1 Introduction

Let $f : D \to E$ be a definable map between definable classes. The following theorem is well known:

**Theorem 1** ([1], [2, V 6.8]) *If $E$ has Morley rank $\beta$ and the Morley rank of all fibers $f^{-1}(e)$ is bounded by $\alpha$. Then*

1. *If $\alpha = 0$ $D$ has Morley rank at most $\beta$.*
2. *If $\alpha > 0$ the Morley rank of $D$ is bounded by $\alpha(\beta + 1)$.*

It seems to be less well known that this theorem gives the optimal bound. We will prove:

**Lemma 2** *For all $\alpha > 0$ and all $\beta$ there is a theory $T$ and (in the monster model of $T$) a definable map $f : D \to E$ such that*

a) *$E$ has Morley rank $\beta$*

b) *the Morley rank of all fibers of $f$ is $\alpha$*

c) *$D$ has Morley rank $\alpha \cdot (\beta + 1)$.*

In section 4 we discuss a bound for the Morley rank of $D$ if the Morley rank of all fibers of $f$ is smaller than a limit ordinal $\alpha$.

In the sequel let $R(F)$ denote the (Morley) rank of the definable set $F$. 

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*Revision*: 1.9

†March 2011: Section 5 added.

‡May 2013: Correction of the first part of the proof of Theorem 1

§April 2014: Further simplification of the first part of the proof of Theorem 1. Typo in the formulation of the problem after Theorem 5.

¶March 2015: Remark 6 added.
2 Proof of Theorem 1

We include a proof of Theorem 1 for the convenience of the reader.

Let \( E \) have rank \( \alpha \) and \( A \) be a set of parameters. We call an element \( e \) of \( E \) generic over \( A \) if it is not contained in any \( A \)-definable set of smaller rank than \( \alpha \). \( E \) has always generic elements (in the monster model). Note that all generics of \( E \) have the same type over \( A \) if \( E \) has degree one.

First we handle case 1, where \( \alpha = 0 \). Let \( f : D \to E \) have finite fibers. We prove

\[
R(D) \leq R(E)
\]

by induction on \( \beta = R(E) \).

We may assume that \( E \) has degree 1. Let \( D_i \) be an infinite family of disjoint definable subsets of \( D \). We have to show that almost all of them have smaller rank than \( \beta \). Let \( e \in E \) be generic over the parameters over which \( f, D, E \) and the \( D_i \) are defined. Almost all of the \( D_i \) do not intersect \( f^{-1}(e) \). So these \( f(D_i) \) do not contain \( e \) and have therefore smaller rank than \( \beta \). So by induction almost all \( D_i \) have smaller rank than \( \beta \).

For the case 2 we need a lemma. If \( E \) has Morley degree one and \( e \in E \) is generic over the relevant parameters we call the (possibly empty) set \( f^{-1}(e) \) a generic fiber of \( f \).

Lemma 3 Let \( E \) have Morley degree one and \( \alpha \) be the rank of the generic fiber of \( f \). If

\[
\gamma + \alpha < R(D),
\]

\( D \) has a definable subset \( D' \) such that \( f \mid D' \to E \) has finite generic fiber and

\[
\gamma < R(D').
\]

Proof: We may assume \( \alpha > 0 \). \( D \) contains then an infinite family \( D_i \) of definable disjoint sets having at least rank \( \gamma + \alpha \). Let \( e \in E \) be generic. Then for one index \( i \) the rank of \( D_i \cap f^{-1}(e) \) is smaller than \( \alpha \). By induction on \( \alpha D_i \) contains a \( D' \) as required.

We prove case 2 of the theorem by induction on \( \beta \). We may assume that \( E \) has degree one. If

\[
\alpha \cdot \beta + \alpha < R(D),
\]

by the last lemma, \( D \) contains a definable \( D' \) of rank bigger than \( \alpha \cdot \beta \) such that the generic fiber of \( f \mid D' \) has finitely many, say \( k \) many, elements. For

\[
E^* = \{ e \in E \mid D' \cap f^{-1}(e) \text{ has cardinality } k \},
\]

the complement \( E \setminus E^* \) has a rank \( \beta' < \beta \). Since (by case 1) \( D' \cap f^{-1}(E^*) \) has at most rank \( \beta \), the rank of \( D'' = D' \cap f^{-1}(E \setminus E^*) \) is bigger than \( \alpha \cdot \beta \geq \alpha (\beta' + 1) \). This contradicts the induction hypothesis applied to \( f \mid D'' : D'' \to E \setminus E^* \).
3 Proof of Lemma 2

We deal only with countable $\alpha$ and $\beta$. (The proof in the uncountable case is essentially the same.) So in the sequel infinite means countably infinite.

For a fixed $\alpha > 0$ and for all $\beta$ we will construct models

$$\mathcal{M}_\beta = (D_\beta, E_\beta, f_\beta),$$

which consist of a two sorts $D_\beta$ and $E_\beta$, a map $f_\beta : D_\beta \to E_\beta$ and unary predicates on $D_\beta$ and $E_\beta$ such that

a) $E_\beta$ has Morley rank $\beta$

b) the Morley rank of all fibers of $f_\beta$ is $\alpha$

c) $D_\beta$ has Morley rank $\alpha \cdot (\beta + 1)$.

d) $\mathcal{M}_\beta$ is saturated and has quantifier elimination.

We start with a structure $\mathfrak{A} = (A, P_i)_{i \in I}$, where $A$ is an infinite set and the $P_i$ are unary predicates which ensure that $\mathfrak{A}$ has rank $\alpha$ (and is saturated). For the model $\mathcal{M}_0$ we take $(A, E_0, f_0, P_i)_{i \in I}$, where $E_0$ consists of one point and $f_0$ is the constant map.

We give the following case a special treatment: Assume that $\alpha$ is finite and $\beta$ is a limit cardinal. We take for $E_\beta$ any set with unary predicates giving it rank $\beta$. Choose a surjection $f_\beta : D_\beta \to E_\beta$ with infinite fibers and sets $X^a \subset D_\beta, (a \in A)$, which intersect each fiber of $f_\beta$ in exactly one point. From the predicates $P_i$ we define the predicates $Q_i = \bigcup_{a \in A} X^a$. This is our $\mathcal{M}_\beta$. The sets $X^a$ inherit rank $\beta$ from $E_\beta$. Whence $D_\beta$ has rank $\beta + \alpha$, which in our case equals $\alpha(\beta + 1)$.

Now assume that $\alpha$ is infinite or $\beta$ is a successor ordinal. Also assume that for all $\beta' < \beta$ the structures $\mathcal{M}_{\beta'}$ are constructed. Let $\alpha'$ be such that $1 + \alpha' = \alpha$ and $\mathfrak{A}' = (A', P'_i)_{i \in I'}$ be the $\alpha'$-version of $\mathfrak{A}$. To construct $\mathcal{M}_\beta$ we take infinite sets $D_\beta$ and and $E_\beta$ and a surjective map $f_\beta : D_\beta \to E_\beta$ with infinite fibers.

On $D_\beta$ and $E_\beta$ we choose two families $(X^a)_{a \in A'}$ and $(E^{a,i})_{a \in A', i \in \omega}$ of disjoint subsets (and introduce predicates for them) such that

1. All intersections $X^a \cap f_\beta^{-1}(e)$ and the differences $f_\beta^{-1}(e) \setminus \bigcup_{a \in A'} X^a$ are infinite.

2. The difference $E_\beta \setminus \bigcup_{a \in A', i \in \omega} E^{a,i}$ is infinite. The cardinality of the $E^{a,i}$ will be specified later.

\[1\text{If } \alpha' = 0 \text{ A' is just any finite set}\]
From the predicates $P'_i$ we define also the predicates

$$Q_i = \bigcup_{a \in P'_i} X^a.$$ 

Let $(\beta_i)_{i \in \omega}$ be an enumeration of the ordinals $\beta' < \beta$ where all $\beta'$ occur infinitely often. In our last step, for each $a \in A'$ and $i \in \omega$ we introduce new predicates for subsets of $E^{\alpha,i}$ and for subsets of $D^{\alpha,i} = f^{-1}_\beta(E^{\alpha,i}) \cap X^a$

such that with the new predicates the structure $f_\beta \restriction D^{\alpha,i} : D^{\alpha,i} \rightarrow E^{\alpha,i}$ looks like $\mathcal{M}_\beta$. This also tells us the right cardinality of the $E^{\alpha,i}$. This completes the construction of $\mathcal{M}_\beta$.

It is easy to check that $\mathcal{M}_\beta$ has quantifier elimination and is saturated.

Since the $E_{a,i}$ have rank $\beta_i$, $E_\beta$ has rank $\beta$.

Without the structure imprinted on the $D_{a,i}$ the fibers look like $\mathcal{M}'$ with each point blown up to an infinite set and have therefore rank $1 + \alpha' = \alpha$. The structure on $D_{a,i}$ adds one set of rank $\alpha$ on the fiber. Whence the fibers have rank $\alpha$.

Each $D^{\alpha,i}$ has rank $\alpha \cdot (\beta_i + 1)$. We have to distinguish two cases:

1. $\beta$ is a successor ordinal: Then $X^\alpha$ has at least rank $\alpha \cdot \beta + 1$ and $D_\beta$ has at least rank $\alpha \cdot \beta + 1 + \alpha' = \alpha(\beta + 1)$.

2. $\beta$ is a limit ordinal and $\alpha$ is infinite: Then $X^\alpha$ has at least rank $\alpha \cdot \beta$ and $D_\beta$ the rank $\alpha \cdot \beta + \alpha' = \alpha \cdot \beta + \alpha = \alpha \cdot (\beta + 1)$.

4 Fiber rank smaller than a limit ordinal

The following problem is left open by Theorem 1: Let $\alpha$ be a limit ordinal. If $E$ has rank $\beta$ and the ranks of all fibers of $f : D \rightarrow E$ are smaller than $\alpha$ can we say more about $R(D)$ than just $R(D) \leq \alpha(\beta + 1)$? The answer is yes:

**Theorem 4** Let $\alpha$ be a limit ordinal and $\beta$ be arbitrary

1. Let $f : D \rightarrow E$ be a definable map between definable classes: Assume $E$ has Morley rank $\beta$ and that the Morley rank of all fibers $f^{-1}(e)$ is smaller than $\alpha$. Then the Morley rank of $D$ is smaller than $\alpha(\beta + 1)$.

2. For all $\gamma < \alpha$ and all $\beta$ there is a theory $T$ and (in the monster model of $T$) a definable map $f : D \rightarrow E$ such that
a) $E$ has Morley rank $\beta$

b) the Morley rank of all fibers of $f$ is smaller than $\alpha$

c) $D$ has Morley rank $\alpha \cdot \beta + \gamma$.

Part 1 has the same proof as Theorem 1. But part 2 needs a modification of the construction in Lemma 2.

Again we deal only with countable $\alpha$ and $\beta$. By recursion on $\beta$ we construct models

$$\mathfrak{M}_\beta = (D_\beta^\gamma, E_\beta^\gamma, f_\beta^\gamma)$$

such that

a) $E_\beta^\gamma$ has Morley rank $\beta$

b) the Morley rank of all fibers of $f_\beta^\gamma$ is smaller than $\alpha$

c) $D_\beta^\gamma$ has Morley rank $\alpha \cdot \beta + \gamma$.

d) $\mathfrak{M}_\beta$ is saturated and has quantifier elimination.

We construct $\mathfrak{M}_\beta$ as in the proof of Lemma 2 from a structure $\mathfrak{A} = (A, P_i)_{i \in I}$ of rank $\gamma$. If $\beta > 0$ assume that for all $\beta' < \beta$ (and all $\gamma < \alpha$) the structures $\mathfrak{M}_{\beta'}$ are constructed. Take infinite sets $D_\beta^\gamma$ and and $E_\beta^\gamma$ and a surjective map $f_\beta^\gamma : D_\beta^\gamma \to E_\beta^\gamma$ with infinite fibers. Then choose two families $(X^a)_{a \in A}$ and $(E^a,i)_{a \in A, i \in \omega}$ of disjoint subsets (and introduce predicates for them) such that

1. All intersections $X^a \cap (f_\beta^\gamma)^{-1}(e)$ and the differences $(f_\beta^\gamma)^{-1}(e) \setminus \bigcup_{a \in A} X^a$ are infinite.

2. The difference $E_\beta^\gamma \setminus \bigcup_{a \in A, i \in \omega} E^{a,i}$ is infinite.

Define again the predicates $Q_i = \bigcup_{a \in P_i} X^a$.

Finally we introduce new unary predicates on $E^{a,i}$ and

$$D^{a,i} = (f_\beta^\gamma)^{-1}(E^{a,i}) \cap X^a$$

such that with the new predicates the structure

$$\mathfrak{N}^{a,i} = (D^{a,i}, E^{a,i}, f_\beta^\gamma \upharpoonright D^{a,i})$$

looks as follows:

Case 1: $\beta = \beta' + 1$ is a successor.

Then choose an enumeration $(\gamma_i)_{i \in \omega}$ of the ordinals below $\alpha$ and let $\mathfrak{N}^{a,i}$ look like $\mathfrak{M}_{\beta'}^{\gamma_i}$. 

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Case 2: $\beta$ is a limit ordinal.

Let $(\gamma_i)_{i \in \omega}$ enumerate the ordinals below $\beta$ and let $\mathfrak{M}^{a,i}$ look like $\mathfrak{M}^{\beta}$. In the successor case $D^{n,i}$ has rank $\alpha \cdot \beta' + \gamma_i$, $X^a$ has rank $\alpha \cdot \beta' + \alpha = \alpha \cdot \beta$. In the limit case $D^{n,i}$ has rank $\alpha \cdot \beta_i$ and it follows again that $X^a$ has rank $\alpha \cdot \beta$. This implies that $D^\gamma$ has rank $\alpha \cdot \beta + \gamma$.

5 A better bound

The following theorem implies both Theorem 1 (2) and Theorem 4 (1):

**Theorem 5 ([3, Exercise 6.4.4])** If $E$ has Morley rank $\beta$, Morley rank of all fibers $f^{-1}(e)$ is bounded by $\alpha > 0$ and the Morley rank of the generic fibers is bounded by $\alpha^{\text{gen}}$, then the Morley rank of $D$ is bounded by

$$\alpha \beta + \alpha^{\text{gen}}.$$  

The proof is a slight variation of the proof of Theorem 1. One proves similarly:

**Remark 6** If $\beta$ is a limit ordinal, $\beta < \alpha \beta$, and $\alpha^{\text{gen}}$ is finite, then the Morley rank of $D$ is smaller than $\alpha \beta + \alpha^{\text{gen}}$.

Slight modifications of the constructions above show that this bounds are optimal: If $\beta$ and $0 \leq \alpha^{\text{gen}} \leq \alpha$ are given, there are two cases:

1. If the conditions of Remark 6 are not satisfied, there is an example $D$ with Morley rank $\alpha \beta + \alpha^{\text{gen}}$.

2. If the conditions of Remark 6 are satisfied, for every $\gamma$ smaller than $\alpha \beta + \alpha^{\text{gen}}$ there is an example with at least Morley rank $\gamma$.

References

