Notes on totally categorical theories

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0 Introduction

Cherlin, Harrington and Lachlan's paper on ω_0-categorical, ω_0-stable theories ([CHL]) was the starting point of geometrical stability theory. The progress made since then allows us better to understand what they did in modern terms (see [PI]) and also to push the description of totally categorical theories further, (see [HR1, AZ1, AZ2]).

The first two sections of what follows give an exposition of the results of [CHL]. Then I explain how a totally categorical theory can be decomposed by a sequence of covers and in the last section I discuss the problem how covers can look like.

I thank the parisian stabilists for their invation to lecture on these matters, and also for their help during the talks.

1 Fundamental Properties

Let T be a totally categorical theory (i.e. T is a complete countable theory, without finite models which is categorical in all infinite cardinalities).

Since T is ω_1-categorical we know that

a) T has finite Morley-rank, which coincides with the Lascar rank U.

b) T is *unidimensional* : All non-algebraic types are non-orthogonal.

The main result of [CHL] is that

c) T is *locally modular.*

A pregeometry X (i.e. a matroid) is *modular* if two subspaces are always independent over their intersection. X is called *locally modular* if two subspaces are independent over their intersection provided this intersection has positive dimension. A pregeometry is a *geometry* if the closure of the empty set is empty and the one-dimensional subspaces are singletons. The one-dimensional subspaces of X form a geometry in a natural way, the *associated* geometry $\tilde{X}$. X is (locally) modular iff $\tilde{X}$ is.

The realization set of a regular type $p \in S(A)$ carries the structure of a pregeometry, where "c is in the closure of C" is interpreted as "$c \underset{A}{\not\perp} C$", which for U-rank 1 types is the same as "$c \in \mathrm{acl}_A(C)$". A theory T is *locally modular* if all stationary types of U-rank 1 are locally modular i.e. have locally modular geometry.

By a) every stationary $p \in S(A)$ of U-rank 1 is strongly minimal. Whence p contains an A-definable strongly minimal set $\mathbf{D}$. "acl_A" defines on $\mathbf{D}$ its *pregeometry over* A which is the same as the pregeometry of p except that $\mathbf{D}$ may contain elements from $\mathrm{acl}(A)$.

If A is finite the closure of the empty set in $\mathbf{D}$ is finite and the one-dimensional subspaces of $\mathbf{D}$ form a uniformly definable family of finite sets. Whence the associated geometry of $\mathbf{D}$ is the pregeometry over A of a certain strongly minimal set $\mathbf{D}_A$, which lives in $\mathbf{C}^{eq}$. $\mathbf{D}_A$ is called the associated *strictly minimal set over* A. It has the same geometry as p. The local modularity of T (at least for types defined over finite sets) follows therefore from

Theorem 1.1 ([CHL]) *Let $\mathbf{D}$ be a strictly minimal set over $\emptyset$. Then the structure of $\mathbf{D}$ is one of the following:*

a) trivial

b) a projective space over a finite field

c) an affine space over a finite field

Some explanations:

The *induced* structure on a 0-definable set $\mathbf{P}$ tells us what the 0-definable relations on $\mathbf{P}$ are. If we are in a countable ω_o-categorical model M the induced structure on P is given by the group of all permutations of P which are induced by automorphisms of M: The 0-definable relations on P are the relations which are invariant under all automorphisms of M. If A is a set of parameters and P is A-definable one can also speak about the induced structure of P over A, giving the A-definable relations on P.

Let X be an infinite-dimensional projective (or affine) space over the field $\mathbf{F}_{p^n}$. Let G be a group of collineations which contains all linear collineations of X. (There are exactly $\phi(n)$ possible choices for G). X with the structure given by G is what we mean by a *projective (affine) space over* $\mathbf{F}_{p^n}$. The acl-closed subsets of X are just the subspaces of X. All these structures (and the trivial structure) are totally categorical and strictly minimal. The projective spaces are (as the trivial structure) modular, the affine spaces locally modular.

Many properties of totally categorical theories are shared by all locally modular theories of finite U-Rank - the LMFR-theories. See [B].

The following Lemma is the main use of modularity. If A is a subset of $\mathbf{D}$ it specializes just to the definition.

Lemma 1.2 *Let T be stable and $\mathbf{D}$ be a 0-definable strongly minimal set which is modular over $\emptyset$. Let B be a subset of $\mathbf{D}$ and A be a set of parameters. Then B and A are independent over* $\mathrm{acl}(B) \cap \mathrm{acl}(A) \cap \mathbf{D}$.

Proof:
We can assume that $\mathrm{acl}(B) \cap \mathrm{acl}(A) \cap \mathbf{D}$ is in the definable closure of the empty set. Using the following easy fact

Lemma *Let T be a stable theory. Then for all sets A and elements b there is a set C of realizations of* $\mathrm{tp}(b/A)$ *such that* $b \underset{C}{\downarrow} A$ *and* $b \underset{A}{\downarrow} C$.

we obtain a subset C of $\mathbf{D}$ such that

(1) $$B \underset{C}{\downarrow} A$$

(2) $$B \underset{A}{\downarrow} C \qquad .$$

If B and A are dependent over $\emptyset$ then by (1) also B and C are dependent over $\emptyset$. Whence by modularity there must be an element $d \in \mathbf{D}$ which is algebraic over B and over C but not algebraic over $\emptyset$. It follows from (2) that d is algebraic over A, whence d belongs to $\mathrm{acl}(B) \cap \mathrm{acl}(A) \cap \mathbf{D}$, a contradiction.□

Lemma 1.2 answers the following question which arises from theorem 1.1. Let $\mathbf{D}$ be an A-definable strongly minimal modular set and B a finite set extending A. The two geometries of $\mathbf{D}$ over A and B are given by theorem 1.1, where we added the elements of A and B respectively. What is the relationship between these two geometries ? By 1.2 the answer is that the pregeometry of $\mathbf{D}$ over B (also for infinite B) is the pregeometry of $\mathbf{D}$ over A relativized by $\mathrm{acl}(B) \cap \mathbf{D}$. We conclude that also the U-rank 1 types which are defined over infinite sets are locally modular.

Corollary 1.3 *(see [C.L] for LMFR-theories)*
All stationary types $p \in S(A)$ of U-rank 1 which are not modular are sound *i.e. if $q \in S(B)$ is a non-forking modular extension of p then p is realized in* $\mathrm{acl}(B)$.

Proof:
We can assume that A is empty. Let $\mathbf{D}$ be a strictly minimal set over $\emptyset$ which is not modular. By theorem 1.1 $\mathbf{D}$ is an affine geometry. Let **D'** be the set of the parallel-classes of $\mathbf{D}$, which is strictly minimal and modular over $\emptyset$. $\mathbf{D} \cup \mathbf{D'}$ is a set of Morleyrank 1 (and degree 2) which has a modular geometry over $\emptyset$ and we can apply 1.2 to it (!). Now assume that $\mathbf{D}$ and $\mathrm{acl}(B)$ have empty intersection. Then the pregeometry of $\mathbf{D}$ over B is the pregeometry of $\mathbf{D}$ relativized by $\mathrm{acl}(B) \cap \mathbf{D'}$. Since an affine geometry relativized by a set of parallel-classes remains affine $\mathbf{D}$ remains be non-modular over B. □

A stable theory is *1-based* if the canonical base of every strong type $\mathrm{st}(A/B)$ is contained in $\mathrm{acl}(A)$, which is equivalent to $A \underset{\mathrm{acl}(A)\cap\mathrm{acl}(B)}{\downarrow} B$. Thus 1-basedness means that A and B can only be dependent over C for the simple reason that there is an element which is algebraic over A and algebraic over B but not algebraic over C.

Corollary 1.4 *T is 1-based*

Proof:
Assume that a and b are dependent over C. We want to find an element c algebraic over a and b but not over C. In a superstable theory $\mathrm{st}(a/C)$ and $\mathrm{st}(b/C)$ are domination equivalent to products of regular types p_i. This means that one can find an extension M of C, independent from ab over C, and for a an M-independent tuple $\overline{a}$ of realizations of regular types over M which is domination equivalent to a over M and similarly for b a tuple $\overline{b}$. Since a and b are dependent over M also $\overline{a}$ and $\overline{b}$ are dependent over M. We can assume that all the occuring regular types are equal or orthogonal. Whence there are subsequences $\overline{a'}$ and $\overline{b'}$ of $\overline{a}$ and $\overline{b}$ whose elements all have the same type p and are still dependent over M. Since a and b have finite U-rank over C, we can assume that p is of U-rank 1 (use [D.L]). p is the non-forking extension to M of a stationary type q, which is defined over a small subset of M. If we assume M to be enough saturated q is realized in M. Whence the local modularity of q implies that p is modular. Now by 1.2 we find a c' realizing p which is algebraic over $M\overline{a'}$ and over $M\overline{b'}$. c' is dominated by a and by b over M, whence algebraic over Ma and over Mb. Let E be the canonical base of $s\mathrm{tp}(Mc'/Cab)$.

Since $Mc' \underset{Ca}{\downarrow} b$, $Mc' \underset{Ca}{\downarrow} b$ and $Mc' \underset{C}{\not\downarrow} a$, E is a subset of $\mathrm{acl}(Ca)$, $\mathrm{acl}(Cb)$ but not of $\mathrm{acl}(C)$. Now choose c as an element from E which is not algebraic over C. □

Corollary 1.5 *("Coordinatization")*
If a is non-algebraic over A there is a b which has U-rank 1 over A and is algebraic over Aa

Proof:
If $n = U(a/A) = 1$ there is nothing to show. If $n > 1$ we choose a c such that $U(a/Ac) = n - 1$.

Since a and c are dependent over A 1-basedness gives a b' which is algebraic over Aa and over Ac but not over A. a is not algebraic over Ac and therefore not algebraic over Ab'. The equation

$$U(a/A) = U(ab'/A) = U(a/Ab') + U(b'/A)$$

shows $U(b'/A) < n$. By induction we find b in the algebraic closure of Ab' which has U-rank 1 over A. □

Lemma 1.6 *For every a there is a sequence $a_1, ...a_n = a$ such that all a_i are in* $\mathrm{acl}(a)$ *and all types* $\mathrm{tp}(a_{i+1}/a_1...a_i)$ *are algebraic or strictly minimal.*

Proof:
If a is algebraic over $\emptyset$ we set $a_1 = a$. Otherwise 1.5 gives us an a_2 in $\mathrm{acl}(a)$ which has U-rank 1 over $\emptyset$. By the arguments preceding 1.1 there is an a_1 such that $\mathrm{tp}(a_1/\emptyset)$ is strictly minimal and a_2 is algebraic over a_1. Since $U(a/a_1a_2) < U(a/\emptyset)$ we can use induction to conclude that there is a decomposition $a_3, ..a_n = a$ of a over the parameters a_1a_2. □

Note that n can be bounded by two times the Morleyrank of a over $\emptyset$.

Example 1 Let p be a prime, n a natural number and M be the totally categorical abelian group $\bigoplus_\omega (\mathbf{Z}/p^n\mathbf{Z})$. For every element a of order $p^m < p^n$

$$D_a = \{b^* \mid pb = a,\, b \neq 0\},$$

where $b^* = b + \mathbf{Z}p^m b$, is strictly minimal over a. A decomposing sequence of a is

$$(p^{m-1}a)^*,\, p^{m-1}a, \ldots a^*, a.$$

Note that $p^i a$ is algebraic over $(p^i a)^*$ and that $(p^{i-1}a)^*$ belongs to $D_{p^i a}$.

Lemma 1.7 *Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be 0-definable strictly minimal modular sets. Then there is a unique 0-definable bijection between them.*

Proof:
If f is a definable bijection between $\mathbf{D}_1$ and $\mathbf{D}_2$, then $f(b_1)$ must be the (unique) element b_2 of $\mathbf{D}_2$ which is algebraic over b_1. Whence f is unique. On the other hand it is enough to find two elements b_i of $\mathbf{D}_i$ (i=1,2) which are interalgebraic. Then the formula $f(x, y) = \mathrm{tp}(b_1, b_2)$ defines the desired bijection:

Since $\mathbf{D}_1$ and $\mathbf{D}_2$ are not orthogonal there are subsets B_i of $\mathbf{D}_i$ (i=1,2) which are dependent over $\emptyset$. Since $\mathbf{D}_2$ is modular, there is an element b_2 of $\mathrm{acl}(B_1) \cap \mathrm{acl}(B_2) \cap \mathbf{D}_2$ which is not algebraic over $\emptyset$. Since B_1 and b_2 are dependent, and $\mathbf{D}_1$ is modular, there is an element b_1 of $\mathrm{acl}(B_1) \cap \mathrm{acl}(b_2) \cap \mathbf{D}_1$ which is not algebraic over $\emptyset$. Now b_1 and b_2 are interalgebraic. □

Corollary 1.8 *There is a strictly minimal modular set $\mathbf{D}$ over $\emptyset$.*

Proof:
By 1.5 there is a 0-definable set $\mathbf{D}$ of Morleyrank 1. We can assume that $\mathbf{D}$ does not contain elements of $\mathrm{acl}(\emptyset)$. The finite equivalence relation theorem gives a 0-definable equivalence relation E which partitions $\mathbf{D}$ into finitely many strongly minimal sets $\mathbf{D}_1, ...\mathbf{D}_n$. The $\mathbf{D}_i$ have over $\emptyset$ the same pregeometry as over $\mathrm{acl}(\emptyset)$ over which set they are defined. Let $\mathbf{D'}_i$ be the strictly minimal set of parallel classes of $\mathbf{D}_i$ and $\mathbf{E}$ the 0-definable disjoint union of the $\mathbf{D'}_i$. By 1.7 there are unique $\mathrm{acl}(\emptyset)$-definable bijections between the $\mathbf{D'}_i$. This shows that in the pregeometry of $\mathbf{E}$ every 1-dimensional subspace meets each of the $\mathbf{D'}_i$ in exactly one point. The set of 1-dimensional subspaces of $\mathbf{E}$ is therefore 0-definable and strictly minimal. □

Corollary 1.9 *If one strictly minimal set over some parameter set has trivial geometry then all strictly minimal sets over arbitrary parameter sets have trivial geometry.*

Proof:
If $\mathbf{D}$ is strongly minimal over a finite set A it is of one of the types a), b) or c) of 1.1. If B extends A then by 1.2 $\mathbf{D}_B$ has the relativized geometry (where in the affine case one has not only to relativized by a subset but also by a set of parallel classes, see the proof of 1.3.) This shows that $\mathbf{D}_B$ is of the same type, except that an affine geometry could have become projective. Now the corollary follows from 1.7. □

2 Envelopes

In this section we fix a totally categorical theory T and a strictly minimal modular set $\mathbf{D}$ over $\emptyset$ as given by 1.8.

Definition 1 *Let B be a set of parameters. A set E is called an* envelope *of B if $E \underset{B}{\downarrow} \mathbf{D}$ and E is maximal with this property.*

If $\mathbf{P}$ is a B-definable class in a stable theory then $A \underset{B}{\downarrow} \mathbf{P}$ is equivalent to $\mathrm{tp}(A/B)$ being almost orthogonal to every type $\mathrm{tp}(\bar{c}/B)$ of a tuple $\bar{c}$ from $\mathbf{P}$. If $\mathbf{P}$ is strongly minimal we have to check this only for tuples of B-independent elements of $\mathbf{P}$. If furthermore $\mathbf{P}$ is modular over B 1.2 tells us that it is enough to test single elements of $\mathbf{P}$ i.e. $A \underset{B}{\downarrow} \mathbf{P}$ is equivalent to $\mathrm{acl}(AB) \cap \mathbf{P} = \mathrm{acl}(B) \cap \mathbf{P}$.

Example 2 In the group M of example 1 the envelopes are -up to an automorphism- the subgroups $\bigoplus_{i<m} (\mathbf{Z}/p^n\mathbf{Z})$.

The following Lemma shows that envelopes are in our situation what for non-totally transcendental unidimensional theories was called "maximal non-modular models" in [PR]. But of course envelopes are not models in general.

Lemma 2.1

1. *Let A be algebraically closed over B. Then $A \underset{B}{\downarrow} \mathbf{D}$ iff no modular type of U-rank 1 over any subset of A containing B is realized in A*
2. *E is an envelope of B iff*
   (a) $E \underset{B}{\downarrow} \mathbf{D}$
   (b) *E is algebraically closed over B*
   (c) *All non-modular stationary types of rank 1 over subsets of E are realized in E. It is enough to know this for types over E i.e. that all U-rank 1 types over E are modular.*

Proof:
1: Assume $A \underset{B}{\downarrow} \mathbf{D}$, let C be a subset of A which contains B and let $p \in S(C)$ be a modular stationary type of U-rank 1.The proof of 1.7 shows that one (and therefore every) realization a of p is interalgebraic over C with some element d of $\mathbf{D}$. $a \underset{C}{\not\downarrow} d$ implies then $Ca \underset{B}{\not\downarrow} d$. Whence

a cannot lie in A. For the converse assume that $A \not\downarrow_B \mathbf{D}$. Since A is algebraically closed in $\mathbf{C}^{eq}$ there is an element a of A with $a \not\downarrow_B \mathbf{D}$. By 1.6 there is a seqence $a_1, ...a_n = a$ of elements of A such that the types $\mathrm{tp}(a_{i+1}/a_1...a_i)$ have at most rank 1. For some i we must have $a_{i+1} \not\downarrow_C \mathbf{D}$ where $C = Ba_1...a_i$. Since $\mathbf{D}$ is modular over C, a_{i+1} is interalgebraic over C with an element of $\mathbf{D}$. But then the geometries of $p = \mathrm{tp}(a_{i+1}/\mathrm{acl}(C))$ and $\mathbf{D}$ must be the same (see 1.7) and p ist modular.
2: Assume first that E is an envelope of B. (a) and (b) are clearly true. So let C be a subset of E and p a non-modular stationary type of rank 1 over C. Let a be a realization of the nonforking extension $p|E$ of p to E. By maximality of E we have $a \not\downarrow_E \mathbf{D}$. The proof of 1 shows that $p|E$ must be modular. By 1.3 p is realized in E. This shows (c). For the converse assume (a) and (b) are true but that E is not maximal with (a). Then there is an a which is not algebraic over E such that $a \downarrow_E \mathbf{D}$. By 1.5 we can assume that $p = \mathrm{tp}(a/E)$ has rank 1. By 1 p is not modular. Whence (c) is false. □

Corollary 2.2 *If* $\mathbf{D}$ *is trivial* T *is* almost strongly minimal *i.e.* $\mathbf{C}$ *is the algebraic closure of a strongly minimal set.*

Proof:
Let a be any element and $B = \mathrm{acl}(a) \cap \mathbf{D}$. We show that $a \in \mathrm{acl}(B)$. By 1.2 $aB \downarrow_B \mathbf{D}$. If a is not algebraic over B we find a subset C of $\mathrm{acl}(a)$ which contains B such that $\mathrm{tp}(a/C)$ is stationary and of U-rank 1. Then part 1 of the lemma shows that $\mathrm{tp}(a/C)$ is not modular. But $\mathrm{tp}(a/C)$ is trivial by 1.9. A contradiction. □.

Corollary 2.3 *Envelopes of finite sets are finite in every sort of* $\mathbf{C}^{eq}$.

Proof:
Let B be finite and E be an envelope of B. We show first that

$$A = \{e \in E | U(e/B) \leq 1\}$$

is finite: Since in every sort there are only finitely many types over $\mathrm{acl}(B)$ we can fix a type $p \in S(\mathrm{acl}(B))$ of rank at most 1 and show that p has only finitely many realizations in E. If p is algebraic this is clear. If p has rank 1 it is locally modular. If $e \in E$ is a realization of p the nonforking extension $p|\mathrm{acl}(B)e$ of p to $\mathrm{acl}(B)e$ is modular. By 2.1 it cannot be realized in E . We conclude that all other realizations of p in E must be algebraic over Be. Whence there are only finitely many.

Every element e of E of a fixed sort has a decomposition $e_1, ...e_n = e$ as in 1.6 for a fixed n where all the e_i belong to E. We show that E contains only finitely many elements in this sort: The case $n = 1$ is already clear. If we add the elements of A to the language E remains an envelope of B and all the e's have a decomposition of length $n - 1$. Now we are finished by induction. □

The proof shows also that the envelope of an infinite set B has the same cardinality as B.

Corollary 2.4 *Any two envelopes of* B *are isomorphic over* B.

Proof:
Let E and F be two envelopes of B. Let $f : C \to F$ be a maximal elementary map over B from

a subset C of E to F. Clearly C is algebraically closed over B. We want to show that $C = E$. If not, there is an element e in $E \setminus C$. By 1.5 we can assume that $p = \mathrm{tp}(e/C)$ has rank 1. Now by 2.1(1) p is not modular and by 2.1(2) $f(p)$ is realized in F. This gives an extension of f to $C \cup \{e\}$, a contradiction. Whence f is defined on E. Since E is maximal f is surjectiv. $\square$

Definition 2 *A set E is called* strongly homogeneous *if every isomorphism between two subsets of E can be extended to an automorphism of E.*

Corollary 2.5 *An envelope of a finite set is strongly homogeneous.*

Proof:
Let E be an envelope of a finite set. Then E envelopes the finite subspace $U = E \cap \mathbf{D}$ of $\mathbf{D}$, whose dimension we denote by n. Let f be an isomorphism between the subsets A_1 and A_2 which we can assume to be algebraically closed. Let m be the dimension of the two subspaces $V_i = A_i \cap \mathbf{D}$ $(i = 1, 2)$ of U. First extend f to a partial map g defined on E. U and $g(U)$ both have dimension $n - m$ over V_2. The remark after 1.2 shows that also in the geometry of $\mathbf{D}$ over A_2 U and $g(U)$ are subspaces of dimension $n - m$. Whence there is an elementary map which fixes the elements of A_2 and maps $g(U)$ onto U. Composing g with this map shows that we can assume that $g(U) = U$.

E is an envelope of both sets $U \cup A_i$ $(i = 1, 2)$. Therefore $g(E)$ is an envelope of $U \cup A_2 = g(U \cup A_1)$. By 2.4 there is an elementary map which fixes the elements of $U \cup A_2$ and maps $g(E)$ onto E. If we compose g with this map we obtain the desired extension of f to an automorphism of E. $\square$

Theorem 2.6 *Every finite set of axioms of T has a finite model.*

Proof:
We can assume that the language of T is relational. Let B be a finite set which realizes all n-types over $\emptyset$. Let E be an envelope of B. Since E is strongly homogeneous every 1-type over every $(n-1)$-element subset of E is realized in E. Now an induction as in the proof of Tarski's criterion shows that for all $k \leq n$, all formulas $\phi(x_1, \ldots x_{n-k})$ of quantifier depth at most k and all $e_1, \ldots e_{n-k} \in E$

$$E \models \phi(e_1, \ldots e_{n-k}) \iff \mathbf{C} \models \phi(e_1, \ldots e_{n-k})$$

. Whence all true axioms of quantifier depth at most n are true in E. $\square$

3 Decomposition into covers

In this section let T be an arbitrary complete theory.
Let $\mathbf{E}$ and $\mathbf{F}$ be A-definable classes. We assume $\mathbf{E}$ to be non-empty and $\mathbf{F}$ to be infinite.

Definition 1 $\mathbf{E}$ *is* $\mathbf{F}$-internal *if there is a set B such that* $\mathbf{E} \subset \mathrm{dcl}(\mathbf{F} \cup B)$.

Example 3 Let $\mathbf{G}$ be a group acting transitively on a class $\mathbf{H}$ in a definable way. Then $\mathbf{H}$ is $\mathbf{G}$-internal.

Lemma 3.1 $\mathbf{E}$ *is* $\mathbf{F}$*-internal iff for some n there is a definable surjection* $\pi : \mathbf{F}^n \to \mathbf{E}$.

Proof:
If π is B-definable then each $\pi(\overline{f})$ is definable over $B\overline{f}$. If conversely $\mathbf{E} \subset \mathrm{dcl}(\mathbf{F} \cup B)$, for every $e \in \mathbf{E}$ there is a B-definable partial function π_e and elements $\overline{f}$ of $\mathbf{F}$ such that $e = \pi_e(\overline{f})$. By compactness we find the π_e among a finite family $\pi_0, \ldots \pi_m$ of k-place functions and if we enlarge

B to include A we can assume that they map into $\mathbf{E}$. Choose an element e_0 from $\mathbf{E}$. We define now the $n = m + 1 + k$-place function π defined on $\mathbf{F}$ as follows: If $f_0, ...f_m, \overline{f}$ are given let i be maximal such that $f_0 = f_i$. If $\pi_i(\overline{f}) = e$ is defined set $\pi(\overline{f}) = e$. Otherwise set $\pi(\overline{f}) = e_0$. π is surjectiv and Be_0-definable. $\square$

Remark 3.2 *Let* $\mathbf{D}_i$ $(i = 1, 2)$ *be two strictly minimal sets over parameter sets* A_i *in a totally categorical theory. If* $A_1 \subset A_2$ *then* $\mathbf{D}_2$ *is* $\mathbf{D}_1$*-internal.*

Proof:
Since there is a A_2-definable surjection from $\mathbf{D}_1$ onto $(\mathbf{D}_1)_{A_2}$ we can assume that $A_1 = A_2$. Furthermore there is an obvious definable surjection from $\mathbf{D}_1$ onto $\mathbf{D'}_1$, the strictly minimal set of parallel classes of $\mathbf{D}_1$. We may therefore assume that $\mathbf{D}_1$ is modular. If $\mathbf{D}_2$ is also modular the two sets are isomorphic (1.7). Otherwise $\mathbf{D}_2$ is an affine space and $\mathbf{D}_1$ is - up to isomorphy - the set of parallel classes of $\mathbf{D}_2$. We fix a line B in $\mathbf{D}_2$. Then every point c in $\mathbf{D}_2$ outside of B can be defined from to points $a, b \in B$ and the two directions from c to a and from c to b. Whence $\mathbf{D}_2 \subset \mathrm{dcl}(\mathbf{D}_1 \cup B)$. $\square$

A type $p \in S(A)$ is $\mathbf{F}$-*internal* if $p(\mathbf{C}) \subset \mathrm{dcl}(\mathbf{F} \cup B)$ for some B, which is by compactness equivalent to the existence of a formula $\mathbf{E}(x)$ in p which is $\mathbf{F}$-internal.

Lemma 3.3 p *is* $\mathbf{F}$*-internal iff there is a realization* e *of* p *and an element* b *such that* $e \underset{A}{\downarrow} b$ *and* $e \in \mathrm{dcl}(\mathbf{F}b)$.

Proof:
If p is $\mathbf{F}$-internal there is an $\mathbf{E}(x) \in p$ and a definable surjection $\pi : \mathbf{F}^n \to \mathbf{E}$. Choose a parameter b over which π is defined and a realization e of p which is independent from b over A. Then $e \in \mathrm{dcl}(\mathbf{F}b)$. If conversely e and b with $e \models p$, $e \underset{A}{\downarrow} b$ and $e \in \mathrm{dcl}(\mathbf{F}b)$ are given take for B the elements of a long Morley sequence $(b_i)_{i \in I}$ of $\mathrm{st}(b/A)$. If e' is any realization of $\mathrm{st}(e/A)$ some b_i is independent from e' over A. Then also $e' \in \mathrm{dcl}(\mathbf{F}b_i)$. Whence $\mathrm{st}(e/A) \subset \mathrm{dcl}(\mathbf{F} \cup B)$ and $\mathrm{st}(e/A)$ is $\mathbf{F}$-internal. Then also all the other extensions of p to $\mathrm{acl}(A)$ are $\mathbf{F}$-internal, which implies that p is $\mathbf{F}$-internal. $\square$

We call a tuple $\overline{e}$ of elements of $\mathbf{E}$ a *fundamental solution* if $\mathbf{E} \subset \mathrm{dcl}(\mathbf{F} \cup A\overline{e})$.

Lemma 3.4 *If* T *is stable and* $\mathbf{E}$ *is* $\mathbf{F}$*-internal there is a fundamental solution* $\overline{e}$*. If* T *is totally transcendental* $\mathrm{tp}(\overline{e}/\mathbf{F} \cup A)$ *is isolated.*

Proof:
There is a definable surjection $\pi : \mathbf{F}^n \to \mathbf{E}$. By stability π is definable from parameters in $A \cup \mathbf{F} \cup \mathbf{E}$. The parameters used from $\mathbf{E}$ form a fundamental solution.
If $\pi : \mathbf{F}^n \to \mathbf{E}$ is an $A\overline{e}$-definable surjection we can write $\pi(x) = \pi(x, \overline{e})$ for an A-definable partial function $\pi(x, \overline{y})$. That π is a surjection $\mathbf{F}^n \to \mathbf{E}$ is an elementary property of $\overline{e}$ which can be expressed by a formula with parameters in A. If T is totally transcendental we can therefore find an $\overline{e}$ which is isolated over $\mathbf{F} \cup A$. $\square$

Let G be a group of permutations of the definable class $\mathbf{P}$. We call G A-*definable* if there is an isomorphism of G with an A-definable group $\mathbf{G}$ such that the induced action of $\mathbf{G}$ on $\mathbf{P}$ is A-definable. If $\mathbf{G}$ is a subclass of $\mathbf{Q}$ we say that G *lives on* $\mathbf{Q}$. For example a definable group which acts regularly on $\mathbf{P}$ lives on $\mathbf{P}^{eq}$.

If $\mathbf{P}$ is A-definable and G is an A-definable regular permutation group on $\mathbf{P}$ then the *opposite* group G^{opp} of all G-invariant permutations of $\mathbf{P}$ is again an A-definable permutation group.

Theorem 3.5 (Groupe de liaison) *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be 0-definable classes. Assume that* $\mathbf{E}$ *is* $\mathbf{F}$*-internal and that* $h_0 = \overline{e}_0$ *is a fundamental solution which is isolated over* $\mathbf{F}$*. Let* $\mathbf{H}$ *be the* $\mathbf{F}$*-definable class of all fundamental solutions which have the same type over* $\mathbf{F}$ *as* h_0*. Then*

1. *The natural map* $\mathrm{Aut}(\mathbf{E}/\mathbf{F}) \to \mathrm{Aut}(\mathbf{H}/\mathbf{F})$ *is an isomorphism.*
2. $\mathrm{Aut}(\mathbf{H}/\mathbf{F})$ *acts regularly on* $\mathbf{H}$ *and is* $\mathbf{F}$*-definable. (This was called the "binding group" in [PO].)*
3. $\mathrm{Aut}(\mathbf{H}/\mathbf{F})^{opp}$ *lives on* $\mathbf{F}^{eq}$.

Proof:
The 0-definable map $\pi(x, y)$ in the proof of 3.4 maps $\mathbf{F}^n \times \mathbf{H}$ to $\mathbf{E}$. $\pi(x, h)$ is surjective for every $h \in \mathbf{H}$.
(1): The map is surjective since by stability every automorphism of $\mathbf{H}$ over $\mathbf{F}$ extends to an automorphism of the monster model $\mathbf{C}$ over $\mathbf{F}$. Fix an element $h_1 \in \mathbf{H}$. Then

$$\alpha(\pi(\overline{f}, h_1)) = \pi(\overline{f}, \alpha(h_1))$$

shows that every $\alpha \in \mathrm{Aut}(\mathbf{E}/\mathbf{F})$ is determined by its value on h_1. Whence our map ist also injective. The formula shows also that $\mathrm{Aut}(\mathbf{E}/\mathbf{F})$ is $\mathbf{F}$-definable: Any pair (h_1, h_2) of elements of $\mathbf{H}$ defines an α by

$$\alpha(\pi(\overline{f}, h_1)) = \pi(\overline{f}, h_2),$$

which is in $\mathrm{Aut}(\mathbf{E}/\mathbf{F})$ since h_1 can be mapped to h_2 by an element of $\mathrm{Aut}(\mathbf{E}/\mathbf{F})$, which must be α. If $\mathbf{H}$ is defined over F_0 then also $\mathrm{Aut}(\mathbf{E}/\mathbf{F})$ is defined over F_0 .
(2): Is clear by the proof of (1).
(3): Let m be the length of h_0. The m-th power of π gives a F_0-definable map $\sigma : \mathbf{F}^{nm} \times \mathbf{H} \to \mathbf{E}^m$. $\sigma(x, h)$ is surjective for every $h \in \mathbf{H}$. Since $\mathbf{H}$ is a complete type over $\mathbf{F}$ the set

$$\mathbf{S} = \{x \in \mathbf{F}^{nm} \mid \sigma(x, h) \in \mathbf{H}\}$$

does not depend on h. We have therefore an F_0-definable map

$$\tau : \mathbf{S} \times \mathbf{H} \to \mathbf{H}.$$

$\tau(x, h)$ and (by completeness) $\tau(s, y)$ are surjective for all $h \in \mathbf{H}$ and $s \in \mathbf{S}$. Now for any pair $(s, t) \in \mathbf{S}$

$$\Phi_{st}(\tau(h, s)) = \tau(h, t) \quad (h \in \mathbf{H})$$

defines a map $\Phi_{st} : \mathbf{H} \to \mathbf{H}$ which commutes with every $\beta \in \mathrm{Aut}(\mathbf{H}/\mathbf{F})$. Since Φ_{ts} is invers to Φ_{st} it belongs to $\mathrm{Aut}(\mathbf{H}/\mathbf{F})^{opp}$. Since $(\tau(h, s), \tau(h, t))$ can be any pair in $\mathbf{H}$ every element of $\mathrm{Aut}(\mathbf{H}/\mathbf{F})^{opp}$ has the form Φ_{st}. This shows that $\mathrm{Aut}(\mathbf{H}/\mathbf{F})^{opp}$ lives on $\mathbf{S}^{eq}$. $\square$

Definition 2 *Let* $\mathbf{P} \subset \mathbf{Q}$ *be 0-definable classes. We call* $\mathbf{Q}$ *a* cover *of* $\mathbf{P}$ *if the following entities exist 0-definably:*

1. *A partition of* $\mathbf{Q} \setminus \mathbf{P}$ *into a family* $(\mathbf{H}_{\overline{a}})_{\overline{a} \in \mathbf{P}}$
2. *A family* $(\mathbf{G}_{\overline{a}})_{\overline{a} \in \mathbf{P}}$ *of groups living on* $\mathbf{P}^{eq}$ *(the* structure groups*)*
3. *A regular action of each* $\mathbf{G}_{\overline{a}}$ *on* $\mathbf{H}_{\overline{a}}$

If all the $(\mathbf{H}_{\overline{a}})_{\overline{a} \in \mathbf{P}}$ live on $\mathbf{D}^{eq}$ for a 0-definable subclass $\mathbf{D}$ of $\mathbf{P}$ we call $\mathbf{Q}$ a $\mathbf{D}$-*cover*.

Theorem 3.6 ([HR2]) *Let T be ω_1-categorical and $\mathbf{D}$ a 0-definable strongly minimal set. Then there is a sequence*

$$\mathbf{D} = \mathbf{P}_0 \subset ... \subset \mathbf{P}_n$$

of 0-definable classes such that each $\mathbf{P}_{i+1}$ is a $\mathbf{D}$-cover of $\mathbf{P}_i$ and the monster model $\mathbf{C}$ is in the definable closure of $\mathbf{P}_n$. The length n can be bounded by the Morleyrank of $\mathbf{C}$.

The groups we will find are opposite binding groups as constructed in 3.5. We have therefore to show that there are enough $\mathbf{D}$-internal sets. This is done by the following lemma.

Lemma 3.7 *Let T and $\mathbf{D}$ be as in 3.6. Then for every a of Morleyrank n over $\emptyset$ there is a sequence $a_0, ...a_n = a$ such that all a_i are in $\mathrm{dcl}(a)$ and all types $\mathrm{tp}(a_{i+1}/a_1...a_i)$ are $\mathbf{D}$-internal.*

Proof:
If $n = 0$ then $\mathrm{tp}(a/\emptyset)$ is obviously $\mathbf{D}$-internal.
Otherwise $\mathrm{tp}(a/\emptyset)$ is not orthogonal to $\mathbf{D}$ and there must be a b and an element d of $\mathbf{D}$ such that a is independent from b over $\emptyset$ but not independent from bd. Choose an element c in the canonical base of $q = \mathrm{st}(bd/a)$ such that $a \not\downarrow c$. Since q is definable over any infinite Morley sequence of q c is definable from an a-independent sequence $b_1d_1, ...b_md_m$ of realizations of q. We have $a \downarrow b_i$ for all i. This implies that $a \downarrow b_1...b_m$ and - because $c \in \mathrm{acl}(a)$ - $c \downarrow b_1...b_m$. By 3.3 we know that $\mathrm{tp}(c/\emptyset)$ is $\mathbf{D}$-internal. If $c_1, ...c_k$ are the conjugates of c over a the types $\mathrm{tp}(c_i/\emptyset)$ are also $\mathbf{D}$-internal. We set $a_1 = \{c_1, ...c_k\}$. Then $\mathrm{tp}(a_1/\emptyset)$ is $\mathbf{D}$-internal and $a_1 \in \mathrm{dcl}(a)$.
Since a is dependent from a_1 the Morleyrank of $\mathrm{tp}(a/a_1)$ is smaller that n. After adding the constant a_1 to the language we can use induction to obtain the desired sequence. □

Proof of 3.6:
We express the content of 3.7 by saying that any a of rank n is *n-analysable*. It is easy to see from the definition that "$\mathrm{tp}(a/b)$ is $\mathbf{D}$-internal" is a $\bigwedge$-definable property of ab i.e. definable by a conjunction of a set of formulas. Whence also "$\mathrm{tp}(a/\emptyset)$ is n-analysable" is $\bigwedge$-definable.

Let $\mathbf{R}$ be a 0-definable class of elements which are all n-analysable. We show by induction on n that $\mathbf{R}$ is in the definable closure of a "covering" sequence of length n. By the above we can apply compactness to

$$\models \forall a \in \mathbf{R}\ \ \exists b\ \ ((\mathrm{tp}(a/b)\quad \mathbf{D} - \mathit{internal})\ \wedge\ (\mathrm{tp}(b/\emptyset)\quad (n-1) - \mathit{analysable}))$$

to obtain a 0-definable class $\mathbf{Q}$ with

> For every element a of $\mathbf{R}$ there is a b in $\mathbf{Q}$ for which $\mathrm{tp}(a/b)$ is $\mathbf{D}$-internal.
>
> All elements of $\mathbf{Q}$ are $(n-1)$-analysable.

Using induction we find a covering sequence $\mathbf{D} = \mathbf{P}_0 \subset \ldots \subset \mathbf{P}_{n-1}$ such that $\mathbf{Q}$ is in $\mathrm{dcl}(\mathbf{P}_{n-1})$. If $\mathrm{tp}(a/b)$ is $\mathbf{D}$-internal and $b \in \mathrm{dcl}(\bar{c})$ then also $\mathrm{tp}(a/\bar{c})$ is $\mathbf{D}$-internal. Whence for all $a \in \mathbf{R}$ there is a $\bar{c} \in \mathbf{P}_{n-1}$ for which $\mathrm{tp}(a/\bar{c})$ is $\mathbf{D}$-internal. We have to find a cover $\mathbf{P}_n$ of $\mathbf{P}_{n-1}$ such that $\mathbf{R} \subset \mathrm{dcl}(\mathbf{P}_n)$.

By 3.1 every $a \in \mathbf{R}$ belongs to a $\mathbf{D}$-internal class $\mathbf{E}$ which is definable over a subset A of $\mathbf{P}_{n-1}$. Let $\mathbf{H}$ be the class of fundamental solutions defined in 3.5 and $\mathbf{G} = \mathrm{Aut}(\mathbf{H}/\mathbf{D} \cup A)^{opp}$. Then for some parameter $\bar{c} \in \mathbf{P}_{n-1}$:

> $\mathbf{H}$,$\mathbf{G}$ and the - regular - action of $\mathbf{G}$ on $\mathbf{H}$ are definable from $\bar{c}$.
>
> $a \in \mathrm{dcl}(\mathbf{H}\bar{c})$.

We can assume that $\mathbf{H} = \mathbf{H}'_{\bar{c}}$ and $\mathbf{G} = \mathbf{G}'_{\bar{c}}$ for 0-definable families

$$(\mathbf{H}'_{\overline{b}}) \,, (\mathbf{G}'_{\overline{b}}) \qquad (\overline{b} \in \mathbf{P}_{n-1} \;)$$

of regular acting groups. The families still depend on a. But by compactness there is a finite set of families

$$(\mathbf{H}^i_{\overline{b}}) \,, (\mathbf{G}^i_{\overline{b}}) \qquad (\overline{b} \in \mathbf{P}_{n-1} \,, i = 1, \ldots m)$$

which can be used for every $a \in \mathbf{R}$.

Finally define

$$\mathbf{H}_{\overline{c}} = \mathbf{H}^1_{\overline{c}} \times \ldots \times \mathbf{H}^m_{\overline{c}}$$
$$\mathbf{G}_{\overline{c}} = \mathbf{G}^1_{\overline{c}} \times \ldots \times \mathbf{G}^m_{\overline{c}} \qquad (\overline{c} \in \mathbf{P}_{n-1}).$$

with the obvious action. We can assume that the $\mathbf{H}_{\overline{c}}$ are pairwise disjoint and disjoint to $\mathbf{P}_{n-1}$ and that still $\mathbf{R} \subset \mathrm{dcl}(\mathbf{P}_{n-1} \cup \bigcup_{\overline{c} \in \mathbf{P}_{n-1}} \mathbf{H}_{\overline{c}})$. Set

$$\mathbf{P}_n = (\mathbf{P}_{n-1} \cup \textstyle\bigcup_{\overline{c} \in \mathbf{P}_{n-1}} \mathbf{H}_{\overline{c}}) \;.$$

This proves 3.6. $\square$

In totally categorical theories we have a better result. Also the theory of internality is not used at all.

Theorem 3.8 *If T is totally categorical and $\mathbf{D}$ is a projective space over the field $\mathbf{F}_q$ we can find a covering sequence*

$$\mathbf{D} = \mathbf{P}_0 \subset \ldots \subset \mathbf{P}_m$$

such that the monster model $\mathbf{C}$ is in the definable closure of $\mathbf{P}_m$ and each $\mathbf{P}_{i+1}$ is a $\mathbf{D}$-cover of $\mathbf{P}_i$ with structure groups $\mathbf{G}_{\overline{a}}$ which are either finite or $\mathbf{F}_q$-vectorspaces (and with no other structure over $\overline{a}$).

Proof:
By a modification of the proof of 3.6. Instead of 3.7 we use 1.6: Also by ω-categoricity there is no need to deal with a finite family of groups at the same time and use the direct product. The price we pay of course is that we cannot compute a bound for the length of the covering sequence. Furthermore no use is made of internality and the groupe de liason.

Thus the only thing we have to prove is the following:
Suppose that $\overline{c}$ is a tuple from $\mathbf{P}_{n-1}$ and that $\mathrm{tp}(a/\overline{c})$ is algebraic or strictly minimal. Then there is a two-step cover $\mathbf{P}_{n+1}$ of $\mathbf{P}_{n-1}$ of the desired kind such that a is definable over $\mathbf{P}_{n+1}$. For this it is enough to find a group $\mathbf{G}$ which lives on $\mathbf{D}^{eq}$ and acts regularly on a set $\mathbf{H}$, everything definable over $\mathbf{P}_n$, such that a is definable over $\mathbf{P}_n \cup \mathbf{H}$.

If a is algebraic over $\overline{c}$ only one cover is needed: set $\mathbf{P}_n = \mathbf{P}_{n-1}$. Let $b_1 \ldots b_m$ be the sequence of conjugates of a over $\overline{c}$ and define $\mathbf{G} = \mathrm{Sym}(1 \ldots m)$ and $\mathbf{H} = \{b_{\sigma(1)} \ldots b_{\sigma(m)} | \sigma \in \mathbf{G}\}$ with the obvious action.

If $\mathrm{tp}(a/\overline{c})$ is strictly minimal, it is modular or affine. If it is modular it is by 1.7 algebraic over $\mathbf{D}\overline{c}$ and we are reduced to the first case. If it is affine the realization set $\mathbf{D}_1$ of $\mathrm{tp}(a/\overline{c})$ is an affine geometry with no other structure over $\overline{c}$. The corresponding $\mathbf{F}_q$-vectorspace V which acts regularly on $\mathbf{D}_1$ is definable from $\overline{c}$. Let $\mathbf{D}_2 = \mathbf{D}_{\overline{c}}$ be the strictly minimal set over $\overline{c}$, associated to $\mathbf{D}$. Then by 1.7 there is a $\overline{c}$-definable bijection between $\mathbf{D}_2$ and the projective space of 1-dimensional subspaces of V. Whence V is algebraic over $\mathbf{P}_{n-1}$ and we can build a cover $\mathbf{P}_n$ with finite structure groups (a *finite* cover) such that all elements of V are definable over $\mathbf{P}_n$ i.e. V

lives in $\mathbf{P}_n^{eq}$. If we now set $\mathbf{H} = \mathbf{D}_1$ and $\mathbf{G} = V$ we are done, besides that $\mathbf{G}$ is only to be known to live on $\mathbf{P}_n^{eq}$ and we want it to live on $\mathbf{D}^{eq}$.

But this is dealt with by finding a $\mathbf{P}_n$-definable injection $\pi : V \to \mathbf{D}_2^3$. We choose three independent elements v_1, v_2, v_3 from V and define

$$\pi(v) = (\mathbf{F}_q(v + v_1), \mathbf{F}_q(v + v_2), \mathbf{F}_q(v + v_3)).$$

Of course the copy of V living on $\mathbf{D}^{eq}$ is now defined using the parameters $\bar{c}v_1v_2v_3$. $\square$

4 The structure of covers

Let M be an arbitrary structure and let a 0-definable family of groups $(G_{\bar{b}})_{\bar{b}\in M}$ be given. A *cover* of M with structure groups $(G_{\bar{b}})$ is a two-sorted structure $N = (M, H; R_1, R_2, \ldots)$ in which H is 0-definably partitioned into sets $(H_{\bar{b}})_{\bar{b}\in M}$ on which the $G_{\bar{b}}$ act regularly in a 0-definable way. Also we want N *not to add new structure* to M.

This means that

1. every in N 0-definable relation on M is 0-definable in M
2. every in N definable relation on M is definable in M

The second clause means that M is *stably embedded* in N in the sense of the next definition. Only by this condition it is that a cover of a totally categorical model is again totally categorical (see 4.4).

Definition 1 *Let T be arbitrary and $\mathbf{P}$ be a 0-definable class. $\mathbf{P}$ is called* stably embedded *if every definable relation on $\mathbf{P}$ is definable with parameters in $\mathbf{P}$. A 0-definable set P in a structure M is called stably embedded* with finite support *if all types $\mathrm{tp}(\bar{a}/P)$ of elements of M over P are definable over a finite subset of P.*

Remark 4.1 $\mathbf{P}$ *is stably embedded iff all types over* $\mathbf{P}$ *are definable over $\mathbf{P}$. Let B be a subset of $\mathbf{P}$. Then $\mathrm{tp}(\bar{a}/\mathbf{P})$ is definable over B iff $\mathrm{tp}(\bar{a}/B) \vdash \mathrm{tp}(\bar{a}/\mathbf{P})$ iff $\mathrm{tp}(\bar{b}/B) \vdash \mathrm{tp}(\bar{b}/B\bar{a})$ for all $\bar{b} \in \mathbf{P}$.*

If T is stable all $\mathbf{P}$ are stably embedded. If in some model M P is stably embedded with finite support then $\mathbf{P}$ is stably embedded. If T is ω-stable then all P are stably embedded with finite support. The following theorem was explained to me by Hrushovski and is also contained in [HP]

Theorem 4.2 *Let T be a countable complete theory and $\mathbf{P}$ be a 0-definable predicate.*

1. *Let M be an uncountable saturated model. Then $\mathbf{P}$ is stably embedded iff every automorphism of $\mathbf{P}$ extends to an automorphism of M.*
2. *Let M be a countable saturated model. Then P is stably embedded with finite support iff every automorphism of P extends to an automorphism of M.*

Proof:
(1): Assume first that $\mathbf{P}$ is stably embedded and that α is an automorphism of P which is already extended to an elementary map $\beta : (P \cup A) \to M$ for a subset A of small cardinality. We want to extend β to an arbitrary element a of M. Let $p = \mathrm{tp}(a/P \cup A)$. By 4.1 there is a small subset B of P with $q = \mathrm{tp}(a/B \cup A) \vdash p$. Then also $\beta(q) \vdash \beta(p)$. Since we can realize $\beta(q)$ we can also realize $\beta(p)$ in M and extend β.

For the converse assume that Q is a definable subset of P. The structure (P,Q) - P with its induced structure and with the predicate Q- is saturated. In $\mathrm{Aut}(M)$ Q has at most $|M|$-many conjugates. If one can extend every automorphism of P to an automorphism of M Q has also at most $|M|$-many conjugates in $\mathrm{Aut}(P)$. Kueker's theorem implies now that Q is definable in P.

(2):Same as (1). But one needs a generalized Kueker theorem: If $(P,Q_i)_{i\in I}$ is a saturated structure such that $(Q_i)_{i\in I}$ hat at most $|M|$-many conjugates in $\mathrm{Aut}(P)$ then there is a finite set of parameters over which all the Q_i are definable in P. □

Remark 4.3 *If P is countable and ω-categorical one has not to assume that (P,Q) is saturated for Kueker's theorem to be true. For the "if"-direction of (2) one has therefore not to assume that M is saturated if P is ω-categorical.*

Lemma 4.4 *Let $N=(M,H)$ be a two-sorted structure where in a 0-definable way H is partitioned into a family of sets $H_{\overline{b}}$ ($\overline{b}\in M$) on which groups $G_{\overline{b}}$ living in M act regularly.*

1. *Assume that M is stably embedded in N. If then M (with its induced structure) is ω-stable or ω_1-categorical then also N is ω-stable or ω_1-categorical respectively.*
2. *Assume that M is stably embedded in N with finite support. Then N is ω-categorical if M is.*

Proof:
(We write the $\overline{b}$ as 1-tuples) (1): The Morleyrank of H is bounded if the ranks of the H_b are uniformly bounded and the rank of the index set M is bounded. Because of the regular action the H_b and the G_b have the same rank. But since M is stably embedded the (H,M)-Morleyrank of definable subsets of M is the same as their M-Morleyrank. Thus if M has Morleyrank as a structure of its own also H has Morleyrank.
Concerning ω_1-categoricity we prove that $Th(N)$ has no Vaughtian pairs if $Th(M)$ has no Vaughtian pairs. Suppose that $Th(M)$ has no Vaughtian pairs and let $N\prec N'$ be a proper elementary extension and $Q=\bigcup_{s\in S}Q_s$, where $Q_s\subset H_s$, a definable subset of H which does not increase in (M',H'). Since each H_b is in definable bijection with G_b M' is a proper extension of M. That Q does not increase means that S does not increase and that all the Q_s do not increase. Since S is definable in M it must be finite. There are definable bijections of the Q_s onto definable subsets of G_s, which are by assumption definable in M. Whence with these sets the Q_s also must be finite. We conclude that Q is finite and that $Th(N)$ has no Vaughtian pairs.
(2): We can assume that N is countable. Since $\mathrm{Aut}(M)$ has only finitely many orbits on M^n by 4.2 also $\mathrm{Aut}(N)$ has only finitely many orbits on M^n. To count the orbits of $\mathrm{Aut}(N)$ on H^n it is therefore enough to count for any fixed sequence $\overline{s}=s_0\ldots s_m\in M$ the types of n-tuples from $H_{\overline{s}}=\bigcup_{i\le m}H_{s_i}$ over $\overline{s}$. We choose a representative b_i from every H_{s_i}. Then there is a $\overline{b}$-definable bijection of $H_{\overline{s}}$ onto the disjoint union $G_{\overline{s}}$ of the G_{s_i}. Whence we can bound the number of types of n-tuples from $H_{\overline{s}}$ over $\overline{s}$ by the number of types of n-tuples of elements of M over $\overline{s}\overline{b}$. Now let $\mathrm{tp}(\overline{s}\overline{b}/M)$ be definable over the finite $B\subset M$. Then we have as many types of elements of M over B as we have types over $B\overline{s}\overline{b}$: Finitely many! □

Corollary 4.5 *Covers of totally categorical structures are totally categorical.*

□

We fix now a countable totally categorical structure M and a 0-definable family of groups $(G_{\overline{b}})_{\overline{b}\in M}$ living in M^{eq}. For notational simplicity we assume the $b=\overline{b}$ be 1-tupels.

All covers of M with structure groups $(G_{\overline{b}})$ are (up to isomorphy) expansions of the *principal* cover $N_0=(M,H)$ where the only relations added to M are a map $H\to M$ to partition H into sets H_b ($b\in M$) and a family of regular actions of the G_b on the H_b.

Corollary 4.6 *The covers of M correspond up to interdefinability to the closed subgroups of* $\mathrm{Aut}(N_0)$ *which induce* $\mathrm{Aut}(M)$ *on* M*.*

Proof:
The automorphism groups of covers are closed subgroups of $\mathrm{Aut}(N_0)$ which induce $\mathrm{Aut}(M)$ by 4.2. Since covers are ω-categorical they are uniquely determined (as an expansion of N_0) by their automorphism groups. If a closed subgroup G of $\mathrm{Aut}(N_0)$ is given it is the automorphism group of an expansion N of N_0. If G induces $\mathrm{Aut}(M)$ by 4.2 M is stably embedded in N and N is a cover of M. $\Box$

We distinguish covers N by their *kernel* $\mathsf{K} = \mathrm{Aut}(N/M)$, which is a closed subgroup of $\mathsf{K}_0 = \mathrm{Aut}(N_0/M) \cong \prod_{b\in M} G_b$. The last isomorphism is not natural if the G_b are not abelian. The natural isomorphism is between K_0 and $\prod_{b\in M}(G_b)^{opp}$.

By [HHLS] countable totally categorical models M have the small index property: The open subgroups of $\mathrm{Aut}(M)$ coincide with the subgroups of countable index. This is the main ingredient of the proof of the following lemma:

Lemma 4.7 ([HP]) *Let* G *be a subgroup of* $\mathrm{Aut}(N_0)$ *which induces* $\mathrm{Aut}(M)$*. Then* G *is closed iff the kernel* $\mathsf{K} = \mathsf{G} \cap \mathsf{K}_0$ *is closed.*

Proof:
Assume the kernel of $\mathsf{G}$to be closed. The small index properties implies that the restriction map from G onto $\mathrm{Aut}(M)$ is open. Thus if O is an open neighbourhood of 1 in $\mathrm{Aut}(N_0)$ then $(O\cap\mathsf{G})|M$ is an open neighhourhood of 1 in $\mathrm{Aut}(M)$. It follows that $O' = \mathsf{K}_0(O \cap \mathsf{G})$ is open in $\mathrm{Aut}(N_0)$.
Let h be an automorphism of N_0 which does not belong to G . We want to show that h is not in the closure of G . Since G contains an element which has the same restriction to M as h we can divide by this element and assume that h belongs to K_0. Now choose an open subgroup O of $\mathrm{Aut}(N_0)$ such that hO is disjoint from K . Then $O' \cap hO$ is a neighbourhood of h which is disjoint from G. $\Box$.

We assume from now on that all G_b are abelian. This is a harmless assumption by 3.8. Then in $\mathrm{Aut}(N_0)$ conjugation with h depends only on hK_0. We obtain therefore a natural action of $\mathrm{Aut}(M)$ on $\mathrm{Aut}(N_0)$. The kernels of subgroups which project onto $\mathrm{Aut}(M)$ are obviously invariant under this action.

Lemma 4.8 *For every closed subgroup* K *of* K_0 *which is invariant under the action of* $\mathrm{Aut}(M)$ *there is a cover of* M *with kernel* K *.*

Proof:
Choose a *transversal* $t = (t_b)_{b\in M}$, where $t_b \in H_b$. Then every element of H_b has the form $t_b g$ for a unique $g \in G_b$. We lift now every automorphism α of M to an automorphism $\tilde{\alpha}$ of N_0 by setting $\tilde{\alpha}(t_b g) = t_{\alpha(b)}\alpha(g)$. Then the automorphism group of the desired cover is $\mathsf{G} = \{h\tilde{\alpha}\,|h \in \mathsf{K}, \alpha \in \mathrm{Aut}(M)\}$. $\Box$

We call the covers constructed in the proof of 4.8 *trivial extensions* of K . They depend on the choice of the transversal t, but they are all *conjugated* by an element of K_0 i.e. their automorphism groups are in $\mathrm{Aut}(N_0)$ conjugated by an element of K_0.

Theorem 4.9 *Let* K *be a closed,* $\mathrm{Aut}(M)$*-invariant subgroup of* K_0*. Then there is is a natural bijection between the conjugacy classes of covers of* M *with structure groups* $(G_b)_{b\in M}$ *and* $H^1(\mathrm{Aut}(M), \mathsf{K}_0/\mathsf{K})$*. The trivial covers correspond to 0.*

Proof:
Let N be a cover with kernel K .Choose a transversal t and define for every $\alpha \in \mathrm{Aut}(M)$ an element $d\alpha$ of K_0 by $\overline{\alpha}(t) = td\alpha$, where $\overline{\alpha}$ is any extension of α to N. Modulo K this defines a

derivation $d : \mathrm{Aut}(M) \rightarrow (\mathsf{K}_0/\mathsf{K})$ whose cohomology is independent of the choice of t and $\overline{\alpha}$. This defines the desired bijection. More details and the computation of $H^1(\mathrm{Aut}(M), \mathsf{K}_0/\mathsf{K})$ in some special cases can be found in [AZ2]. $\square$

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5 Errata

- In the proof of 1.5 no induction is needed. The Lascar–equation yields $U(b'/A) = 1$ immediately.
- Remark about 2.1: Define the *coordinates* of A to be the set $A_D = \mathrm{acl}(A) \cap D$. Let B be a subset of A. It is easy to see that $A \underset{B}{\downarrow} \mathbf{D}$ iff $B_D = A_D$. If A is algebraically closed this is also equivalent to the non–existence of a modular Rank–1 type over $\mathrm{acl}(B)$ realized in A. (Take the type of an element of A_D over $\mathrm{acl}(B)$.

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