Divisible Uniserial Modules over Valuation Domains

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1 Introduction

Let R be a valuation domain with quotient field K. An R–module M is *uniserial* if for all $x, y \in M$ there is an $r \in R$ such that

$$rx = y \quad \text{or} \quad ry = x.$$

M is *divisible* if for all $x \in M$ and all non–zero $r \in R$ there is a $y \in M$ such that $ry = x$.

A non–zero torsionfree divisible uniserial R–module is obviously isomorphic to the quotient field K of R. We call a divisible uniserial R–module *proper* if it is not torsionfree.

The main theorem 4.3 of this paper classifies proper divisible uniserial R–modules by two invariants:

- an element of the cohomology group

  $$\varprojlim^1_{\alpha \in \Gamma} U/U_\alpha,$$

  where Γ is the value semigroup of R, U is the group of units of R and U_α is the group of units, which differ from 1 by an element of larger value than α.

- an element of the ideal class semigroup of R

The first invariant of M is zero iff M is a *standard*, i.e. if M is a quotient of K. In section 2 we give an easy example of a valuation domain R for which $\varprojlim^1_{\alpha \in \Gamma} U/U_\alpha$ does not vanish, which shows that R has a nonstandard divisible uniserial module M. The part of 4.3 needed to obtain M is given an extra proof in section 2, which thereby contains a self–contained new construction of a nonstandard divisible uniserial module over a valuation domain.

Shelah showed in [7, p.135-150] that the existence of nonstandard divisible uniserial modules is consistent with ZFC. This result was then improved by Fuchs and Salce [2] and Franzen and Göbel [1], who showed that the existence of nonstandard divisible uniserial modules follows from $\diamondsuit_{\omega_1}$ and even from $2^{\omega_0} < 2^{\omega_1}$.

The existence of nonstandard divisible uniserial modules was finally proved without set–theoretical hypotheses by Fuchs and Shelah [3] using a model theoretic transfer principle and later by B. Osofsky [6] by purely algebraic means.

If M is a nonstandard divisible uniserial module the matrix ring

$$S = \left\{ \left(\begin{array}{cc} r & m \\ 0 & r \end{array} \right) \middle| \, r \in R, m \in M \right\}$$

is a counter example for an old conjecture of Kaplansky, according to which every valuation ring should be a quotient of a valuation domain.

2 Construction of a nonstandard uniserial module

Let I be a totally ordered set. A *projective system* is an I-indexed family (A_α) of abelian groups together with a commutative system of homomorphisms

$$\pi_{\alpha\beta} : A_\beta \to A_\alpha\ , \quad (\alpha < \beta \in I).$$

A 0–*cochain* is a sequence $(e_\alpha)_{\alpha\in I}$ of elements e_α of A_α. A 1–*cochain* is a family $(c_{\alpha\beta})_{\alpha<\beta\in I}$ of elements $c_{\alpha\beta}$ of A_α. The *coboundary* $\delta(e)$ of a 0-cochain e is the 1–cochain defined by

$$\delta(e)_{\alpha\beta} = \pi_{\alpha\beta}(e_\beta) - e_\alpha.$$

Coboundaries are *cocycles* in the following sense: A 1–cochain c is called a 1–cocycle if

$$\pi_{\alpha\beta}(c_{\beta\gamma}) - c_{\alpha\gamma} + c_{\alpha\beta} = 0$$

for all $\alpha < \beta < \gamma$. Let us denote the quotient group (1–cocycles)/(coboundaries) by

$$\underset{\longleftarrow\,\alpha\in I}{\lim{}^1} A_\alpha.$$

Now let R be valuation domain and $\mathrm{v} : R \to \Gamma \cup \{\infty\}$ the valuation of R. For every $\alpha \in \Gamma$ we define

$$U_\alpha = \{u \in R \mid v(1-u) > \alpha\}.$$

U_α is a multiplicative subgroup of U, the group of units of R.

Theorem 2.1 *R has a nonstandard uniserial divisible module iff* $\underset{\longleftarrow\,\alpha\in\Gamma}{\lim{}^1} U/U_\alpha$ *is non–trivial.*

Proof:
That a nonstandard uniserial divisible module gives rise to a non–trivial element of $\underset{\longleftarrow\,\alpha\in\Gamma}{\lim{}^1} U/U_\alpha$ is not needed for our construction. We will prove this in Lemma 4.3.

Now assume that the units $(u_{\alpha\beta})$ represent a cocycle of the projective system (U/U_α). We are going to construct a uniserial divisible module M, which can only be standard if the $(u_{\alpha\beta})$ represent a coboundary.

First let us fix some notation. For every $\alpha \in \Gamma$ we choose an element r_α with value α. If A is an R–module the multiple $r_\alpha A$ does not depend on the choice of r_α, so we denote it by αA. If P is the maximal ideal of R we have then $U_\alpha = 1 + \alpha P$.

For all $\alpha < \beta$ multiplication by $r_\alpha^{-1} r_\beta$ defines an embedding from $R/(\alpha P)$ into $R/(\beta P)$. The direct limit of this system is isomorphic to K/P. To obtain a more interesting limit we use the embeddings

$$u_{\alpha\beta} r_\alpha^{-1} r_\beta : R/(\alpha P) \to R/(\beta P).$$

This is a commutative system since $(u_{\alpha\beta} r_\alpha^{-1} r_\beta)(u_{\beta\gamma} r_\beta^{-1} r_\gamma)$ and $u_{\alpha\gamma} r_\alpha^{-1} r_\gamma$ differ only by the factor $u_{\beta\gamma} u_{\alpha\gamma}^{-1} u_{\alpha\beta} \in U_\alpha$ and define therefore the same map $R/(\alpha P) \to R/(\gamma P)$. Clearly the direct limit M is uniserial. M is divisible since every element of $R/(\alpha P)$ is divisible by $r_\alpha^{-1} r_\beta$ in $R/(\beta P)$. We will use the notation x_α for the coset of 1 in $R/(\alpha P)$.

Now suppose $M \cong K/I$ and let $y_\alpha + I$ be the image of x_α under this isomorphism. Multiplication by y_0^{-1} shows that we can assume that $y_0 = 1$ and $I = P$. Then y_α has value $-\alpha$ and $e_\alpha = r_\alpha y_\alpha$ is a unit. $x_\alpha = u_{\alpha\beta} r_\alpha^{-1} r_\beta x_\beta$ implies $y_\alpha \equiv u_{\alpha\beta} r_\alpha^{-1} r_\beta y_\beta \pmod{P}$. If we multiply this equation by r_α we obtain $e_\alpha \equiv u_{\alpha\beta} e_\beta \pmod{\alpha P}$. This shows that u is a coboundary. $\square$

To find a valuation domain with the property of 2.1 we make use of the following Lemma.

Lemma 2.2 (Todorcevic) *Let $(B_\xi)_{\xi\in\omega_1}$ be a family of infinite abelian groups. For the projective system $(A_\xi) = \bigoplus_{\eta<\xi} B_\eta \quad (\xi \in \omega_1)$ with the the obvious projection maps we have*

$$\varprojlim_{\xi\in\omega_1}{}^1 A_\xi \neq 0.$$

Proof:
The standard construction of an Aronszajn tree ([5, p.70]) yields a sequence $(f_\xi)_{\xi<\omega_1}$ of injective functions $f_\xi : \xi \to \omega$ such that for all $\xi < \zeta$ the two functions f_ξ and $f_\zeta \restriction \xi$ differ only for finitely many arguments. In each B_ζ we choose a copy of ω. Then f_ξ defines an element of $A'_\xi = \prod_{\eta<\xi} B_\eta$. Define

$$c_{\xi\zeta} = f_\xi - f_\zeta \in A_\xi.$$

Then c is a 1-cocycle, which is not a coboundary. Otherwise, there would be a sequence $d_\xi \in A_\xi \quad (\xi \in \omega_1)$ such that $c_{\xi\zeta} = d_\xi - d_\zeta$. But then the functions $f_\xi - d_\xi$ form an ascending sequence. The union f of this sequence is a map defined on ω_1, which differs from each f_ξ only on finitely many values in ξ. Since the f_ξ have values in ω, the preimage $f^{-1}(\omega)$ is cofinal in ω_1. Since the f_ξ are injective all $f^{-1}(n)$ are finite. This is impossible. $\square$

Fix a field F. Choose indeterminates t_ξ, $(\xi < \omega_1)$ and let K be the rational function field $F(\,t_\xi\,)_{\xi<\omega_1}$. Order $\Gamma = \bigoplus_{\xi\in\omega_1} \mathbb{Z}$ lexicographically and let

$$\mathrm{v} : K \to \Gamma \cup \{\infty\}$$

be the (uniquely determined) valuation of K which is trivial on F and maps t_ξ to 1_ξ, the 1 of the ξ–th copy of $\mathbb{Z}$. (Note that the 1_ξ form an ascending sequence.)

Theorem 2.3 *The valuation ring of (K, v) has a nonstandard uniserial divisible module.*

Proof:
We start the computation of the group U of units with the observation that

$$U = F^{\cdot} \times U_0,$$

where $F^{\cdot}$ is the multiplicative group of F. In order to compute U_0 we enlarge the linear order Γ by cuts

$$\alpha_\xi = \sup\{n \cdot 1_\eta \mid \eta < \xi,\, n \in \mathbb{N}\}$$

to $\tilde{\Gamma}$. By the next lemma we have then

$$U_0 = \prod_{\xi<\omega_1} V_\xi,$$

where $V_\xi = \{x \in F(t_\eta)_{\eta \leq \xi} \mid \mathrm{v}(1-x) > \alpha_\xi\}$. It results that

$$U/U_{\alpha_\xi} = F^{\cdot} \times \left(\prod_{\eta < \xi} V_\eta \right).$$

By Lemma 2.2 $\varprojlim^1_{\xi<\omega_1} U/U_{\alpha_\xi}$ is not trivial. By proposition 3.2 we conclude that

$$\varprojlim^1_{\alpha<\Gamma} U/U_\alpha \cong \varprojlim^1_{\alpha<\tilde{\Gamma}} U/U_\alpha \cong \varprojlim^1_{\xi<\omega_1} U/U_{\alpha_\xi}$$

is not trivial. Now apply Theorem 2.1. $\square$

Avoiding the use of 3.2 one can construct M directly as follows: Choose a family $(u_{\xi\zeta})_{\xi<\zeta<\omega_1}$ of units which represents a non–trivial element of $\varprojlim^1_{\xi<\omega_1} U/U_{\alpha_\xi}$. Let M be the direct limit of the system $(R/(t_\xi P))_{\xi<\omega_1}$ with maps

$$u_{\xi+1,\zeta+1} t_\xi^{-1} t_\zeta : R/(t_\xi P) \to R/(t_\zeta P).$$

Lemma 2.4 *Let* $\mathrm{v} : H \to G \cup \{\infty\}$ *be a valued field. Order* $G \times \mathbb{Z}$ *lexicographically and let* α *be the cut* $\sup G$. *Extend* v *to a valuation* $\mathrm{v} : H(t) \to G \times \mathbb{Z}$ *with* $\mathrm{v}(t) = (0,1)$. *Then the group* U_0 *of* 1*-units of* $H(t)$ *is the direct product of the* 1*–units of* H *and of* U_α.

Proof: Easy.
$\square$

3 The right derived functors of the inverse limit functor

Let I be a totally ordered set. A *projective system* is an I-indexed family (A_α) of abelian groups together with a commutative system of homomorphisms

$$\pi_{\alpha\beta} : A_\beta \to A_\alpha \ , \quad (\alpha < \beta \in I).$$

Projective systems forms an abelian category in a natural way. $\varprojlim$ is a right exact functor to the category of abelian groups. Since the category of projective systems has enough injectives $\varprojlim$ has right derived functors

$$\varprojlim = \varprojlim{}^0, \ \varprojlim{}^1, \ \varprojlim{}^2 \dots$$

Fix a projective system $(A_\alpha, \pi_{\alpha\beta})_{\alpha<\beta\in I}$ and a number $n \geq 0$. We call a family

$$c = (c_{\alpha_0\dots\alpha_n}),$$

indexed by ascending sequences $\alpha_0 < \dots < \alpha_n$ of elements of I, an *n–cochain* if each $c_{\alpha_0\dots\alpha_n}$ is an element of A_{α_0}. The set of n–chains form an abelian group C^n under component–wise addition. The coboundary homomorphisms

$$\delta : C^{n+1} \to C^n,$$

defined by $(\delta c)_{\alpha_0\dots\alpha_{n+1}} = \pi_{\alpha_0\alpha_1}(c_{\alpha_1\dots\alpha_n}) + \sum_{i=1}^{n+1}(-1)^i c_{\alpha_0\dots\widehat{\alpha_i}\dots\alpha_{n+1}}$, make $C = (C^n)_{n\geq 0}$ into a cochain complex.

Theorem 3.1 ([4, Théorème 4.1])

$$\varprojlim{}^n_{\alpha\in I} A_\alpha = \mathrm{H}^n(C)$$

Readers who don't like derived functors can take $\mathrm{H}^n(C)$ as the definition of $\varprojlim{}^n_{\alpha\in I} A_\alpha$. The content of the last theorem is then that the $\varprojlim{}^n$ has the characterizing properties of the derived functors: They are trivial on injective projective systems and there is a natural long cohomology sequence.

If J is a subset of I there is an obvious restriction map

$$res : \varprojlim{}^n_{\alpha\in I} A_\alpha \to \varprojlim{}^n_{\alpha\in J} A_\alpha.$$

Jensen proved in [4, p.12] that res is an isomorphism if J is cofinal in I. (As a special case, we have for all I with a last element that $\varprojlim{}^n_{\alpha\in I} A_\alpha = 0$ for all $n \geq 1$.) The inverse map can be obtained as follows: One chooses a function $\phi : I \to J$ such that always $\alpha \leq \phi(\alpha)$. Then to every n–cochain d over J assign the n–cochain $\phi^*(d)$ over I defined by

$$\phi^*(d)_{\alpha_0\dots\alpha_n} = \pi_{\alpha_0\beta} d_{\phi(\alpha_0)\dots\phi(\alpha_n)},$$

where β is the smallest element of $\phi(\alpha_0)\dots\phi(\alpha_n)$. (We use here the convention that $d_{\beta_0\dots\beta_n}$ is zero if there is an double index, and is changed by the sign of permutation if the β_i are not in ascending order.) It is easy to see that the maps $\phi^* : C^n \to C^n$ commute with δ and induce a homomorphism

$$\phi^* : \varprojlim_{\alpha\in J}^n A_\alpha \to \varprojlim_{\alpha\in I}^n A_\alpha.$$

The next proposition shows that the composition

$$\varprojlim_{\alpha\in I}^n A_\alpha \overset{res}{\to} \varprojlim_{\alpha\in I}^n A_\alpha \overset{\phi^*}{\to} \varprojlim_{\alpha\in J}^n A_\alpha$$

is the identity, giving another proof of Jensen's result.

Proposition 3.2 *Let $(A_\alpha)_{\alpha\in I}$ be a projective system and $\phi : I \to I$ a function with $\alpha \leq \phi(\alpha)$. Let d be an n–cocycle and let c be the n–cochain defined by $c_{\alpha_0\ldots\alpha_n} = \pi_{\alpha\beta} d_{\phi(\alpha_0)\ldots\phi(\alpha_n)}$, where β is the smallest element of $\phi(\alpha_0)\ldots\phi(\alpha_n)$. Then d and c differ by a coboundary.*

Proof:
Let C be the functor which assigns to every projective system (A_α) the corresponding cochain complex (C^n). ϕ^* defines a natural transformation $C \to C$ and therefore a family of natural transformations $\phi^* : \varprojlim^n \to \varprojlim^n$ which for every short exact sequence $0 \to (A_\alpha) \to (B_\alpha) \to (C_\alpha) \to 0$ commute with the connecting homomorphisms $\delta : \varprojlim^n(C_\alpha) \to \varprojlim^{n+1}(A_\alpha)$. By the general theory of derived functors ϕ^* is therefore determined by what it does on $\varprojlim^0$. Since ϕ^* is the identity on $\varprojlim^0$, as one can easily check, it is the identity on all $\varprojlim^n$. □

Jensen proved in [4, Corollaire 3.2] that for all I of cofinality ω_k

$$\varprojlim_{\alpha\in I}^n A_\alpha = 0 \quad (\text{for all } n \geq k+2).$$

Furthermore he proved that the result is optimal: For every $n \geq 2$ there is a projective system $(A_\alpha)_{\alpha\in\omega_{n-1}}$ such that $\varprojlim_{\alpha\in\omega_{n-1}}^n A_\alpha \neq 0$ ([4, Proposition 6.2]).

If we look at *epimorphic* systems $(A_\alpha, \pi_{\alpha\beta})_{\alpha<\beta\in I}$, where all the $\pi_{\alpha\beta}$ are surjective, we have a better result:

Theorem 3.3 *For epimorphic systems of cofinality ω_k we have*

$$\varprojlim_{\alpha\in I}^n A_\alpha = 0 \quad (\textit{for all } n \geq k+1).$$

Proof:
We use induction on n and begin with the case $n = 1$, where we can assume that $I = \mathbb{N}$. Let a 1–cocycle c be given. We choose recursively elements $d_i \in A_i$ such that $\pi_{i,i+1}(d_{i+1}) = d_i - c_{i,i+1}$. The relation $\delta c = 0$ entails now $\delta d = c$.

Now assume $n > 1$. We begin with a general observation. Fix an element $\lambda \in I$ and denote by C^n_λ the set of n–cochains over $I_\lambda = \{\alpha \in I \mid \alpha < \lambda\}$. Define two homomorphisms, the restriction

$$\mathrm{t} : C^n \to C^n_\lambda$$

and

$$\mathrm{h}: C^n \to C^{n-1}_\lambda$$

by $\mathrm{h}(c)_{\alpha_0<\ldots<\alpha_{n-1}} = c_{\alpha_0<\ldots<\alpha_{n-1}\lambda}$. h does not commute with δ, but we have for $c \in C^n$

$$\mathrm{h}\delta(c) = (-1)^{n+1}\mathrm{t}(c) + \delta\mathrm{h}(c).$$

We may assume that I is isomorphic to ω_k. Let c be an n–cocycle. We want to write c as the coboundary of an $(n-1)$–cochain d. We construct the components $d_{\alpha_0<\ldots<\alpha_{n-1}}$ by recursion on α_{n-1}.

Fix $\lambda \in I$ and assume that d is already constructed up to λ. This means that a $d' \in C^{n-1}_\lambda$ is given such that $\delta(d') = \mathrm{t}(c)$. To extend d' to a suitable $(n-1)$–cochain d defined on $\{\alpha \in I \mid \alpha \leq \lambda\}$ means that $\mathrm{t}(d) = d'$ and that $\mathrm{t}\delta(d) = \mathrm{t}(c)$ and $\mathrm{h}\delta(d) = \mathrm{h}(c)$. But I_λ either has a last element or has a cofinality smaller that ω_k, which gives us $\varprojlim^{n-1}_{\alpha\in I_\lambda} A_\alpha = 0$. On the other hand $\delta(c) = 0$ implies $(-1)^{n+1}\mathrm{t}(c) + \delta\mathrm{h}(c) = 0$. Therefore $(-1)^{n+1}d' + \mathrm{h}(c)$ is a cocycle, which we may write as δe for some $(n-2)$–chain e on I_λ. Now extend d' to d such that $\mathrm{t}(d) = d'$ and $\mathrm{h}(d) = e$. Then $\mathrm{t}\delta(d) = \delta\mathrm{t}(d) = \delta(d') = \mathrm{t}(c)$ and

$$\begin{aligned}
\mathrm{h}\delta(d) &= (-1)^n\mathrm{t}(d) + \delta\mathrm{h}(d) \\
&= (-1)^n d' + \delta e \\
&= (-1)^n d' + (-1)^{n+1}d' + \mathrm{h}(c) \\
&= \mathrm{h}(c).
\end{aligned}$$

□

4 The classification

Let R be a valuation domain. We use the notations of section 2.

Lemma 4.1 *Let A be a cyclic non-zero R–module and α a positive element of the valuation semigroup Γ. Then αA is a proper submodule of A.*

Proof:
This is Nakayama's Lemma. For a short proof assume that A is generated by a and that $\mathrm{v}(r) = \alpha$. $A = rA$ would imply that $a = sra$ for some $s \in R$. Then $(1 - sr)a = 0$, which implies $a = 0$ since $1 - sr$ is a unit. $\square$

Let M be a proper divisible uniserial R-module. We want to associate to it an element $\mu(M)$ of $\varprojlim^1_{\alpha\in\Gamma} U/U_\alpha$.

Fix a non–zero cyclic submodule Z_0. Then for every $\alpha \in \Gamma$ there is a unique cyclic submodule Z_α with $\alpha Z_\alpha = Z_0$. This defines a bijection between Γ and the set of all cyclic submodules which contain Z_0. Note that $(\beta - \alpha)Z_\beta = Z_\alpha$ for $\alpha \leq \beta$. If Z_0 is isomorphic to R/I, Z_α is isomorphic to $R/(\alpha I)$.

Now fix isomorphisms $f_\alpha : R/(\alpha I) \to Z_\alpha$. If $\alpha < \beta$ the induced embedding $R/(\alpha I) \to R/(\beta I)$

$$\begin{tikzcd}
R/(\beta I) \arrow[r, "f_\beta"] & Z_\beta \\
R/(\alpha I) \arrow[u, "r_{\alpha\beta}"] \arrow[r, "f_\alpha"'] & Z_\alpha \arrow[u]
\end{tikzcd}$$

is given by multiplication with an element $r_{\alpha\beta}$ of value $\beta - \alpha$.

We define $\mu(M)$ from the $r_{\alpha\beta}$ as follows: Choose elements r_α of value α and write $r_{\alpha\beta} = u_{\alpha\beta} r_\alpha^{-1} r_\beta$ for units $u_{\alpha\beta}$. Since $(u_{\alpha\beta} r_\alpha^{-1} r_\beta)(u_{\beta\gamma} r_\beta^{-1} r_\gamma)$ and $u_{\alpha\gamma} r_\alpha^{-1} r_\gamma$ define the same map $R/(\alpha I) \to R/(\beta I)$ they differ by a factor from $1 + \alpha I \subset U_\alpha$. This shows that $(u_{\alpha\beta})$ is a 1–cocycle. We let $\mu(M)$ be the class determined by this cocycle.

Lemma 4.2 *$\mu(M)$ does not depend on the choice of Z_0, of the isomorphisms f_α and the choice of the $r_{\alpha\beta}$ and r_α.*

Proof:
We treat first the case where Z_0 (and therefore all Z_α) remains the same but we have new f'_α, $r'_{\alpha\beta}$ and r'_α. Then there are units v_α and w_α such that $f_\alpha = f'_\alpha v_\alpha$ and $r_\alpha = r'_\alpha w_\alpha$.

$$\begin{tikzcd}
R/(\beta I) \arrow[r, "v_\beta"] & R/(\beta I) \\
R/(\alpha I) \arrow[u, "r_{\alpha\beta}"] \arrow[r, "v_\alpha"'] & R/(\alpha I) \arrow[u, "r'_{\alpha\beta}"']
\end{tikzcd}$$

Since $r'_{\alpha\beta}v_\alpha$ and $v_\beta r_{\alpha\beta}$ define the same map $R/(\alpha I) \to R/(\beta I)$ we have $v_\alpha v_\beta^{-1} r'_{\alpha\beta} r_{\alpha\beta}^{-1} \in 1 + \alpha I \subset U_\alpha$. Therefore

$$u'_{\alpha\beta}u^{-1}_{\alpha\beta} = r'_\alpha r'^{-1}_\beta r'_{\alpha\beta} r_\alpha^{-1} r_\beta r^{-1}_{\alpha\beta} = w_\alpha^{-1} w_\beta r'_{\alpha\beta} r^{-1}_{\alpha\beta} = (w_\alpha v_\alpha)^{-1}(w_\beta v_\beta)(v_\alpha v_\beta^{-1} r'_{\alpha\beta} r^{-1}_{\alpha\beta})$$

If we define $e_\alpha = w_\alpha v_\alpha$ we have

$$u' u^{-1}(\delta(e))^{-1} \in U_\alpha,$$

which shows that $\mu = \mu'$.

Now assume that we have chosen Z'_0 to be Z_{α_0}. For the computation of $u'_{\alpha\beta}$ we can use $f'_\alpha = f_{\alpha_0+\alpha}$, $r'_{\alpha\beta} = r_{\alpha_0+\alpha,\,\alpha_0+\beta}$ and $r'_\alpha = r_\alpha$. It results that $u'_{\alpha\beta} = u_{\alpha_0+\alpha,\,\alpha_0+\beta}$. By Proposition 3.2 we have again $\mu = \mu'$. □

The next theorem will describe a proper uniserial divisible module M by two invariants: $\mu(M)$ and its ideal class $\mathrm{C}(M)$, which is defined as the class of any annihilator of a non-zero element of M in the ideal class semigroup, the multiplicative semigroup of all non-zero ideals of R modulo the principal ideals. Observe that the annihilators of all non-zero elements of M are in the same class.

If R is a field there are no proper divisible uniserial modules. We assume from now on that R is not a field.

Theorem 4.3 $M \mapsto (\mu(M), \mathrm{C}(M))$ *defines a bijection between proper divisible uniserial modules up to isomorphy and pairs of elements of* $\varprojlim^1_{\alpha\in\gamma} U/U_\alpha$ *and ideal classes of* R. *M is standard iff* $\mu(M) = 1$.

Proof:
Let us first assume that M and M' have the same invariants. M and M' are (isomorphic to) the direct limits of two systems $(R/(\alpha I), r_{\alpha\beta})$ and $(R/(\alpha I'), r'_{\alpha\beta})$. Since $\mathrm{C}(M) = \mathrm{C}(M')$ we can assume that $I = I'$. Choose ring elements r_α with value α. That $\mu(M) = \mu(M')$ means that $(r_\alpha r_\beta^{-1} r_{\alpha\beta})$ and $(r_\alpha r_\beta^{-1} r'_{\alpha\beta})$ differ by a coboundary i.e. there is a family (v_α) of units such that $r'_{\alpha\beta} r^{-1}_{\alpha\beta}$ differ from $v_\alpha^{-1} v_\beta$ by a factor from U_α. Since the $(r_{\alpha\beta})$ and $(r'_{\alpha\beta})$ form commuting systems and are therefore cocycles mod$I$ in the sense of the next lemma by part 2 of this lemma this factor belongs to $1 + \alpha I$. Then the last diagram is commutative for all $\alpha < \beta$ and the family of maps $v_\alpha : R/(\alpha I) \to R/(\alpha I)$ defines an isomorphism between M and M'.

Now assume that an element μ of $\varprojlim^1_{\alpha\in\Gamma} U/U_\alpha$ and an ideal class C is given. Let μ be represented by $(u_{\alpha\beta})$. Since R is not a field C can be represented by a proper ideal I. By part 1 of the next lemma we can assume that u is a cocycle mod$I$. The direct limit of the commutative system

$$(R/(\alpha I), u_{\alpha\beta} r_\alpha^{-1} r_\beta),$$

(r_α elements with value α) is then a proper uniserial divisible module with the desired invariants.

The ideal class of the standard module K/I is the class of I. $\mu(K/I) = 1$ since it is the direct limit of the system $(R/(\alpha I), r_\alpha^{-1} r_\beta)$. This proves the last part of the theorem. $\square$

Lemma 4.4 *Let the family of units $u = (u_{\alpha\beta})$ represent a 1–cocycle for the projective system (U/U_α) and I be a non–zero proper ideal. We call u a* cocycle $\mathrm{mod} I$ *if $u_{\beta\gamma} u_{\alpha\gamma}^{-1} u_{\alpha\beta} \in 1 + \alpha I$ for all $\alpha < \beta < \gamma$ and a* coboundary $\mathrm{mod} I$ *if there is a family (v_α) of units such that $v_\alpha v_\beta^{-1} u_{\alpha\beta} \in 1 + \alpha I$ for all $\alpha < \beta$.*

1. *The class of u in $\varprojlim_{\alpha\in\Gamma} U/U_\alpha$ can be represented by a cocycle* $\mathrm{mod} I$.
2. *If u is a cocycle* $\mathrm{mod} I$ *and represents a coboundary it is a coboundary* $\mathrm{mod} I$.

Proof:
Let α_0 be the value of a non–zero element of I. We will use Proposition 3.2 with the map $\phi(\alpha) = \alpha_0 + \alpha$.

Proof: of (1)
Define u' by $u'_{\alpha\beta} = u_{\alpha_0+\alpha,\, \alpha_0+\beta}$. u' is a cocycle $\mathrm{mod} I$ and in the same class as u by Proposition 3.2.

Proof: of (2)
By 1 we can assume that $(u_{\alpha\beta})$ is a cocycle $\mathrm{mod}\, \alpha_0 P$, which implies that u' defined by $u'_{\alpha\beta} = u_{-\alpha_0+\alpha,\, -\alpha_0+\beta}$ is a cocycle of the projective system $(U/U_\alpha)_{\alpha_0 \leq \alpha}$. By Proposition 3.2 the class of u' corresponds to the class of u in the isomorphism between $\varprojlim^1_{\alpha\in\Gamma} U/U_\alpha$ and $\varprojlim^1_{\alpha_0\leq\alpha} U/U_\alpha$. Since u is a coboundary u' is also a coboundary. So we have a family v'_α of units, such that $v'_\alpha v'^{-1}_\beta u'_{\alpha\beta} \in U_\alpha$ for all $\alpha_0 \leq \alpha < \beta$. If we set $v_\alpha = v'_{\alpha_0+\alpha}$ we have $v_\alpha v_\beta^{-1} u_{\alpha\beta} \in 1 + \alpha I$ for all $\alpha < \beta$. $\square$

Finally we show that in Theorem 4.3 we can replace $\varprojlim^1_{\alpha\in\Gamma} U/U_\alpha$ by $\varprojlim^1_{\alpha\in\Gamma} U_0/U_\alpha$

Proposition 4.5 *The natural map*

$$\varprojlim^1_{\alpha\in\Gamma} U_0/U_\alpha \to \varprojlim^1_{\alpha\in\Gamma} U/U_\alpha$$

is an isomorphism

Proof:
One checks easily that $\varprojlim^1_{\alpha\in I} A_\alpha = 0$ if $A_\alpha = A$ is a constant sequence. (In fact $\varprojlim^n_{\alpha\in I} A_\alpha = 0$ for all $n \geq 1$). Look at the following part of the long exact cohomology sequence:

$$\varprojlim^0_{\alpha\in\Gamma} U/U_\alpha \to \varprojlim^0_{\alpha\in\Gamma} U/U_0 \to \varprojlim^1_{\alpha\in\Gamma} U_0/U_\alpha \to \varprojlim^1_{\alpha\in\Gamma} U/U_\alpha \to \varprojlim^1_{\alpha\in\Gamma} U/U_0 = 0.$$

Since the first arrow is surjective the third arrow is an isomorphism. $\square$

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