mathrm{L}}(T)$ is bounded by $2^{|T|}$.*

PROOF: The class of f in $\mathrm{Gal}_{\mathrm{L}}(T)$ is determined by the type of $f(M)$ over N. If M and N are chosen to be of cardinality T, there are at most $2^{|T|}$ possible types. $\square$

3 Digression: Lascar strong types and thick formulas

Definition *Let $\theta(x, y)$ be a formula in two tuples of variables x and y having the same length. $\theta(x, y)$ is thick, if it has no infinite* antichain*, that is a sequence of tuples $a_0, a_1, \ldots$ such that $\mathbb{C} \models \neg\,\theta(a_i, a_j)$ for all $i < j$.*

Clearly $\theta(x, y)$ is thick iff there is no indiscernible sequence $a_0, a_1, \ldots$ such that $\mathbb{C} \models \neg\,\theta(a_0, a_1)$. With this description it is easy to see that the intersection of two thick formulas is thick again and that a formulas remains thick if one interchanges the role of x and y.

Lemma 6 *Let $\Theta(x, y)$ be the set of all thick formulas in x and y and let a and b two tuples of the same length. Then the following are equivalent:*

a) $\mathbb{C} \models \Theta(a, b)$

b) a and b belong to an infinite indiscernible sequence.

PROOF: Assume $\mathbb{C} \models \Theta(a, b)$. Then, if $\psi(x, y)$ is satisfied by ab, $\neg\psi$ is not thick, so there is an infinite sequence of indiscernibles $a_0, a_1, \ldots$ such that $\psi(a_0, a_1)$ is true. Whence, by compactness, there is one infinite sequence of indiscernibles such that $a_0 a_1$ has the same type as ab.
If conversely a, b are the first two elements of an infinite indiscernible sequence they have to satisfy all thick formulas □

Lemma 7

1. *If $\mathbb{C} \models \Theta(a, b)$, there is a model over which a and b have the same type.*
2. *If a and b have the same type over some model, the pair ab satisfies the relational product $\Theta \circ \Theta$. I.e. there is a tuple a' such that $\mathbb{C} \models \Theta(a, a')$ and $\mathbb{C} \models \Theta(a', b)$.*

PROOF:
1. Let I be an infinite sequence of indiscernibles and M any small model. Then there are indiscernibles I' over M of the same type as I. Whence there is a model M' of the same type as M over which I is indiscernible. Therefore, if a, b are the first elements of some I, they have the same type over some model M'. Now apply Lemma 6.

A more direct proof, which avoids Lemma 6, uses the observation that two sequences a and b of the same length have the same type over a model iff ab satisfies all formulas of the form

$$\exists z\, \varphi(z) \rightarrow \exists z \left(\varphi(z) \wedge \bigwedge_{i=1}^{n} \psi_i(x, z) \leftrightarrow \psi_i(y, z)\right) \tag{1}$$

for all finite variable tuples z and formulas $\varphi(z), \psi_1(x, z), \ldots, \psi_n(x, z)$. All formulas (1) are thick, antichains have length at most 2^n.

2. Assume that a and b have the same type over M. If θ is a thick formula, consider a maximal antichain $a_1, \ldots, a_n$ for θ in M. Then, since M is an elementary substructure, $a_1, \ldots, a_n$ is also a maximal antichain in $\mathbb{C}$. Whence $\mathbb{C} \models \theta(a_i, a)$ for some i. Since b has the same type over M, we have $\mathbb{C} \models \theta(a_i, b)$. This proves that for every finite subset Θ_0 of Θ there is an a' such that $\mathbb{C} \models \Theta_0(a', a)$ and $\mathbb{C} \models \Theta_0(a', b)$. This proves the claim using compactness and the observation that Θ defines a symmetric relation. □

Corollary 8 *The relation of having the same Lascar strong type is the transitive closure of the relation defined by Θ.* □

Let π be a type defined over the empty set. A formula $\theta(x, y)$ is *thick on* π if θ has no infinite antichain in $\pi(\mathbb{C})$. Let Θ_π be the set of all formulas which are thick over π.

Corollary 9 *Two realizations of π, a and b, have the same Lascar strong type if the pair (a,b) is in the transitive closure of the relation defined by Θ_π.*

Proof: Assume that a and b have the same type over a model M. The proof of Lemma 7 (1) shows that we can assume that M is ω–saturated. If θ is thick on π, let $a_1,\dots,a_n$ be a maximal antichain for θ in $\pi(M)$. Then, since is ω–saturated, $a_1,\dots,a_n$ is also maximal in $\pi(\mathbb{C})$. Now proceed as in Lemma 7 (2). $\square$

4 The topology

Let M and N be two small submodels of $\mathbb{C}$. Assign to every automorphism f of $\mathbb{C}$ the type of $f(M)$ over N. This defines a surjective map μ from $\mathrm{Aut}(\mathbb{C})$ to $\mathrm{S}_M(N)$, the set all types over N of conjugates of M. By Lemma 1 the projection $\mathrm{Aut}(\mathbb{C}) \to \mathrm{Gal_L}(T)$ factors through μ:

$$\mathrm{Aut}(\mathbb{C}) \xrightarrow{\mu} \mathrm{S}_M(N) \xrightarrow{\nu} \mathrm{Gal_L}(T).$$

$\mathrm{S}_M(N)$, as a closed subspace of $\mathrm{S}_I(N)$, is a boolean space. We give $\mathrm{Gal_L}(T)$ the quotient topology with respect to ν.

To show that this does not depend on the choice of M and N we consider another pair M' and N'. We may assume that $M \subset M'$ and $N \subset N'$. The map $\mathrm{S}_{M'}(N') \longrightarrow \mathrm{Gal_L}(T)$ then factors as

$$\mathrm{S}_{M'}(N') \longrightarrow \mathrm{S}_M(N) \xrightarrow{\nu} \mathrm{Gal_L}(T),$$

where the first map is restriction of types. Since restriction is continuous and the spaces are compact, $\mathrm{S}_M(N)$ carries the quotient topology of $\mathrm{S}_{M'}(N')$, which implies that on $\mathrm{Gal_L}(T)$ the two topologies, coming from $\mathrm{S}_{M'}(N')$ and $\mathrm{S}_M(N)$, are the same.

A quotient of a quasicompact space remains quasicompact. So we have

Lemma 10 $\mathrm{Gal_L}(T)$ *is quasicompact.* $\square$

Let p and q be types in $\mathrm{S}_M(N)$. Two realizations M' and M'' of p and q have the same Lascar strong type iff $\nu(p) = \nu(q)$. Whence, by Corollary 8, the equivalence relation

$$p \approx q \;\Leftrightarrow\; \nu(p) = \nu(q)$$

is the transitive closure of the relation D, where $\mathrm{D}(p,q)$ holds if p and q have realizations M' and M'' with $\mathbb{C} \models \Theta(M', M'')$.

Lemma 11

1. D *is a closed subset of* $\mathrm{S}_M(N) \times \mathrm{S}_M(N)$

2. $\approx$ is a F_σ–set, i.e. a countable union of closed sets.

Proof:
1. This is clear, because

$$\mathrm{D}(p,q) \;\Leftrightarrow\; p(x) \cup q(y) \cup \Theta(x,y) \text{ consistent.}$$

2. $\approx$ is the union of all powers

$$\mathrm{D}^n = \underbrace{\mathrm{D} \circ \cdots \circ \mathrm{D}}_{n \text{ times}}.$$

So, is suffices to show that all D^i are closed. This follows from the fact that, in compact spaces, the product of two closed relations is closed again. To see this, note that, for binary relations R and S, $R \circ S$ is the projection of $\{(p,q,r)|R(p,q) \wedge S(q,r)\}$ onto the first and third variable. □

In general the map $\mathrm{S}_M(N) \xrightarrow{\nu} \mathrm{Gal}_\mathrm{L}(T)$ is not open.[4] But it has a property that comes close to openness. Define for $p \in \mathrm{S}_M(N)$

$$\mathrm{D}[p] = \{q \in \mathrm{S}_M(N) \mid \mathrm{D}(p,q)\}.$$

Lemma 12 *If $\mathrm{D}[p]$ is contained in the interior of some subset $O \subset \mathrm{S}_M(N)$, then $\nu(p)$ is an inner point of $\nu(O)$.*

Proof: $\mathrm{D}[p]$ is the intersection of all

$$\mathrm{D}_\delta[p] = \big\{q \in \mathrm{S}_M(N) \bigm| p(x) \cup q(y) \cup \{\delta(x,y)\} \text{ consistent}\big\}, \quad (\delta \in \Theta).$$

By compactness some $\mathrm{D}_\delta[p]$ is contained in (the interior of) O.

Claim 1: p is an inner point of $\mathrm{D}_\delta[p]$.

Proof: Since δ is thick, there is a finite set $\{H_1, \ldots, H_n\}$ of realizations of p such that for every other realization H we have $\mathbb{C} \models \delta(H_i, H)$ for some i. By compactness this is true for every realization H of any p' contained in a small enough neighborhood C of p, which implies that C is contained in $\mathrm{D}_\delta[p]$.

After replacing O by $\nu^{-1}(\nu O)$ we can assume that O is closed under $\approx$ (i.e. is a union of $\approx$–classes.) We set

$$U = \{q \in \mathrm{S}_M(N) \mid \mathrm{D}_\delta[q] \subset O \text{ for some } \delta \in \Theta\}.$$

[4] If $\mathrm{Aut}(\mathbb{C})$ is endowed with the topology of point–wise convergence, μ becomes continuous (see Lemma 29). If ν were always open, $\mathrm{Aut}(\mathbb{C}) \to \mathrm{Gal}_\mathrm{L}(T)$ would be open too: If a, b are two (finite) tuples, choose N, M in such a way that $a, b \in M = N$. Then the basic open set $\{f \in \mathrm{Aut}(\mathbb{C})|f(a) = b\}$ will be mapped onto an open subset of $\mathrm{S}_M(N)$ and whence, by assumption, onto an open subset of $\mathrm{Gal}_\mathrm{L}(T)$. Whence, the closedness of $\mathrm{Autf}_\mathrm{L}(\mathbb{C})$ would imply that $\mathrm{Gal}_\mathrm{L}(T)$ is hausdorff. That this is not true shows one of the examples in [2] ($Th(M^*)$ in Proposition 4.5).

U contains p.

Claim 2: U is closed under $\approx$.

Proof: Let q be in U, witnessed by $\mathrm{D}_\delta[q] \subset O$, and $q \approx r$. Then a realization H of q is mapped by a Lascar strong automorphism f to a realization $f(H) = K$ of r. In order to show that r belongs to U we fix an element r' of $\mathrm{D}_\delta[r]$. We have then a realization K' of r' such that $\mathbb{C} \models \delta(K, K')$. Let q' be the type of $H' = f^{-1}(K')$ over M. Since $\mathbb{C} \models \delta(H, H')$, q' belongs to $\mathrm{D}_\delta[q]$ and therefore to O. Since $q \approx q'$ and O is closed under $\approx$, we have $q' \in O$. It follows $\mathrm{D}_\delta[r] \subset O$.

Claim 3: U is open.

Proof: U is a subset of the interior of O by Claim 1. Since U is closed under $\approx$, it is contained in the open set

$$U' = \{q \in \mathrm{S}_M(N) \mid \mathrm{D}[q] \subset \mathrm{interior}(O)\},$$

which, by compactness, equals

$$U'' = \{q \in \mathrm{S}_M(N) \mid \mathrm{D}_\delta[q] \subset \mathrm{interior}(O) \text{ for some } \delta \in \Theta\}.$$

But U'' is contained in U, which shows that $U = U'$.

By Claims 2 and 3 the projection of U is an open subset of $\nu(O)$ and contains $\nu(p)$. This completes the proof of Lemma. □

Corollary 13 *If L is countable,* $\mathrm{Gal_L}(T)$ *has a countable basis.*

Proof: If L is countable we can choose countable M and N. $\mathrm{S}_M(N)$ has then a countable base, $\mathcal{B}$. We can assume that $\mathcal{B}$ is closed under finite unions. Let us show that the set of all $\nu(B)^\circ$, $(B \in \mathcal{B})$, is a basis of $\mathrm{Gal_L}(T)$. Let Ω be open and $\alpha \in \Omega$. Choose a preimage p of α and a basic open set B, such that $\mathrm{D}[p] \subset B \subset \nu^{-1}(\Omega)$. This is possible, since B is compact and $\mathcal{B}$ closed under finite unions. Then $\nu(B)^\circ \subset \Omega$ is an open neighborhood of p. □

The following corollary is a reformulation of Corollary 3.5 in [2].

Corollary 14 *Let X be a subset of* $\mathrm{Gal_L}(T)$*. Then*

$$\overline{X} = \nu(\overline{\nu^{-1}(X)}).$$

Proof: Since ν is continuous the right hand side lies inside $\overline{X}$. Let $\nu(p)$ be an element of $\mathrm{Gal_L}(T)$ which does not belong to $\nu(\overline{\nu^{-1}(X)})$. Then the whole $\approx$–class of p, which contains $\mathrm{D}[p]$, is disjoint from $\overline{\nu^{-1}(X)}$. By Lemma 12 the complement of $\overline{\nu^{-1}(X)}$ is mapped to a neighborhood of $\nu(p)$, which is disjoint from X. This shows $\nu(p) \notin \overline{X}$. □

Corollary 15 $\mathrm{Gal_L}(T)$ *is hausdorff iff* $\approx$ *is closed.*

PROOF: "$\mathrm{Gal_L}(T)$ hausdorff $\Rightarrow$ $\approx$ closed" is an easy consequence of the continuity of ν.

Now assume that $\approx$ is closed. Consider two different elements x, y of $\mathrm{Gal_L}(T)$. Since $\approx$ is closed, we can separate each element of $\nu^{-1}(x)$ from each element of $\nu^{-1}(y)$ by a pair of neighborhoods which projects onto disjoint subsets of $\mathrm{Gal_L}(T)$. But $\nu^{-1}(x)$ and $\nu^{-1}(y)$ are compact. This implies that there is one pair of open sets, O and U, which separate $\nu^{-1}(x)$ and $\nu^{-1}(y)$ and have disjoint projections $\nu(O)$ and $\nu(U)$, which are, by the lemma, neighborhoods of x and y. $\square$

We will see in section 7 (Theorem 28) that $\mathrm{Gal_L}(T)$ need not to be hausdorff.

5 The topological group

Theorem 16 (Lascar) $\mathrm{Gal_L}(T)$ *is a topological group.*

For the proof we fix again two small submodels M and N and consider the natural mappings

$$\mathrm{Aut}(\mathbb{C}) \xrightarrow{\mu} \mathrm{S}_M(N) \xrightarrow{\nu} \mathrm{Gal_L}(T).$$

Lemma 17 *The projections of multiplication*

$$\mathcal{M} = \big\{ \big(\mu(f), \mu(g), \mu(fg)\big) \bigm| f, g \in \mathrm{Aut}(\mathbb{C}) \big\}$$

and of inversion

$$\mathcal{I} = \big\{ \big(\mu(f), \mu(f^{-1})\big) \bigm| f \in \mathrm{Aut}(\mathbb{C}) \big\}$$

are closed subset of $\mathrm{S}_M(N) \times \mathrm{S}_M(N) \times \mathrm{S}_M(N)$ *and of* $\mathrm{S}_M(N) \times \mathrm{S}_M(N)$*, respectively.*

PROOF: We introduce two unary function symbols F and G and express the fact that F are G automorphisms by the $L \cup \{F, G\}$–theory $A(F, G)$. Then (p, q, r) belongs to $\mathcal{M}$ iff there are are functions $f, g : \mathbb{C} \to \mathbb{C}$ which satisfy the theory

$$B(F, G, p, q, r) = A(F, G) \cup p(F(M)) \cup q(G(M)) \cup r(F(G(M))).$$

Since $\mathbb{C}$ is saturated, $B(F, G, p, q, r)$ can be satisfied in $\mathbb{C}$ if it is consistent with the theory of $\mathbb{C}_{M,N}$. This is a closed condition on p, q, r.

The closedness of $\mathcal{I}$ is similar. $\square$

The graphs of the multiplication and inversion in $\mathrm{Gal_L}(T)$ are the projections of $\mathcal{M}$ and $\mathcal{I}$. If $\mathrm{Gal_L}(T)$ is hausdorff, the projections are closed, which, by compactness, implies that multiplication and inversion are continuous in $\mathrm{Gal_L}(T)$.

For the general case we need the following notation: For two subsets of A and B of $\mathrm{S}_M(N)$ define

$$A * B = \bigl\{r \in \mathrm{S}_M(N) \bigm| (p, q, r) \in \mathcal{M} \text{ for a pair } (p, q) \in A \times B\bigr\}.$$

Lemma 18 *If A and B are closed and $A * B$ is contained in the open set W, there are neighborhoods U and V of A and B such that $U * V \subset W$.*

Proof: Let W' be the complement of W. $A \times B$ is disjoint from the projection C of

$$\mathcal{M} \cap \bigl(\mathrm{S}_M(N) \times \mathrm{S}_M(N) \times W'\bigr)$$

on the first two coordinates. Since C is closed (and A and B are compact) there are neighborhoods U and V of A and B such that $U \times V$ is disjoint from C. It follows that $U * V \subset W$. $\square$

We can now prove that multiplication in $\mathrm{Gal_L}(T)$ is continuous. Let $\alpha = \nu(p)$ and $\beta = \nu(q)$ be elements of $\mathrm{Gal_L}(T)$ and Ω an open neighborhood of $\alpha\beta$. Then

$$\mathrm{D}[p] * \mathrm{D}[q] \;\subset\; \nu^{-1}(\alpha) * \nu^{-1}(\beta) \;\subset\; \nu^{-1}(\alpha\beta) \;\subset\; \nu^{-1}(\Omega).$$

By the last lemma there neighborhoods U and V of $\mathrm{D}[p]$ and $\mathrm{D}[q]$, respectively, such that $U * V \subset \nu^{-1}(\Omega)$. This implies $\nu(U)\nu(V) \subset \Omega$. Finally, we remark that, by Lemma 12, $\nu(U)$ and $\nu(V)$ are neighborhoods of α and β.

The continuity of inversion is proved in the same manner, which completes the proof of the theorem.

6 Two subgroups

$\mathrm{Gal_L}(T)$ has two canonical normal subgroups:

- $\Gamma_1(T)$, the closure of $\{1\}$.
- $\mathrm{Gal}^0_\mathrm{L}(T)$, the connected component of 1.

Since $\mathrm{Gal_L}(T)$ is quasicompact, we have

Lemma 19

1. *The quotient* $\mathrm{Gal}^\mathrm{c}_\mathrm{L}(T) = \mathrm{Gal_L}(T)/\Gamma_1(T)$ *is a compact group, the* closed Galois group *of T.*
2. $\mathrm{Gal}^0_\mathrm{L}(T)$ *is the intersection of all closed (normal) subgroups of finite index.*

Proof: $\mathrm{Gal}_{\mathrm{L}}^{\mathrm{c}}(T)$ is quasicompact and hausdorff, i.e. compact. For the second part, note that the quotient $\mathrm{Gal}_{\mathrm{L}}(T)/\mathrm{Gal}_{\mathrm{L}}^{0}(T)$ is totally disconnected ([12, §2]) and compact, whence a profinite group. In a profinite group the intersection of all normal closed subgroups of finite index is the identity. □

An *imaginary* element of $\mathbb{C}$ is a class of a $\emptyset$–definable equivalence relation on a cartesian power $\mathbb{C}^n$. Automorphisms of $\mathbb{C}$ act in a natural way on imaginaries. An imaginary with only finitely many conjugates under $\mathrm{Aut}(\mathbb{C})$ is called *algebraic*.

Let us prove that algebraic imaginaries are fixed by Lascar strong automorphisms: Let a/E be an algebraic imaginary with k conjugates. This means that E partitions the set of all conjugates of a into k classes. It follows that the type of a contains a formula $\varphi(x)$ whose realization set meets exactly k equivalence classes. Let f fix the model M. Then $\varphi(M)$ meets the same classes as $\varphi(\mathbb{C})$, which implies that a/E contains an element b of M, which must also belong to $f(a)/E$. It follows that $a/E = f(a)/E$.

This result extends easily to *hyperimaginaries*. Hyperimaginaries are equivalence classes of type–definable equivalence relations E, which are defined by a set of formulas Φ without parameters:

$$E(a, b) \Leftrightarrow \mathbb{C} \models \Phi(a, b).$$

a and b are, possible infinite, tuples of elements of $\mathbb{C}$, of length smaller than $|\mathbb{C}|$. A hyperimaginary is *bounded* if it has less than $|\mathbb{C}|$ conjugates.

Lemma 20 *Bounded hyperimaginaries are fixed by Lascar strong automorphisms.*

Proof: Let a/E be a bounded hyperimaginary and E defined by $\Phi(x, y)$. Then $\Phi \subset \Theta_\pi$, where $\pi = \mathrm{tp}(a)$, since otherwise some $\theta \in \Phi$ would have antichains in $\pi(\mathbb{C})$ of arbitrary length, contradicting the assumption that a/E is bounded. If f is Lascar strong, a and $f(a)$ have the same Lascar strong type. By Corollary 9, $E(a, f(a))$. □

If, conversely, a hyperimaginary h is fixed by all Lascar strong automorphisms, $f(h)$ is determined by the class of f in $\mathrm{Gal}_{\mathrm{L}}(T)$. Whence h has no more than $2^{|T|}$–many conjugates and is bounded.

We conclude that $\mathrm{Gal}_{\mathrm{L}}(T)$ acts on bounded hyperimaginaries in a well defined way.

Theorem 21

1. *$\Gamma_1(T)$ is the set of all elements of $\mathrm{Gal}_{\mathrm{L}}(T)$ which fix all bounded hyperimaginaries.*

2. $\mathrm{Gal}^0_{\mathrm{L}}(T)$ *is the set of all elements of* $\mathrm{Gal}_{\mathrm{L}}(T)$ *which fix all algebraic imaginaries.*

Proof:
1. Let a/E be a bounded hyperimaginary and $\Gamma \leq \mathrm{Gal}_{\mathrm{L}}(T)$ the stabilizer of a/E. The preimage of Γ in $\mathrm{S}_M(N)$ is

$$\nu^{-1}(\Gamma) = \{\mathrm{tp}(f(M)/N) \mid f \in \mathrm{Aut}(\mathbb{C}),\ E(f(a), a)\}.$$

Choose M containing a, let $N = M$ and E be axiomatized by Φ. Then

$$\nu^{-1}(\Gamma) = \{p(x) \in \mathrm{S}_M(N) \mid \Phi(x', a) \subset p(x)\},$$

where the variables x' are a subtuple of x, as a is a subtuple of (m_i), the enumeration of M. Whence Γ is closed and we conclude $\Gamma_1(T) \subset \Gamma$. This shows that the elements of $\Gamma_1(T)$ fix all bounded imaginaries.

For the converse consider the inverse image G_1 of $\Gamma_1(T)$ in $\mathrm{Aut}(\mathbb{C})$. For $|T|$–tuples a, b let $E(a, b)$ denote the equivalence relation of being in the same G_1–orbit. Since the index of G_1 is bounded by $2^{|T|}$, E has at most $2^{|T|}$ classes. Since $\Gamma_1(T)$ is closed, E is type–definable. To see this, write the closed set $\nu^{-1}(\Gamma_1(T))$ as $\{p(x) \in \mathrm{S}_M(N) \mid \Psi(x) \subset p(x)\}$ for a set $\Psi(x)$ of $L(N)$–formulas. Then

$$E(a, b) \ \Leftrightarrow \ \text{for some } f \in \mathrm{Aut}(\mathbb{C}) \ \ \mathbb{C} \models f(a) = b \wedge \Psi(f(M)).$$

This shows, by an argument similar to that in the proof of Lemma 17, that E is can be defined by a set of formulas with parameters from M and N. Since $\Gamma_1(T)$ is a normal subgroup, G_1 is a normal subgroup of $\mathrm{Aut}(\mathbb{C})$. This implies that E is invariant under automorphisms, and whence can be defined by a set of formulas without parameters.

Now assume that $\alpha \in \mathrm{Gal}_{\mathrm{L}}(T)$ fixes all bounded hyperimaginaries. Take a model K of cardinality $|T|$ and consider it as a $|T|$–tuple. Then K/E is a bounded hyperimaginary and fixed by α. This means that α is represented by an automorphism which agrees on K with an automorphism f from G_1. Since K is a model, this implies that α is represented by f and belongs to $\Gamma_1(T)$.

2. Let i be an algebraic imaginary and Γ the stabilizer of i in $\mathrm{Gal}_{\mathrm{L}}(T)$. Γ is closed and has finite index, since the index equals the number of conjugates of i. It follows that $\mathrm{Gal}^0_{\mathrm{L}}(T) \subset \Gamma$. Thus the elements of $\mathrm{Gal}^0_{\mathrm{L}}(T)$ fix all algebraic imaginaries.

For the converse it suffices to show that every normal closed $\Gamma \leq \mathrm{Gal}_{\mathrm{L}}(T)$ of finite index is the stabilizer of an algebraic imaginary. The first part of the proof shows that Γ, being a normal[5] closed subgroup, is the stabilizer of a bounded

[5] A slight variation of the argument shows that normality is not necessary: Let G be the preimage of Γ, and K a model of size $|T|$. Define $E(a, b)$ to be true if $a = b$ or, for some $f \in \mathrm{Aut}(\mathbb{C})$ and $g \in G$, $f(K) = a$ and $fg(K) = b$. Then K/E is a bounded hyperimaginary with Γ as its stabilizer. See [7, 4.12].

hyperimaginary a/E. Since Γ has finite index, a/E has only a finite number of conjugates. We will show that a/E has the same stabilizer as an algebraic imaginary a/F. (If a is an infinite tuple, we can replace it by the finite subtuple of elements which occur in F.)

Let E be defined by Φ and let $a_1/E, \dots, a_n/E$ be the different conjugates of a/E. By compactness there is a symmetric formula[6] $\theta \in \Phi$ such that no pair (a_i, a_j) $(i \neq j)$ satisfies $\theta^2 = \theta \circ \theta$.[7] This means that the sets $\theta(a_i, \mathbb{C})$ are disjoint. Since they cover the set of conjugates of a, there is a formula $\varphi(x)$ satisfied by a such that the intersections

$$D_i = \varphi(\mathbb{C}) \cap \theta(a_i, \mathbb{C})$$

form a partition of $\varphi(\mathbb{C})$. In order to ensure that this partition is invariant under automorphisms, we choose $\theta \in \Phi$ so small that no pair (a_i, a_j) satisfies θ^4. This implies that $\theta^2(c, d)$ is never true for $c \in D_i$ and $d \in D_j$ and, therefore, that

$$F(x,y) \;=\; (\neg\varphi(x) \wedge \neg\varphi(y)) \vee (\varphi(x) \wedge \varphi(y) \wedge \theta^2(x,y))$$

defines an equivalence relation, with classes $\neg\varphi(\mathbb{C}), D_1, \dots, D_n$. Thus a/F is an algebraic imaginary. Since a/E and a/F contain the same conjugates of a, they have the same stabilizer. □

Corollary 22

1. $\mathrm{Gal}^{\mathrm{c}}_{\mathrm{L}}(T)$ *is the automorphism group of the set of all bounded hyperimaginaries of length* $|T|$.
2. $\mathrm{Gal}_{\mathrm{L}}(T)/\mathrm{Gal}^{0}_{\mathrm{L}}(T)$ *is the automorphism group of the set of all algebraic imaginaries.*

□

It was shown in [7] that every bounded hyperimaginary has the same (pointwise) stabilizer as a set of bounded hyperimaginaries of finite length. So $\mathrm{Gal}^{\mathrm{c}}_{\mathrm{L}}(T)$ is the automorphism group of the set of all bounded hyperimaginaries of finite length.

The set of algebraic imaginaries is often called $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. The group

$$\mathrm{Gal}_{\mathrm{L}}(T)/\mathrm{Gal}^{0}_{\mathrm{L}}(T) = \mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$$

is the Galois group introduced by Poizat in [9].

For stable T two tuples a and b which have the same *strong* type (i.e. the same type over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$) have the same type over any model which is independent from ab. It follows that $\mathrm{Gal}^{0}_{\mathrm{L}}(T) = 1$. This was extended to supersimple theories in [1]. Whether this is true for all simple[8] theories is an open problem. All we know is Kim's result ([5]) that $\Gamma_1(T) = 1$ for simple T.

[6] Assume Φ closed under conjunction.

[7] Recall that $\theta(x, y)$ is the formula $\exists z\, \theta(x, z) \wedge \theta(z, y)$.

[8] See [11] for an introduction to simple theories.

7 Two Examples

The first part of this section is concerned with the proof of the following unpublished result of E. Bouscaren, D. Lascar and A. Pillay:

Theorem 23 *Any compact Lie group is the Galois group of a countable complete theory.*

First we need a lemma on O-minimal structures. Recall that a structure M with a distinguished linear order $<$ is O-minimal if every definable subset of M is a union of finitely many points and intervals with endpoints in M. Note that every structure elementarily equivalent to an O-minimal structure is itself O-minimal.

Lemma 24 *Every automorphism of a big saturated O-minimal structure is Lascar strong.*

PROOF: Let $\mathbb{C}$ be a big saturated O-minimal structure. We prove that any two small submodels M, N of the same type have the same type over some model K. This implies, as in the proof of Lemma 1, that every automorphism which maps M to N is the product of an automorphism which fixes M and an automorphism which fixes K.

It is enough (and equivalent, see the proof of Lemma 7 (1)) to show the following : Every consistent formula $\varphi(z)$ has a realization c over which M and N have the same type.

We prove this by induction on the length of z. Assume that z consists of a tuple z_1 and a single variable z_2. By induction there is a realization c_1 of $\exists z_2\varphi(z_1, z_2)$ over which M and N have the same type. Let $\psi(m, c_1, z_2)$ be any formula over Mc_1, and let the tuple $n \in N$ correspond to m. By O-minimality, and since m and n have the same type over c_1, either both $\psi(m, c_1, \mathbb{C})$ and $\psi(n, c_1, \mathbb{C})$ contain a non–empty final segment of $\varphi(c_1, \mathbb{C})$ or $\neg\psi(m, c_1, \mathbb{C})$ and $\neg\psi(n, c_1, \mathbb{C})$ contain a non–empty segment. If we choose c_2 in the intersection of all these segments, $c = c_1c_2$ realizes $\varphi(z)$ and M and N have the same type over c. □

Now fix a compact Lie group G. The group G together with its structure of a real analytic manifold can be defined inside an expansion $\mathcal{R}$ of the field $\mathbb{R}$ by a finite number of analytic functions which are defined on bounded rectangles. By a result of van den Dries $\mathcal{R}$ is O–minimal[9] (see [10]).

Let $\mathcal{R}^*$ a big saturated extension of $\mathcal{R}$ and G^* the resulting extension of G. The intersection μ of all $\emptyset$–definable neighborhoods of the unit element of G^* is

[9] As A. Pillay has told me, compact Lie groups are semi–algebraic. This means that here (and in the proof of Corollary 26) one can actually assume that $\mathcal{R}$ is the field of reals with a finite tuple of named parameters.

the normal subgroup of *infinitesimal* elements. The compactness of G implies that every element of G^* differs by an infinitesimal from some element of G. Whence G^* is the semi–direct product of G and μ.

Lemma 25 *μ is the set of all commutators $[\varphi, h] = h^{-1}\varphi(h)$, where $h \in G^*$ and $\varphi \in \mathrm{Aut}(\mathcal{R}^*)$.*

PROOF: Let φ be an automorphism of $\mathcal{R}^*$ and let h differ from $h_0 \in G$ by an infinitesimal ε. Since φ fixes $\mathbb{R}$, it fixes h_0. Whence $h^{-1}\varphi(h) = (h_0\varepsilon)^{-1}\varphi(h_0\varepsilon) = \varepsilon^{-1}\varphi(\varepsilon)$ is infinitesimal.

Let conversely $\varepsilon \in \mu$ be given. Consider a *generic* type $p \in \mathrm{S}(\emptyset)$ of G (cf. [8]). This means that p can be axiomatized by formulas which define (non–empty) open subsets $O(G)$ of G. Each $O(G^*)$ contains two elements h and $h\varepsilon$ (pick any $h \in O(G)$). Whence, by saturation, p has two realizations h and $h\varepsilon$. Choose an automorphism φ with $\varphi(h) = h\varepsilon$. Then $\varepsilon = h^{-1}\varphi(h)$.[10] $\square$

Consider the two–sorted structure

$$\mathcal{M} = (\mathcal{R}, X, \cdot)$$

where $\cdot$ is a regular action of G on the set X. We will show that G is the Galois group of the complete theory of $\mathcal{M}$.

Let $\mathcal{M}^* = (\mathcal{R}^*, X^*)$ be a big saturated elementary extension of $\mathcal{M}$. To describe the automorphisms of $\mathcal{M}^*$ we fix a base point $x_0 \in X^*$. Any element of X^* can then uniquely be written as

$$x = h \cdot x_0$$

for some $h \in G^*$. We extend each automorphism φ of $\mathcal{R}^*$ to $\mathcal{M}^*$ by

$$\overline{\varphi}(x) = \varphi(h) \cdot x_0.$$

The automorphisms which leave $\mathcal{R}^*$ fixed have the form $\overline{g}$, where

$$\overline{g}(x) = hg^{-1} \cdot x_0.$$

This implies that every automorphism of $\mathcal{M}^*$ is a product

$$\Phi = \overline{g}\,\overline{\varphi}.$$

Note the commutation rule $\overline{\varphi}\,\overline{g} = \overline{\varphi(g)}\,\overline{\varphi}$.

Elementary substructures of $\mathcal{M}^*$ have the form $(\mathcal{R}', G' \cdot x)$, where $\mathcal{R}'$ is an elementary substructure of $\mathcal{R}^*$ and $x = h \cdot x_0$ is any element of X^*. Therefore an automorphism fixes a submodel iff it can be written as $\overline{h}^{-1}\overline{\varphi}\,\overline{h}$, for some φ

[10] A variant of the proof shows that one can find a φ which fixes an elementary submodel of $\mathcal{R}^*$.

which fixes an elementary submodel of $\mathcal{R}^*$. It follows that an automorphism is Lascar strong iff it is a product of conjugates of automorphisms of the form $\overline{\varphi}$, for Lascar strong φ.

By Lemma 24 all φ are Lascar strong. The formula

$$\overline{h}^{-1}\overline{\varphi}\,\overline{h} = \overline{[\varphi, h]}\,\overline{\varphi},$$

together with the last Lemma, implies that $\Phi = \overline{g}\,\overline{\varphi}$ is Lascar strong iff g is infinitesimal. We conclude that

$$g \mapsto \text{class of } \overline{g}$$

defines an isomorphism $\iota : G \to \mathrm{Gal_L}(\mathcal{M}^*)$.[11]

Finally we have to prove that ι is a homeomorphism. Let $U(G)$ be a $\emptyset$–definable neighborhood of $1 \in G$. Consider the map $\nu : \mathrm{S}_{\mathcal{M}}(\mathcal{M}) \to \mathrm{Gal_L}(\mathcal{M}^*)$. Then $\nu^{-1}\iota(U(G))$ consists of those $\mathrm{tp}(f(\mathcal{M})/\mathcal{M})$ for which $\mathcal{M}^* \models U(f(1))$. Whence, if 1 has index 1 in the enumeration of $\mathcal{M}$,

$$\nu^{-1}\iota(U(G)) = \{p \in \mathrm{S}_{\mathcal{M}}(\mathcal{M}) \mid U(x_1) \in p\}.$$

This proves that $\iota(U_n)$ is open. So ι is an open map. Since $\mathrm{Gal_L}(\mathcal{M}^*)$ is quasicompact and G is hausdorff, ε must also be continuous. *This completes the proof of Theorem 23.*

Corollary 26 *Every compact group is the Galois group of a complete theory.*

Proof: Let G be a compact group. G is the direct limit of a directed system $(G_i, f_{i,j})_{i\leq j\in I}$ of compact Lie groups ([12, §25]). Again let $\mathcal{R}$ be an expansion of the reals by bounded analytic functions, in which all the G_i and the maps f_{ij} can be defined. The elements of G are then given by certain infinite tuples $g = (g_i)_{i\in I}$ from the direct product of the G_i.

G will be the Galois group (of the complete theory) of the many–sorted structure

$$\mathcal{M} = (\mathcal{R}, X_i, f'_{ij})_{i\leq j\in I},$$

where the directed system of sets $(X_i, f'_{i,j})_{i\leq j\in I}$ is a copy of $(G_i, f_{i,j})_{i\leq j\in I}$ and each G_i operates (regularly) on X_i as it operates on itself by left multiplication.

Let again $\mathcal{M}^*$ be a big saturated elementary extension of $\mathcal{M}$ and G^* the inverse limit of the G_i^*. We call an element $\varepsilon = (\varepsilon_i)$ of G^* infinitesimal if all its components are infinitesimal. Let μ the subgroup of all infinitesimals. It is easy to see that G is isomorphic to the quotient G^*/μ.

Fix a base point $x_0 = (x_{0i})_{i\in I}$ in the (non–empty) inverse limit of the X_i^*. Then every automorphism of $\mathcal{M}^*$ has the form $\Phi = \overline{g}\,\overline{\varphi}$ for $\varphi \in \mathrm{Aut}(\mathcal{R}^*)$ and

[11] The proof shows that two elements of X^* differ by an infinitesimal if they have the same Lascar strong type. It is easy to verify that this happens iff they have the same type over a submodel of $\mathcal{M}^*$.

$g \in G^*$, where $\overline{g}$ and $\overline{\varphi}$ are defined as in the proof of the theorem. Thus, it suffices to show that Φ is Lascar strong iff g is infinitesimal.

Assume first that Φ is Lascar strong. Then each g_i is infinitesimal, since Φ restricted to $(\mathcal{R}^*, X_i)$ is Lascar strong. Conversely, if g is infinitesimal, we find for every i an $h_i \in G_i$ such that h_i and $h_i g_i$ have the same type. A compactness argument shows that we can find the sequence (h_i) in G^*. Then h and hg have the same type. Let ψ be an automorphism of $\mathcal{R}^*$ with $\psi(h) = hg$. As in the proof of the theorem, it is easy to see that $\overline{g}\overline{\psi} = \overline{h}^{-1}\overline{\psi}\,\overline{h}$ is Lascar strong. Whence also $\Phi = (\overline{h}^{-1}\overline{\psi}\,\overline{h})\psi^{-1}\overline{\varphi}$ is Lascar strong. □

We construct our second example from the circle group S, the unit circle in the complex number plane. Let us fix some notation: λ_s denotes multiplication by s. R is the cyclic ordering on S, where $R(r, s, t)$ holds if s comes before t in the counter–clockwise ordering of $\mathrm{S} \setminus \{r\}$.

Fix a natural number N, write σ_N for $\lambda_{\frac{2\pi \mathrm{i}}{N}}$ and consider the structure

$$\mathcal{S}_N = (\mathrm{S}, R, \sigma_N).$$

Let $\mathbb{C}_N$ a big saturated elementary extension and f an automorphism of $\mathbb{C}_N$. If f is Lascar strong, let $|f|$ be the smallest n such that f is the product of n automorphisms which fix elementary submodels. If f is not Lascar strong, write $|f| = \infty$.

We will make use of the following lemma, which can be proved from Lemma 24 (see [2] for details).

Lemma 27

1. *Every automorphism of $\mathbb{C}_N$ is the product of some σ_N^n and some f with $|f| \leq 2$.*
2. *$|\sigma_N^n| = |n| + 2$, whenever $0 < |n| \leq \frac{N}{2}$.*

Let $\mathcal{S}_\infty$ be the disjoint union of the $\mathcal{S}_1, \mathcal{S}_2, \ldots$ viewed as a many–sorted structure[12] with saturated extension $\mathbb{C}_\infty = (\mathbb{C}_1, \mathbb{C}_2, \ldots)$.

Theorem 28 ([2]) *For each N let C_N be the N–element cyclic group with generator c_N. Let B be the group of all sequences $(\mathrm{c}_N^{e_N})$ with a bounded sequence (e_N) of exponents. Then*

$$\mathrm{Gal}_{\mathrm{L}}(\mathbb{C}_\infty) \cong \prod_N \mathrm{C}_N / B.$$

$\mathrm{Gal}_{\mathrm{L}}(\mathbb{C}_\infty)$ carries the indiscrete topology.

[12] We take also the disjoint union of the languages.

Proof: The map $(\mathrm{c}_N^{e_N}) \mapsto (\sigma_N^{e_N})$ defines a map from $\prod_N \mathrm{C}_N$ to $\mathrm{Aut}(\mathbb{C}_\infty)$, which yields a homomorphism

$$\mu : \prod_N \mathrm{C}_N \to \mathrm{Gal}_{\mathrm{L}}(\mathbb{C}_\infty).$$

Let (f_N) be any automorphism of $\mathbb{C}_\infty$. If we apply part 1 of the Lemma to each component we see that we can write (f_N) as a product of some $(\sigma_N^{e_N})$ and two automorphisms which fix a model. This shows that μ is surjective.

Let $(\mathrm{c}_N^{e_N})$ be an arbitrary element of $\prod_N \mathrm{C}_N$. We can assume that $|e_N| \leq \frac{N}{2}$. Then by part 2 of the lemma it is immediate that $(\sigma_N^{e_N})$ is Lascar strong iff (e_N) is bounded, which means that B is the kernel of μ.

It remains to show that the topology of $\mathrm{Gal}_{\mathrm{L}}(\mathbb{C}_\infty)$ is indiscrete, or

$$\mathrm{Gal}_{\mathrm{L}}(\mathbb{C}_\infty) = \Gamma_1(\mathbb{C}_\infty).$$

The preimage of $\Gamma_1(\mathbb{C}_\infty)$ in $\mathrm{Aut}(\mathbb{C}_\infty)$ is, by the next Lemma, a closed subgroup, which contains $\mathrm{Autf}_{\mathrm{L}}(\mathbb{C}_\infty)$. The automorphisms which fix almost every $\mathbb{C}_N$ are Lascar strong and form a dense subset of $\mathrm{Aut}(\mathbb{C}_\infty)$. Thus the preimage of $\Gamma_1(\mathbb{C}_\infty)$ is the whole $\mathrm{Aut}(\mathbb{C})$ group. $\square$

We conclude with a general lemma. Let M be a model of T and consider the topology of *point–wise convergence* on $\mathrm{Aut}(M)$, with basic open sets

$$U_{a,b} = \{f \mid f(a) = b\},$$

where a, b are finite tuples from M.

Lemma 29 *The natural map* $\mathrm{Aut}(M) \to \mathrm{Gal}_{\mathrm{L}}(T)$ *is continuous.*

Proof: Let Ω be a neighborhood of the image of f in $\mathrm{Gal}_{\mathrm{L}}(T)$. The preimage of Ω under[13] $\nu : \mathrm{S}_M(N) \to \mathrm{Gal}_{\mathrm{L}}(T)$ contains a basic neighborhood

$$O = \{p \mid \varphi(x) \in p\}$$

of $\mathrm{tp}(f(M)/N)$. Let a be the tuple of elements of M which are enumerated by the free variables of φ. Then

$$O = \{\mathrm{tp}(g(M)/N) \mid \mathbb{C} \models \varphi(g(a))\} \subset \{\mathrm{tp}(g(M)/N) \mid g(a) = f(a)\}.$$

Whence $U_{a,f(a)}$ is a neighborhood of f which is mapped into Ω. $\square$

[13] N can be any small model.

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