Strong Fraïssé Limits

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Let $\mathcal{K}$ be a countable class of L-structures and $\mathcal{S}$ a countable class of embeddings between elements of $\mathcal{K}$ which is closed under composition. We call the elements of $\mathcal{S}$ *strong* embeddings. We assume that the empty structure 0 is in $\mathcal{K}$ and all $0 \to A$ are strong. (This is a pure convention).

Definition 1. *1. We call a sequence $A_0 \to A_1 \to A_2 \to \ldots$ rich, if the A_i are in $\mathcal{K}$, the maps are strong and the following holds: For all i and for all strong $f : A_i \to B$ there is a $j \geq i$ and a strong $g : B \to A_j$ such that gf is the given map $A_i \to A_j$.*

2. A Fraïssé limit of $(\mathcal{K}, \mathcal{S})$ is a direkt limit of a rich sequence.[1]

Theorem 2. *If $\mathcal{K}$ has the amalgamation property with respect to strong embeddings, rich sequences exist and the Fraïssé limits are all isomorphic.*

Proof. Assume amalgamation. The existence of rich sequences is obvious. Now let $A_0 \to A_1 \to \ldots$ and $A'_0 \to A'_1 \to \ldots$ be rich with direkt limits M and M'. We call a partial isomorphisms bewetween M and M' good if it is given by a pair of strong embeddings $e : C \to A_i$ and $e' : C \to A'_{i'}$. We try to extend such an partial isomorphism so that its image contains A_i[2]. This will show the forth property of the family of good isomorphism. By symmetry we will have also the back property.

We amalgamate first e and e' and obtain strong embeddings $f : A_i \to B$ and $f' : A'_{i'} \to B$, such that $fe = f'e'$. Then we apply richness and get $j \geq i$ and $j' \geq i'$ and strong embeddings $h : B \to A_j$ and $h' : B \to A'_{j'}$ such that hf is the given map $A_i \to A_j$ and $h'f'$ is the given map $A'_{i'} \to A'_{j'}$. This is our prolongation. □

The assumption that § is closed under composition is not necessary by the following simple remark.

Remark 3. *Let $\mathcal{S}$ be a countable set of embeddings between elements of $\mathcal{K}$ such that $(\mathcal{K}, \mathcal{S})$ has the amalgamation property. Let $\mathcal{S}'$ be the closure of $\mathcal{S}$ under composition. Then $\mathcal{S}'$ is countable and $(\mathcal{K}, \mathcal{S}')$ still has the amalgamation property*

[1] Note that the direkt limit of a sequence is only defined up to isomorphism.

[2] or rather the image of A_i in M. Note also that we can alway prolong e to a strong map from C to A_k for arbitrary $k > i$