MORSE THEORY AND HIGHER TORSION INVARIANTS II

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Abstract. Let $p: M \to B$ be a family of compact manifolds equipped with a unitarily flat vector bundle $F \to M$. We generalize Igusa's higher Franz-Reidemeister torsion $\tau(M/B; F)$ to the case that the fibre-wise cohomology $H^*(M/B; F) \to B$ carries a parallel metric. If moreover M admits a fibre-wise Morse function, we compute the difference of $\tau(M/B; F)$ and the higher analytic torsion $\mathcal{T}(M/B; F)$. We also generalise the examples given in [G].

This is the second of two papers devoted to a comparison of Igusa's and Klein's higher Franz-Reidemeister torsion τ with Bismut's and Lott's higher analytic torsion $\mathcal{T}$. In the first part [G], we evaluated the Bismut-Lott torsion form $\mathcal{T}$ for families that carry a fibre-wise Morse function, thus extending earlier work with Bismut ([BG]). Here, we relate τ and $\mathcal{T}$ for families with fibre-wise Morse functions in those situations where both invariants are defined.

We start by recalling some notation and results of [G]. Let $p: M \to B$ be a family of compact manifolds, let $F \to M$ be a flat vector bundle, and let $h: M \to \mathbb{R}$ be a fibre-wise Morse function. Then the fibre-wise critical points of h form a covering $\hat{p}: C \to B$. Let $o(T^u X) \to C$ denote the orientation bundle of the unstable vertical tangent bundle at C, and construct a vector bundle

$$V = \hat{p}_*\big(F|_C \otimes o(T^u X)\big) .$$

Then V carries a flat connection ∇^V induced by ∇^F. In [G], we defined a family Thom-Smale complex by constructing a flat superconnection A' on V. We also constructed a generalised "integration over the fibre", i.e., an $\Omega^*(B)$-linear cochain map $I: (\Omega^*(M; F), \nabla^F) \to (\Omega^*(B; V), A')$. This cochain map naturally identifies the fibre-wise cohomology $H = H^*(M/B; F) \to B$, equipped with the flat Gauß-Manin connection ∇^H, with the bundle $H^*(V, a_0') \to B$ as follows. Both $\Omega^*(M; F)$ and $\Omega^*(B; V)$ can be filtered by horizontal degree. The cochain map I respects these filtrations and induces isomorphisms of the E_p-terms of the associated spectral sequences for $p \geq 1$. The E_1-term of both spectral sequences is isomorphic to $(\Omega^*(B; H), \nabla^H)$.

A metric g^F on F induces a metric g^V on V, and the Morse function h induces an endomorphism h^V of V. Using these data, we defined an analytic torsion form

$$T(A', g^V, h^V) \in \Omega^*(B) .$$

The metric g^V induces a metric g^H_V on H by finite-dimensional Hodge theory. Let $\mathrm{ch}^o(\nabla^F, g^F)$ denote the characteristic form of $F \to M$ defined in [BL], written in the normalisation of [BG]. Then

$$dT(A', g^V, h^V) = \mathrm{ch}^o(\nabla^V, g^V) - \mathrm{ch}^o(\nabla^H, g^H_V).$$

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In particular, $T(A', g^V, h^V)$ generalises the analytic torsion forms on flat families of chain complexes defined in [BL].

In order to state the main theorem of [G], let $T^H M \subset TM$ be a horizontal complement of the vertical tangent bundle $TX \to M$ of M, and let g^{TX} be a vertical metric on TX. Then there exists a unique natural connection g^{TX} on TX which generalises the Levi-Civita connection on a single manifold X. Let

$$\mathcal{T}(T^H M, g^{TX}, \nabla^F, g^F) \in \Omega^*(B) .$$

be the analytic torsion form of Bismut and Lott ([BL]). The metrics g^{TX} and g^F induce an L_2-metric on the space $\Omega^*(M/B; F)$ of fibre-wise differential forms. By Hodge theory, we get a metric $g^H_{L_2}$ by identifying $H \to B$ with the bundle of fibre-wise harmonic forms. Let $\widetilde{\mathrm{ch}}^{\mathrm{o}}(\nabla^H, g^H_{L_2}, g^H_V)$ be the Chern-Simons class associated to the characteristic class $\mathrm{ch}^{\mathrm{o}}(\nabla^H)$. Let $\psi(\nabla^{TX}, g^{TX})$ denote the Matthai-Quillen current on the bundle $TX \to M$. Finally, we define an additive characteristic class

$$^0J = \frac{1}{2} \sum_{k \geq 1} \zeta'(-2k)\ \mathrm{ch}^{[4k]}$$

as in [BG], where ζ is the Riemann zeta function and $\mathrm{ch}^{[4k]}$ is the homogeneous part of the Chern class with coefficients in $\mathbb{R}$ of degree $4k$. The following theorem generalises the main theorem of [BG], where we assumed in addition that h admits a fibre-wise Smale gradient field.

0.1. Theorem ([G]). *Modulo exact forms on B,*

$$\begin{aligned}
\mathcal{T}\big(T^H M, g^{TX}, \nabla^F, g^F\big) - T\big(A', g^V, h^V\big) - \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H_{L_2}, g^H_V\big) \\
= -\int_{M/B} \mathrm{ch}^{\mathrm{o}}\big(\nabla^F, g^F\big)\, \big(\nabla^{TX} h\big)^* \psi\big(\nabla^{TX}, g^{TX}\big) + \hat{p}_*\big((-1)^{\mathrm{ind}_h}\ {}^0J(T^s X - T^u X)\big)\ \mathrm{rk}\, F\ .
\end{aligned}$$

As application, we used analytic torsion forms to detect infinite families of pairwise homeomorphic but not diffeomorphic fibre bundles, which were constructed using Hatcher's example of a topologically trivial, but smoothly non-trivial bundle over $4k$ with fibre S^{2l-1}. The version we present here is slightly more general, since in the second statement, we can now work with an arbitrary unitarily flat F in place of the trivial bundle.

0.2. Theorem (cf. [G]). *For each integer $k > 0$ there exists $l_0 > 0$ such that the following holds. Let $p\colon M \to B$ be a smooth fibre bundle with compact, connected, oriented base B of dimension $4k$ and with compact, connected fibres of dimension $2l-1$, where $l \geq l_0$. Then there exist smooth fibre bundles $p_j\colon M \to B$ for all $j \in Z$ with $p_0 = p$, which are all homeomorphic to p as fibre bundles and have diffeomorphic fibres, but which are pairwise not diffeomorphic as fibre bundles.*

If M moreover carries a unitarily flat vector bundle F such that the fibre-wise cohomology bundle $H^*(M/B; F) \to B$ admits a parallel metric with respect to the Gauß-Manin connection, then the p_j are pairwise not diffeomorphic as maps.

In the present paper, we restrict our attention to the case that both bundles $F \to M$ and $H \to B$ carry parallel metrics. In this case, we can use the higher analytic torsion $\mathcal{T}(T^H M, g^{TX}, \nabla^F, g^F)$ to define a cohomology class $\mathcal{T}(M/B; F) \in H^*(B; \mathbb{R})$, see [G], (2.93). We compare this class to a generalisation $\tau(M/B; F) \in H^*(B; \mathbb{R})$ of Igusa's higher Franz Reidemeister torsion in a normalisation adapted to [BG].

0.3. Theorem. *Let $M \to B$ be a family of compact manifolds, and let $(F, \nabla^F) \to M$ be a flat complex vector bundle with a parallel Hermitian metric, such that (H, ∇^H) also admits a parallel metric. Assume that there exists a function $h: M \to \mathbb{R}$ that is fibre-wise Morse, then*

$$\mathcal{T}(M/B; F) = \tau(M/B; F) + \int_{M/B} e(TX)\ {}^0J(TX)\ \mathrm{rk}\, F \quad \in H^*(B; \mathbb{R})\ .$$

We also prove that $\tau(M/B; F)$ and $\mathcal{T}(M/B; F)$ induce the same universal class $\tau(\mathcal{M}(\mathbb{C}), U)$ on the unitary Whitehead space $Wh^u(\mathcal{M}(\mathbb{C}), U)$, which contains a copy of Igusa's acyclic Whitehead space $Wh^h(\mathcal{M}(\mathbb{C}), U)$.

Note that the fibre-wise Morse function does not appear in the conclusion of Theorem 0.3. We conjecture that the theorem holds without the assumption that such a function exists. By [I1], after taking the fibre-wise product with $\mathbb{R}P^{2N}$ for N sufficiently large, we get the same higher torsion and there exists a function $h: M \to \mathbb{R}$ whose fibre-wise singularities are either of Morse type or birth-death singularities. In order to treat the general case, it thus remains to incorporate birth-death singularities into Theorems 0.1 und 0.3.

The present paper contains the proof of Theorem 0.3 and the new version of Theorem 0.2, together with a generalisation of Igusa's approach to the situation that the two flat bundles $F \to M$ and $H \to B$ admit a parallel metric. It is organised as follows.

We show in Section 6 that the construction of (V, A', h^V) in Section 1 of [G] gives rise to a map ξ from the base space B to Igusa's Whitehead space $Wh(\mathcal{M}(\mathbb{C}), GL(\mathbb{C}))$ up to homotopy, which equals Igusa's map $\xi_{p,h,F}: B \to Wh(\mathcal{M}(\mathbb{C}), GL(\mathbb{C}))$. Here, $GL(\mathbb{C}) = \lim_{\to} GL_n(\mathbb{C}) = \mathcal{M}(\mathbb{C})^\times$ denotes the stabilised general linear group. Conversely, each such map $\xi_{p,h,F}$ up to homotopy determines a $\mathbb{Z}$-graded flat vector bundle $V \to B$ with a flat superconnection $A' = \nabla^V + a'$ and an endomorphism h^V. Thus, $Wh(\mathcal{M}(\mathbb{C}), GL(\mathbb{C}))$ is a universal space for flat vector bundles with the extra structure described above.

We give a different describtion of the finite dimensional torsion $T(A', g^V, h^V)$ in Section 7, where we show that $T(A', g^V, h^V) = T(\nabla^V, A', g^V)$ depends only on the existence of local filtrations, not on h^V. If g^V is parallel and the fibre-wise cohomology $H \to B$ vanishes, then $T(\nabla^V, A', g^V)$ is the pull-back $\xi^*_{p,h,F} T(\mathcal{M}(\mathbb{C}), U)$ of a universal higher torsion class $T(\mathcal{M}(\mathbb{C}), U)$ on the acyclic Whitehead space $Wh^h(\mathcal{M}(\mathbb{C}), U)$. We also decompose the higher torsion $T(\nabla^V, A', g^V)$ of a globally filtered family of complexes into the higher torsions of the steps of the associated spectral sequence, mimicking Ma's result in [Ma].

We show in Section 8 that the Igusa's universal torsion class $\tau(\mathcal{M}(\mathbb{C}), U)$ on the Whitehead space $Wh^h(\mathcal{M}(\mathbb{C}), U)$ coincides up to normalisation with our universal class $T(\mathcal{M}(\mathbb{C}), U)$. By Igusa's two-index theorem, it is enough to compare the two classes on the stabilised acyclic two-index Whitehead space $Wh^{h,\{0,1\},s}(\mathcal{M}(\mathbb{C}), U)$, which is homotopy equivalent to $Wh^h(\mathcal{M}(\mathbb{C}), U)$. The construction of $\tau(\mathcal{M}(\mathbb{C}), U)$ relies heavily on cyclic homology, whereas $T(A', g^V, h^V)$ is constructed using a double transgression of the Chern character. The comparison of both classes proceeds by showing that the components of Igusa's class $\tau(\mathcal{M}(\mathbb{C}), U)$ have the same transgression properties as the corresponding terms in $T(\mathcal{M}(\mathbb{C}), U)$, and can thus be identified in suitable abstract acyclic cochain complexes. In the fibre-wise acyclic case, Theorem 0.3 now follows from Igusa's framing principle and Theorem 0.1.

We discuss the case that the two flat bundles $F \to M$ and $H \to B$ admit a parallel metrics. In Section 9, we construct another Whitehead space $Wh^u(\mathcal{M}(\mathbb{C}), U)$ which is the universal space of flat bundles $V \to B$ with the additional structures (A', g^V) described above, together with a parallel metric g^H on the fibre-wise cohomology $H \to B$. The space $Wh^u(\mathcal{M}(\mathbb{C}), U)$ is homotopy equivalent to a covering of Igusa's original $Wh(\mathcal{M}(\mathbb{C}), U)$. The universal torsion class $T(\mathcal{M}(\mathbb{C}), U)$ can be extended naturally to $Wh^u(\mathcal{M}(\mathbb{C}), U)$. We also give an acyclic cone construction, by which the the torsion form $T(\mathcal{M}(\mathbb{C}), U)$ can be pulled back from a fibre-wise acyclic situation.

Moreover, the class $T(M/B; F, h)$ of Definition 2.64 in [G] pulls back from a universal class on $Wh^u(\mathcal{M}(\mathbb{C}), U)$. Combining this with a natural lift of Igusa's map ξ to the unitary Whitehead space $Wh^u(\mathcal{M}(\mathbb{C}), U)$, we obtain a new class $\tau(M/B; F) \in H^*(B; \mathbb{R})$ for all families $M \to B$ of smooth compact manifolds with boundary $\partial M = M_0 \cup M_1$ which carry unitarily flat vector bundles $F \to M$ whose relative homology $H^*(M/B, M_0/B; F)$ is again unitarily flat.

Finally in Section 10, we show that Igusa's framing principle of [I2] extends to the case that $F \to M$ and $H \to B$ carry parallel metrics. We then use Igusa's framed function theorem of [I1] and the framing principle to finally establish Theorem 0.3. We prove a cut-and-paste formula for $\tau(M/B; F)$ and apply it to extend the examples considered in Section 5 of [G] and prove the generalised second statement of Theorem 0.2. Note that a more precise quantitative statement is given in Theorem 10.20. The proof of Theorem 10.20 uses Theorem 0.3 to switch from $\tau(M/B; F)$ to $\mathcal{T}(M/B; F)$ as needed.

We will generously cite [G]. Therefore, the numbering of the sections in this paper start with 6. Sections 1 to 5 always refer to [G], as well as all theorems, propositions, equations ... whose first number is 1–5.

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6. The universal torsion class on the Whitehead space

In [I2], Igusa defines the Whitehead space as the homotopy fibre of a natural map from the stable homotopy of the classifying space BG of a subgroup G of the group of units $R^\times$ of a ring R into the algebraic K-theory space $BGL(R)^+$ of R itself. In particular

$$Wh^h(R, G) \longrightarrow Q(BG_+) \longrightarrow \overline{\mathbb{Z}} \times BGL(R)^+$$

is a homotopy fibre sequence. Here BG_+ is the universal classifying space of G with an extra base point, and Q denotes the stable homotopy functor $\Omega^\infty S^\infty$. The space $\overline{\mathbb{Z}} \times BGL(R)^+ \subset K(R)$ gives the algebraic K-theory of R in degrees ≥ 1, where $\overline{\mathbb{Z}}$ denote the image of $\mathbb{Z} = K_0(\mathbb{Z})$ in $K_0(R)$, $BGL(R)$ is the classifying space for the general linear group $GL(R) = \varinjlim GL(n, R)_\delta$, and $(\,\cdot\,)^+$ denotes Quillen's plus construction. Note that all groups and rings here carry the discrete topology.

In Chapter 3 of [I2], Igusa constructs a model for the Whitehead space out of "systems of higher homotopies", which are precisely the "combinatorial flat superconnections" that we encountered in Proposition 1.33. In Subsections 6.a and 6.b, we review Igusa's construction of the simplicial Whitehead space as the geometric realisation of a simplicial category $\mathcal{W}h^{\mathrm{simp}}(R, G)$.

Let $p: M \to B$ be a proper submersion equipped with a fibre-wise Morse function $h: M \to \mathbb{R}$ and a flat vector bundle $F \to M$. We show in Section 6.c that our construction of a combinatorial flat superconnection in Section 1.c gives a simplicial functor $\Xi = \Xi_{p,h,F}$ from a certain simplicial category $\mathcal{S}$ associated to B to $\mathcal{W}h^{\mathrm{simp}}(R, G)$, which is homotopy equivalent to the corresponding functor constructed in [I2], Chapter 4.

If the ring R is an $\mathbb{R}$-subalgebra of the ring $\mathcal{M}_n(\mathbb{C})$ of $(n \times n)$-matrices over $\mathbb{C}$, and $G \subset R^\times$ is any subgroup, we develop two more models for the Whitehead space in Section 6.d, given by differential superconnections and by mixed combinatorial and differential superconnections. We show that all these models are homotopy equivalent, so that we may speak of "the" Whitehead space $Wh(R, G)$. We want to point out that this equivalence of Whitehead categories has been investigated independently by K. Igusa in [I3], where he interprets both the map into the Whitehead space and the bundle of the fibre-wise cohomology as A_∞-functors.

In Section 7, we will show that the form $T_f(A', g^V, h^V)$ of Definition 2.46 is given by a universal cohomology class on a Whitehead space if the family of cochain complexes $(V, a_0) \to B$ is fibre-wise acyclic. Since we may choose any of the models above for this Whitehead space, this justifies calling the form $T_f(A', g^V, h^V)$ a "combinatorial torsion form", although it is defined in terms of a differential superconnection. In Section 9, this construction will be generalised to the case that $H = H^*(V, a_0) \to B$ carries a parallel metric.

6.a. Simplicial sets and categories. We recall the definitions of simplicial sets and simplicial categories as well as some elementary facts and examples needed later on.

6.1. Definition. Let Δ denote the small category whose objects are the sets $[k] = \{0, \dots, k\}$, and whose morphisms are all mappings $f\colon [j] \to [k]$ with $f(i+1) \geq f(i)$ for all $0 \leq i < j$. If $\mathcal{C}$ denotes any category, then a *simplicial object* in $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$, and a *simplicial morphism* between to simplicial objects in $\mathcal{C}$ is a natural transformation.

Simplicial sets and *simplicial maps* are simplicial objects and morphisms in the category of sets and set maps. *Simplicial categories* and *simplicial functors* are simplicial objects and morphisms in the category of small categories and covariant functors.

If S is a simplicial set, then $S_k = S[k]$ is the *set of elements of degree* k, and $Sf\colon S_k \to S_j$ a *(generalised) face operator* for each morphism $f\colon [j] \to [k]$ in Δ. This terminology is compatible with Definition 1.11 in [G] of a smooth simplicial complex, where S_k was a set of smooth k-simplices $\sigma\colon \Delta_k \to B$. To comply with Definition 6.1, we must add degenerated simplices and degeneracy maps to the complex defined in Definition 1.11.

If $\mathcal{S}$ is a simplicial category, we still write $\mathcal{S}_k$ for the set of objects of degree k. The morphisms of $\mathcal{S}_k$ commute with the face operators $\mathcal{S}_k \to \mathcal{S}_l$.

The geometric realisation is a functor from the category of simplicial sets to the category of topological spaces, cf. [Lo]. The geometric realisation of a small category is the realisation of its nerve, which is again a simplicial set. Similarly, the geometric realisation of a simplicial category is the realisation of its nerve, which is now a bisimplicial set.

6.2. Example. Let S be a smooth simplicial complex on B as in Definition 1.11. Then we define a simplicial category $\mathcal{S}$:

(1) the set of objects of degree k is

$$\mathcal{S}_k = \left\{ (\sigma, \sigma') \;\middle|\; \sigma' \in S_k \text{ is a generalised face of } \sigma \in S_l \text{ for some } l \right\} ;$$

(2) there is a unique morphism from (σ, σ') to (τ, τ') iff σ is a generalised face of τ and $\sigma' = \tau'$, and none otherwise;

(3) the generalised face operators f act by

$$(\sigma, \sigma') \longmapsto \bigl(\sigma, \sigma'_{f(0),\dots,f(j)}\bigr) .$$

If S is sufficiently fine, then the geometric realisation of $\mathcal{S}$ is homeomorphic to the set $\overline{B} \subset B \times B$ of (1.39). In particular, $B \cong |S|$ is then a deformation retract of $|\mathcal{S}|$.

Note that we have implicitly worked with $\mathcal{S}$ and $|\mathcal{S}|$ in Sections 1.d and 1.e. We will describe the various complexes and superconnections constructed in Section 1 as simplicial functors from $\mathcal{S}$ to several simplicial categories that are models for Igusa's Whitehead space.

6.b. The simplicial Whitehead category. We start by recalling Igusa's simplicial Whitehead category $\mathcal{W}h^{\mathrm{simp}}(R,G)$ of [I2], Chapter 3. The typical object of this category is a system of coefficients $a_{\alpha\beta}(i_0,\dots,i_j)$ as in Proposition 1.33. In this subsection, R is an associative ring with a unit, and $G\subset R^\times$ is any subgroup.

6.3. Definition. A *graded poset* (*graded* partially ordered *set*) is a finite $\mathbb{Z}$-graded set $P=\bigcup_n P^n$ that is partially ordered by a relation $\prec$ which is independent of the $\mathbb{Z}$-grading. For $\alpha\in P^n$, we set $\operatorname{ind}(\alpha)=n$. If P, Q are graded posets, a *graded closed map from P to Q* is a map $f\colon P\to Q$ such that for all α, $\beta\in P$, $\operatorname{ind}(f(\alpha))=\operatorname{ind}(\alpha)$, and $f(\alpha)\prec_Q f(\beta)$ implies $\alpha\prec_P\beta$.

In particular, a graded bijection P to Q is closed iff the partial relation $\prec_P$ is finer than the pullback of $\prec_Q$.

6.4. Definition. Let P be a graded poset, and let V be a left R-module. Let $V^P=\bigoplus_{\alpha\in P}V_\alpha$ be a direct sum of left R-modules with $V_\alpha\cong V$ for all $\alpha\in P$. A *system of V-valued P-vectors v in degree k* assigns to each sequence $0\le i_0\le\cdots\le i_l\le k$ an element $v(i_0,\dots,i_l)\in V$. A *system of P-matrices a in degree k* assigns to each sequence $0\le i_0\le\cdots\le i_l\le k$ a matrix $a(i_0,\dots,i_l)$ with coefficients in R, with rows and columns indexed by P. We call a system of matrices (*lower*) *triangular* iff $a_{\alpha\beta}(i_0,\dots,i_l)=0$ unless $\beta\prec\alpha$. Let $V(k)$ and $\mathcal{M}_P(R;k)$ be the spaces of systems of P-vectors and of P-matrices in degree k, and let $T_P(R;k)\subset\mathcal{M}_P(R;k)$ be the subspace of triangular systems.

We define a simplicial coboundary operator δ on $V(k)$ by

$$(\delta v)(i_0,\dots,i_l)=\sum_{j=0}^{l}(-1)^j\,v(i_0,\dots,\widehat{i_j},\dots,i_l)\;, \tag{1}$$

and similarly on $\mathcal{M}_P(R;k)$. We let $\mathcal{M}_P(R;k)$ act on $V(k)$ by multiplication

$$(a*v)_\alpha(i_0,\dots,i_l)=\sum_{\beta\in P}\sum_{j=0}^{l}(-1)^{(l-j)\,(\operatorname{ind}\alpha-\operatorname{ind}\beta)}a_{\alpha\beta}(i_0,\dots,i_j)\,v_\beta(i_j,\dots,i_l)\;, \tag{2}$$

and similarly, we define a multliplication on $\mathcal{M}_P(R;k)$.

6.5. Remark. The sign factor indicates that we have a "horizontal" grading, given by the degree of the simplices we evaluate on, and a "vertical" grading, given by the grading of P. This is compatible with our orientation convention in Section 1.c. We leave it to the reader to check that "$*$" is associative and that "δ" is an odd derivation with respect to the total grading:

$$\begin{aligned}(a*b)*v&=a*(b*v)\;, &&& \delta(a*v)&=(\delta a)*v+(-1)^{\deg a}\,a*(\delta v)\;,\\ (a*b)*c&=a*(b*c)\;, &&\text{and}& \delta(a*b)&=(\delta a)*b+(-1)^{\deg a}\,a*(\delta b)\;,\end{aligned}$$

where we set

$$\bigl((-1)^{\deg a}a\bigr)_{\alpha\beta}(i_0,\dots,i_j)=(-1)^{j+\operatorname{ind}\alpha-\operatorname{ind}\beta}\,a_{\alpha\beta}(i_0,\dots,i_j)\;.$$

6.6. Definition. If a is an element of $\mathcal{M}_P(R;l)$, we call $\delta+a$ a *simplicial P-superconnection of degree l*. We say that $\delta+a$ is flat if

$$(\delta+a\,*)^2=0 \tag{1}$$

as an operator on $\mathcal{M}_P(R;l)$. We say that $\delta+a$ is *of total degree* 1 if

$$a_{\alpha\beta}(i_0,\dots,i_j)=0\qquad\text{unless}\qquad\operatorname{ind}\alpha+j=\operatorname{ind}\beta+1\;.$$

We say that $\delta + a$ is *triangular* if a is lower triangular with respect to $\prec_P$.

Let $\delta + a$ be a flat simplicial P-superconnection. If $V = \bigoplus_{\alpha \in P} V_\alpha$ is a direct sum of left R-modules and (C^*, ∂) is a cochain complex, then a *system of cochain maps* $I\colon (C, \partial) \to (V(l), \delta + a)$ is an element $I \in (\operatorname{Hom}_{\mathbb{Z}}(C, V))^P(l)$, such that

$$(\delta + a *) \circ I = I \circ \partial \,. \tag{2}$$

We say that I is *of degree* k, if $I_\alpha(i_0, \dots, i_j)|_{C^n} = 0$ unless $\operatorname{ind} \alpha + j = n + k$.

6.7. Example. Let S be a smooth simplicial complex on B, and let $a_{\alpha\beta}(\sigma)$ be defined as in Section 1.c. For each simplex $\sigma \in S_l$, let P be the set of leaves of $\hat{p}\colon C(|\sigma|) \to |\sigma|$, which is partially ordered by $\prec_\sigma$. If we set

$$a_\sigma(i_0, \dots, i_k) = \begin{cases} a(\sigma_{i_0, \dots, i_k}) & \text{if } 0 \le i_0 < \dots < i_k \le l, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{6.8}$$

then $a \in T_P(\mathbb{Z}; l)$, and $\delta + a$ is a flat triangular simplicial superconnection $\delta + a_\sigma$ of total degree 1 by Proposition 1.33 (1) and (3).

Assume that the bundle $T^u X \to C$ is oriented, and define $I_\sigma\colon \Omega^*(M) \to \mathbb{R}^P(l)$ by

$$I_{\sigma,\alpha}(i_0, \dots, i_j)\, \omega = \begin{cases} \int_{M_\alpha(\sigma_{i_0, \dots, i_j})} \omega & \text{if } 0 \le i_0 < \dots < i_k \le l, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{6.9}$$

Then $I_\sigma\colon (\Omega^*(M), d^M) \to (\mathbb{R}^P(l), \delta + a)$ is a system of cochain maps of degree 0 by Stokes' theorem and Proposition 1.33 (2).

Let G be a subgroup of the multiplicative group $R^\times$. As a model for the Whitehead space mentioned at the beginning of this section, we construct a simplicial category $\mathcal{W}h^{\mathrm{simp}}(R, G)$. In fact, we first define the pre-Whitehead category $\mathcal{W}h'^{\mathrm{simp}}(R, G)$. The Whitehead category itself is obtained from $\mathcal{W}h'^{\mathrm{simp}}(R, G)$ by adding the so-called stabilisation morphisms, and the (pre-)Whitehead space is just the geometric realisation of the (pre-)Whitehead category. This category is defined precisely as in [I2], Chapter 3. However, we use a different, but equivalent sign convention, cf. Remark 1.34.

6.10. Definition. The *simplicial pre-Whitehead category* $\mathcal{W}h'^{\mathrm{simp}}(R, G)$ is defined as follows.

(1) The objects $\mathcal{W}h_k'^{\mathrm{simp}}(R, G)$ of degree k are pairs (P, a) where P is a graded poset, and $\delta + a$ is a flat triangular simplicial superconnection of total degree 1 with $a \in T_P(R; k)$, such that $a(i_0, \dots, i_l) = 0$ unless $i_0 < \dots < i_l$.

(2) For (P, a), $(Q, b) \in \mathcal{W}h_k'^{\mathrm{simp}}(R, G)$, the morphisms from (P, a) to (Q, b) are pairs (f, g) of a graded closed bijection $f\colon P \to Q$ and a map $g\colon P \to G$, such that

$$b_{f(\alpha), f(\beta)}\big(i_0, \dots, i_l\big) = g_\alpha\, a_{\alpha\beta}(i_0, \dots, i_l)\, g_\beta^{-1}$$

for all α, $\beta \in P$ and all $0 \le i_0 < \dots < i_l \le k$.

(3) face operators are given by re-indexing:

$$F_{j_0 \dots j_m}(P, a) = (P, b) \in \mathcal{W}h_m'^{\mathrm{simp}}(R; G) \qquad \text{with} \qquad b(i_0, \dots, i_l) = a\big(j_{i_0}, \dots, j_{i_m}\big)$$

for all sequences $0 \le j_0 \le \dots \le j_m \le k$ and $0 \le i_0 \le \dots \le i_l \le m$.

The *simplicial pre-Whitehead space* $Wh'^{\mathrm{simp}}(R,G)$ is the geometric realisation of the simplicial pre-Whitehead category.

In order to keep the category $\mathcal{W}h'^{\mathrm{simp}}(R,G)$ small, we agree that all our posets P are subsets of some large fixed set. Since the poset P is mainly a book-keeping device, this convention will not affect the homotopy type of any space that we construct.

6.11. Remark. Fix a complex representation $\rho\colon R \to \mathcal{M}_r(\mathbb{C})$. Then the simplicial pre-Whitehead space carries two natural $\mathbb{Z}$-graded flat complex vector bundles.

(1) There is a natural $\mathbb{Z}$-graded flat bundle $\mathcal{V} \to Wh'^{\mathrm{simp}}(R,G)$ with fibre $(\mathbb{C}^r)^P$ over each point of $|(P,a)|$ for each k and each $(P,a) \in \mathcal{W}h_k'^{\mathrm{simp}}(R,G)$. The structure group of $\mathcal{V}$ is the group of G-monomial matrices.

(2) Because $a(i)^2 = 0$, we also have cohomology groups $H^*(V^P,a(i))$ for each $0 \leq i \leq k$. Flatness of $\delta + a$ implies that $\mathrm{id} + a(i_0,i_1)$ induces a cochain isomorphism from $(V^P,a(i_1))$ to $(V^P,a(i_0))$, so we get isomorphisms

$$\big(\mathrm{id} + a(i_0,i_1)\big)_*\colon\ H^*\big(V^P,a(i_1)\big) \ \longrightarrow\ H^*\big(V^P,a(i_0)\big)$$

for $0 \leq i_0 \leq i_1 \leq l$. For $i_0 < i_1 < i_2$, the maps $a(i_0,i_1,i_2)$ are cochain homotopies between $(\mathrm{id} + a(i_0,i_1)) \circ (\mathrm{id} + a(i_1,i_2))$ and $(\mathrm{id} + a(i_0,i_2))$, so all spaces $H^*(V^P,a(i))$ can be identified in a compatible way. Morphisms $(f,g)\colon (P,a) \to (Q,b)$ induce maps

$$(f,g)_*(i)\colon\ H^*\big(V^P,a(i)\big) \ \longrightarrow\ H^*\big(V^Q,b(i)\big)$$

that are compatible with the isomorphisms $(\mathrm{id} + a(i_0,i_1))_*$ above. This way, we obtain a natural $\mathbb{Z}$-graded flat bundle

$$\mathcal{H} = H^*(\mathcal{V},a) \ \longrightarrow\ Wh'^{\mathrm{simp}}(R,G)$$

of cohomology groups. Note that the structure group of $\mathcal{H}$ is a subgroup of $GL(R)$, which is in general not related to G.

So far, each connected component of $Wh'^{\mathrm{simp}}(R,G)$ is built from graded posets P having the same number n_k of elements degree k for each k. Using Igusa's stabilisation construction, we add more morphisms to the underlying categories $\mathcal{W}h'^{\mathrm{simp}}(R,G)$ between connected components of $Wh'^{\mathrm{simp}}(R,G)$. This gives us the (stabilised) Whitehead space $Wh^{\mathrm{simp}}(R,G)$. These extra morphisms are necessary to account for the changes of the family Thom-Smale complex that occur at birth-death singularities. We will check that all our constructions are compatible with stabilisation. In particular, we prove that the universal torsion class $T(R,G)$ we are going to define in Section 7.b extends to $Wh(R,G)$.

The only new ingredients in the final construction of $Wh(R,G)$ are elementary expansions of simplicial superconnections. Let $P^{[j]}$ denote the poset consisting of two elements α^- in degree j and α^+ in degree $j+1$, with $\alpha^- \prec \alpha^+$. Let $a^{[j]} \in \mathcal{M}_{P^{[j]}}(R,k)$ denote the system of $P^{[j]}$-matrices given by

$$a^{[j]}(i_0,\dots,i_l) = \begin{cases} \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} & \text{if } l = 0, \text{ and} \\ \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} & \text{if } l \geq 1. \end{cases} \tag{6.12}$$

Then $\big(P^{[j]},a^{[j]}\big)$ is an object in $\mathcal{W}h_k'^{\mathrm{simp}}(R,G)$, which we call the *elementary object of vertical degree j in $\mathcal{W}h_k'^{\mathrm{simp}}(R,G)$*.

We can define a direct sum on the objects of $\mathcal{W}h_k'^{\mathrm{simp}}(R,G)$ by setting

$$(P,a) \oplus (P',a') = (P \cup P', a \oplus a') .$$

Here, the partial ordering on $P \cup P'$ is the union of the partial orderings on P and P', so $\alpha \prec \beta$ iff either α, $\beta \in P$ and $\alpha \prec \beta$ in P, or α, $\beta \in P'$ and $\alpha \prec \beta$ in P'.

6.13. Definition. Let (P,a) be an object of $\mathcal{W}h_k^{\prime\mathrm{simp}}(R,G)$. Then the object $(P,a)\oplus\left(P^{[j]},a^{[j]}\right)$ is called the *elementary expansion of* (P,a) *in degree* j.

We can now define the stabilised simplicial Whitehead category.

6.14. Definition. The *(stabilised) simplicial Whitehead category* $\mathcal{W}h^{\mathrm{simp}}(R,G)$ has the same objects and face operators as $\mathcal{W}h^{\prime\mathrm{simp}}(R,G)$. Its morphisms are iterated formal compositions of $\mathcal{W}h^{\prime\mathrm{simp}}(R,G)$-morphisms and elementary expansions

$$(P,a)\ \longrightarrow\ (P,a)\oplus\left(P^{[j]},a^{[j]}\right)\ .$$

The *simplicial Whitehead space* $Wh^{\mathrm{simp}}(R,G)$ is the geometric realisation of the simplicial Whitehead category.

6.15. Remark. Note that there is a natural inclusion

$$Wh^{\prime\mathrm{simp}}(R,G)\ \longrightarrow\ Wh^{\mathrm{simp}}(R,G)\ ,$$

since $\mathcal{W}h^{\mathrm{simp}}(R,G)$ has more morphisms than $\mathcal{W}h^{\prime\mathrm{simp}}(R,G)$, and thus $Wh^{\mathrm{simp}}(R,G)$ has additional bi-simplices. We will call a flat bundle or a cohomology class on $\mathcal{W}h^{\prime\mathrm{simp}}(R,G)$ *compatible with stabilisation* if it is the pull-back of a corresponding object on $Wh^{\mathrm{simp}}(R,G)$.

The morphisms of the stabilised category still induce isomorphisms on the level of cohomology, because expansion pairs do not contribute to the cohomology. In particular, we still have a well-defined bundle $\mathcal{H}\to Wh^{\mathrm{simp}}(R,G)$ as in Remark 6.11 (2), which pulls back to $\mathcal{H}$. However, the vector bundle $\mathcal{V}\to Wh^{\prime\mathrm{simp}}(R,G)$ extends to $Wh^{\mathrm{simp}}(R,G)$ only as an equivalence class of vector bundles up to elementary expansions. This means that we will not define higher torsion classes directly on $Wh^{\mathrm{simp}}(R,G)$, but we will instead look for classes on $\mathcal{H}\to Wh^{\prime\mathrm{simp}}(R,G)$ that are compatible with stabilisation.

We also recall the definition of the acyclic Whitehead space, which plays an important role in [I2].

6.16. Definition. The *acyclic simplicial Whitehead category* $\mathcal{W}h^{\mathrm{simp},h}(R,G)$ is the full subcategory of $\mathcal{W}h^{\mathrm{simp}}(R,G)$ containing all objects (P,a) such that the complex $(V^P,a(0))$ is acyclic. The *acyclic simplicial Whitehead space* $Wh^{\mathrm{simp},h}(R,G)=\left|N\,\mathcal{W}h^{\mathrm{simp},h}(R,G)\right|$ is the geometric realisation of this category.

We remark that $Wh^{\mathrm{simp},h}(R,G)$ is the union of all connected components of $Wh^{\mathrm{simp}}(R,G)$ for which the bundle $\mathcal{H}$ vanishes.

6.c. A functor to the Whitehead category. Let $M\to B$ be a bundle of compact manifolds, and let $F\to M$ be a flat bundle with structure group $G\subset R^\times$, such that $\rho|_G\colon G\to GL_r(\mathbb{C})$ is faithful (injective). E.g., if F is a complex vector bundle, we may take $R=\mathcal{M}_r(\mathbb{C})$ to be the ring of complex $r\times r$-matrices, and $G\subset GL_rR$, where r is the complex rank of F. If F is unitarily flat, then $G\subset U(r)$. In general, if F is associated to an r-dimensional complex representation of $\pi_1(M)$ that factors over a group G, so

$$\pi_1(M)\to G\xrightarrow{\rho'}GL_r(\mathbb{C})\ ,$$

we may take $R=\mathbb{Z}(G)$ the group ring of G, and the representation $\rho\colon\mathbb{Z}(G)\to\mathcal{M}_r(\mathbb{C})$ induced by ρ'.

Let h be a fibre-wise Morse function on M with fibre-wise critical set C, and let S be a smooth simplicial structure on B as in Definition 1.11, which is fine enough to satisfy (1.36). We want to interpret the construction in Sections 1.c and 1.d of [G] as the construction of a functor from the simplicial category $\mathcal{S}$ of Example 6.2 to $\mathcal{W}h^{\prime\mathrm{simp}}(R,G)$. Throughout this subsection, we fix a ring R and a complex representation $\rho\colon R\to\mathcal{M}_r(\mathbb{C})$. Then there are natural flat complex vector bundles $\mathcal{V}$, $\mathcal{H}\to Wh'^{\mathrm{simp}}(R,G)$ as in Remark 6.11. Note that unless $T^uX\to C$ is oriented, we have to make sure that $-1\in\rho'(G)$ by enlarging G if necessary.

Supposing that M is connected, let us fix a basepoint $o\in M$. For each $\sigma\in S$, let $P(\sigma)$ denote the set of connected components of $C|_{\mathring{M}_\alpha(\sigma)}$, which we order by the partial relation $\prec_\sigma$ of (1.14). If τ is a face of σ, then the relation $\prec_\sigma$ is finer than $\prec_\tau$, so the natural identification

$$f_{\tau,\sigma}\colon\ P(\sigma)\ \longrightarrow\ P(\tau)$$

is a closed bijection.

We take the CW structure on M with cells $M_\alpha(\sigma)$ for each simplex $\sigma\in S$ and each fibre-wise critical level C_α of h over $|\sigma|$ as in Section 1.c. For each $\alpha\in P(\sigma)$, we fix a path $\gamma_\alpha(\sigma)$ from o to some point in $\mathring{M}_\alpha(\sigma)$. Let us identify the space $\mathcal{O}_F(M_\alpha(\sigma))$ of parallel sections of $F|_{\mathring{M}_\alpha(\sigma)}$ with F_o by parallel translation along $\gamma_\alpha(\sigma)$. We also fix an orientation $o_\alpha(\sigma)$ of $T^uX|_{C_\alpha(\mathring{\sigma})}$ and set

$$V(\sigma)=\bigoplus_\alpha V_\alpha(\sigma)\ .$$

Working with the simplicial set $\mathcal{S}$ of Example 6.2, for each $\sigma\in\mathcal{S}_k$ we have to construct an element $a\in T_{P(\sigma)}(R;k)$ such that $\delta+a$ is a flat simplicial superconnection of total degree 1. Therefore, we fix orientations on the bundles $T^uX|_{C_\alpha(\mathring{\sigma})}$ and regard the coboundary operator $\delta+a$ constructed in Proposition 1.33 for trivial coefficients, and in Corollary 1.45 for the general case. The coefficient $a_{\alpha\beta}(\sigma_{i_0\dots i_j})$ arises as a sum of contributions of flow lines of W going from $C_\alpha(\sigma_{i_0})$ to $C_\beta(\sigma_{i_j})$. Together with $\gamma_\alpha(\sigma)$, $\gamma_\beta(\sigma)$ and paths inside $C_\alpha(|\sigma|)$ and $C_\beta(|\sigma|)$, these connecting lines give rise to a loop in M. Since parallel translation along this loop acts by an element of G, we may regard $a_{\alpha\beta}(\sigma_{i_0\dots i_j})$ as an element of $\mathbb{Z}(G)$, or of any ring R containing G. By Proposition 1.33 (1), (3) and by Corollary 1.45 (2), (3), these coefficients give rise to a flat triangular simplicial superconnection $\delta+a_\sigma$ of total degree 1.

Now to each pair $(\sigma,\sigma')\in\mathcal{S}$, where $\sigma'=\sigma_{i_0\dots i_j}$ is a face (or degeneracy) of σ, we associate the graded poset $P(\sigma)$ and the restriction of the simplicial superconnection $\delta+a_\sigma$ to σ' following Definition 6.10 (3). Set

$$\Xi^{\mathrm{simp}}_{p,h,F}(\sigma,\sigma')=\bigl(P(\sigma),a_\sigma|_{\sigma'}\bigr)\quad\in\mathcal{W}h'^{\mathrm{simp}}_j(R,G)\ . \tag{6.17}$$

Face operators in $\mathcal{S}$ map (σ,σ') to (σ,σ''), where σ'' is a face of σ'. This implies that $\Xi^{\mathrm{simp}}_{p,h,F}$ commutes with generalised face operators.

6.18. Proposition. *The assigment $\Xi^{\mathrm{simp}}_{p,h,F}$ constructed above is a simplicial functor from $\mathcal{S}$ to the simplicial pre-Whitehead category $\mathcal{W}h'^{\mathrm{simp}}(R,G)$. The flat bundles $V\to B$ and $H\to B$ are naturally isomorphic to the pull-backs by $|\Xi_{p,h,F}|$ of the bundles $\mathcal{V}$, $\mathcal{H}\to Wh'^{\mathrm{simp}}(R,G)$ of Remark 6.11. If S' is another sufficiently fine smooth simplicial complex on B, $\mathcal{S}'$ is the corresponding simplicial category, and $\Xi^{\mathrm{simp}\prime}_{p,h,F}\colon\mathcal{S}'\to\mathcal{W}h(R;G)$ is another functor constructed as above, possibly with different choices than before, then the diagramme*

$$\begin{tikzcd}[column sep=small]
 & {|\mathcal{S}|} \arrow[dr, "\Xi^{\mathrm{simp}}_{p,h,F}"] & \\
B \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & \mathcal{W}h'^{\mathrm{simp}}(R;G) \\
 & {|\mathcal{S}'|} \arrow[ur, "\Xi^{\mathrm{simp}\prime}_{p,h,F}"'] &
\end{tikzcd}$$

commutes up to homotopy, where the maps on the left are constructed using partitions of unity as in (1.63).

6.19. Remark. Conversely, given S and $\mathcal{S}$ as before, a functor $\Xi\colon \mathcal{S} \to \mathcal{W}h'^{\mathrm{simp}}$ induces

(1) a flat $\mathbb{Z}$-graded complex vector bundle $(V, \nabla^V) \to B$ of the form $V = \hat{p}_* \widehat{V}$, where $\widehat{V} \to C$ is a vector bundle of rank r with structure group G on some finite cover $\hat{p}\colon C \to B$ of B; and
(2) a flat simplicial superconnection $\delta + a$ of total degree 1 acting on the space of V-valued simplicial cochains, which is triangular over each simplex σ with respect to some partial ordering on the leaves of $C|_{\mathring{\sigma}}$.

In other words, the coefficients $a_{\alpha\beta}(\sigma)$ satisfy Proposition 1.33 (1) and (3). Note that property (2) in Proposition 1.33 is meaningless here because we cannot reconstruct the space M from V and a.

This means that $Wh^{\mathrm{simp}}(R, G) = |\,\mathcal{W}h^{\mathrm{simp}}(R; G)|$ is a "universal space" for the flat triangular simplicial superconnections of total degree 1 described above. The geometric realisation

$$\xi^{\mathrm{simp}}_{p,h,F} = \left|\Xi^{\mathrm{simp}}_{p,h,F}\right|\colon B \longrightarrow Wh^{\mathrm{simp}}(R, G)$$

is well-defined up to homotopy; it is the "classifying map" for the given flat triangular simplicial superconnection.

Proof of Proposition 6.18. Recall that there is an $\mathcal{S}$-morphism from (σ, σ') to (τ, τ') iff τ is a face of σ and $\sigma' = \tau'$, and that this morphism is unique. We noted above that the natural identification $f_{\tau,\sigma}\colon P(\sigma) \to P(\tau)$ is a closed bijection of graded posets.

For $\alpha \in P(\sigma)$, the trivialisations of $V_\alpha(\sigma)$ and $V_{f_{\tau,\sigma}\alpha}(\tau)$ differ by an element g of G. If the orientations on $C_\alpha(\sigma)$ chosen above restricts to the one chosen on $C_{f_{\tau,\sigma}\alpha}(\sigma)$, we set $g_{\tau,\sigma;\alpha} = g$, otherwise we set $g_{\tau,\sigma;\alpha} = -g$.

Note that by Proposition 1.33 (1), $a_{\sigma;\alpha,\beta}(\sigma'') = 0$ unless $f_{\tau,\sigma}\beta \prec_\tau f_{\tau,\sigma}\alpha$. Now it is easy to check that

$$a_{\tau; f_{\tau,\sigma}\alpha, f_{\tau,\sigma}\beta}(\sigma'') g_{\tau,\sigma;\alpha} \circ a_{\sigma;\alpha,\beta}(\sigma'') \circ g^{-1}_{\tau,\sigma;\beta}$$

for all generalised faces σ'' of σ'. Thus, the pair

$$\Xi^{\mathrm{simp}}_{p,h,F}\big((\sigma, \sigma') \longrightarrow (\tau, \sigma')\big) = \big(f_{\tau,\sigma}, g_{\tau,\sigma}\big)$$

is a morphism in $\mathcal{W}h'^{\mathrm{simp}}(R; G)$. Functoriality of $\Xi^{\mathrm{simp}}_{p,h,F}$ is now trivial.

To show that $\xi^{\mathrm{simp}}_{p,h,F}$ is well-defined up to homotopy, we regard two smooth simplicial complexes on $B \times \{0\}$ and $B \times \{1\}$. These can be combined to a smooth simplicial complex on $\overline{B} = B \times [0, 1]$ with corresponding simplicial category $\overline{\mathcal{S}}$. We may now construct a functor

$$\overline{\Xi}^{\mathrm{simp}}_{p,h,F}\colon \overline{\mathcal{S}} \longrightarrow \mathcal{W}h'^{\mathrm{simp}}(R, G)$$

that restricts to given functors on simplices in $B \times \{0\}$ and $B \times \{1\}$. Our claim follows from the relativity of the constructions in Section 1, see e.g. the proof of Proposition 1.70. $\square$

Next, we want to show that the functor $\Xi^{\mathrm{simp}}_{p,h,F}\colon \mathcal{S} \to \mathcal{W}h'^{\mathrm{simp}}(R, G)$ constructed above is related to the one constructed in [I2], Section 4.3, by a natural transformation. This will then imply that the realisations are homotopic.

We therefore go back to Section 1.c, where we constructed a cell complex consisting of one cell $M_\alpha(\sigma)$ for each simplex σ of a simplicial decomposition S of B and each fibre-wise critical level α of h over $\mathring{\sigma} \subset B$. We may assume that S is sufficiently fine for the constructions both in Section 1 and in Chapter 4 of [I2], e.g., we may take S to be the first barycentric subdivision of a simplicial complex satisfying (1.13) and (1.36). This implies that any sequence of simplices $\sigma \supset \tau_0 \supset \cdots \supset \tau_k$ in [I2] corresponds to a single simplex in Section 1.

The next technical issue is the choice of partial ordering. In Section 1, we work with the partial order $\prec_\sigma$ of (1.14), while Igusa uses a "standard" and a "minimal order. The standard order is defined by saying that

$$\alpha <^s_\sigma \beta \qquad \text{iff} \qquad h_\alpha < h_\beta \qquad \text{on } |\sigma|.$$

The minimal order is generated by $\alpha <^{\min} \beta$ if there exists a negative gradient flow line from $C_\beta(\sigma)$ to $C_\alpha(\sigma)$ for a fixed vertical metric g^{TX}.

Note that both $\prec_\sigma$ and $<^{\min}_\sigma$ are refinements of the corresponding orders on the faces of σ, while $<^s_\sigma$ behaves the other way. Also, the standard order is always finer than $\prec$, which is in turn finer than the minimal order by Remark 1.15. If (P,a) is an object of $\mathcal{W}h^{\mathrm{simp}}(R,G)$, then there is a morphism from (P,a) to the object $(P^{\min},a)$, where $P^{\min}$ is the same set as P, but equipped with the minimal ordering with respect to a. Let $\Xi^{\min}_{p,h,F}\colon \mathcal{S} \to \mathcal{W}h'^{\mathrm{simp}}(R,G)$ be the corresponding functor, then there is a natural transformation from $\Xi^{\mathrm{simp}}_{p,h,F}$ to $\Xi^{\min}_{p,h,F}$, so their geometric realisations are homotopic.

But the functor $\Xi^{\min}_{p,h,F}$ gives an admissible choice of elements $\Xi^{\min}(\sigma) \in \mathcal{M}_{P^{\min}(\sigma)}(R;k)$ for the construction leading to Igusa's Theorem 4.3.8 (recall that σ is a simplex of the barycentric subdivision, so it corresponds to a sequence of simplices in [I2]). In particular, the realisation of the functor $\Xi^{\mathrm{simp}}_{p,h,F}\colon \mathcal{S} \to \mathcal{W}h'^{\mathrm{simp}}(R,G)$ is homotopic to the pullback of the realisation of Igusa's $\Xi^{\min}$ to $|\mathcal{S}|$ by Theorem 4.3.8 in [I2]. Let us summarize this.

6.20. Theorem. *The constructions in Chapter 4 of [I2] and in Section 1 of this paper give rise to homotopic maps from $|\mathcal{S}| \sim B$ to $Wh'^{\mathrm{simp}}(R,G)$.*

6.d. The differential and the mixed model of the Whitehead space. We introduce two more models $\mathcal{W}h^{\mathrm{diff}}(R,G)$ and $\mathcal{W}h^{\mathrm{mix}}(R,G)$ of the Whitehead space (note that our notion of $\mathcal{W}h^{\mathrm{diff}}(R,G)$ differs from Igusa's). Because we have to interpolate and differentiate the coefficients of mixed and differential superconnections, the ring R must now be an $\mathbb{R}$-algebra (typically, $R = \mathcal{M}_r(\mathbb{k})$ for $\mathbb{k} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). However, we may still work with any suitable group $G \subset R^\times$. We still fix a complex representation $\rho\colon R \to \mathcal{M}_r(\mathbb{C})$.

Let Δ_k denote the standard affine l-simplex, and let $f_{i_0 \dots i_l}\colon \Delta_l \to \Delta_k$ be the unique affine map sending the j-th vertex of Δ_l to the i_j-th vertex of Δ_k. Let $T_P(R) \subset \mathcal{M}_P(R)$ denote the algebra of P-triangular matrices, cf. Definition 6.4. For a superconnection A' on the trivial bundle $V^P \times \Delta_k \to \Delta_k$, we write

$$A' = d + \sum_j a'_j$$

with $a'_j \in \Omega^j(\Delta_k; \operatorname{End} V^P)$ as usual. Recall that superconnections of total degree 1 were defined in Definition 2.21.

6.21. Definition. The *differential pre-Whitehead category* $\mathcal{W}h'^{\mathrm{diff}}(R,G)$ is defined as follows.

(1) The objects $\mathcal{W}h'^{\mathrm{diff}}_k(R,G)$ of degree k are pairs (P,a') where P is a graded poset, and $d+a'$ is a flat superconnection of total degree 1 on the bundle $\Delta_k \times (\mathbb{C}^r)^P \to \Delta_k$ with $a' \in \Omega^*(\Delta_k, T_P(R))$.

(2) For (P,a') and $(Q,b') \in \mathcal{W}h'^{\mathrm{diff}}_k(R,G)$, the morphisms from (P,a') to (Q,b') are pairs (f,g) of a graded closed bijection $f\colon P \to Q$ and a map $g\colon P \to G$, such that

$$b'_{f(\alpha),f(\beta)} = g_\alpha\, a'_{\alpha\beta}\, g_\beta^{-1} \in \Omega^*\big(\Delta_k, R\big) \qquad \text{for all } \alpha,\, \beta \in P.$$

(3) Generalised face operators $F_{i_0\dots i_l}$ act by pulling $d+a'$ back by the corresponding affine map $f_{i_0\dots i_l}\colon\Delta_l\to\Delta_k$, so

$$F_{i_0\dots i_l}(P,a')=\bigl(P,f_{i_0\dots i_l}^*a'\bigr)\;.$$

The *differential pre-Whitehead space* $Wh^{\mathrm{diff}}(R,G)$is the geometric realisation of the differential pre-Whitehead category.

6.22. Remark. As in Remark 6.11, the complex representation ρ gives rise to a flat bundle $\mathcal{V}\to Wh'^{\mathrm{diff}}(R,G)$. Over each geometric simplex of $Wh'^{\mathrm{diff}}(R,G)$, we consider the space of smooth $\mathcal{V}$-valued forms. Define $\Omega^*(Wh'^{\mathrm{diff}}(R,G),\mathcal{V})$ to be the set of functions that assign a smooth $\mathcal{V}$-valued form $\omega(\sigma)$ to each simplex σ of $Wh'^{\mathrm{diff}}(R,G)$, such that $\omega(\sigma')=\omega(\sigma)|_{\sigma'}$ for each face σ' of σ. Then $d+a'$ induces a piece-wise smooth flat superconnection $\mathcal{A}'$ on $\Omega^*(Wh'^{\mathrm{diff}}(R,G),\mathcal{V})$. Let $\mathcal{H}=H^*(\mathcal{V},a_0')\to Wh'^{\mathrm{diff}}(R,G)$ be the flat bundle of the fibre-wise cohomology of $(\Omega^*(Wh'^{\mathrm{diff}};\mathcal{V}),\mathcal{A}')$, then $\mathcal{H}$ also carries a natural flat connection, which is induced by $\mathcal{A}'$.

Arguing as in Section 6.c, the differential pre-Whitehead space is universal for the kind of S-piece-wise smooth flat superconnections on vector bundles over simplicial complexes that we discussed at the beginning of Section 1.f. Similarly, we now define the mixed pre-Whitehead space, which is universal for the mixed superconnections constructed in Proposition 1.51. Let $\Delta_{k\setminus j}$ denote the span of the vertices j, $j+1$, … , k of Δ_l.

6.23. Definition. Let P be a graded poset. A *mixed system of triangular P-matrix-valued forms* on Δ_k is a pair (a,a'), where $a\in T_P(R;k)$ is a triangular system of P-matrices, and a' assigns a differential form $a'(i_0,\dots,i_{l-1})\in\Omega^*(\Delta_{k\setminus i_{l-1}};T_P(R))$ to each sequence $0\le i_0\le\dots\le i_{l-1}\le k$ with $0\le l$, with

$$a'(i_0,\dots,i_{l-1})|_{\Delta_k(i_l)}=a(i_0,\dots,i_l) \tag{1}$$

if $i_{l-1}\le i_l\le k$. Let $M_P^{\mathrm{mix}}(\Delta_k;R)$ denote the space of all mixed systems of triangular P-matrix-valued forms with coefficients in R.

We define two operators δ and d on $M_P^{\mathrm{mix}}(\Delta_k;R)$ by $\delta(a,a')=(\delta a,\delta'a')$ and $d(a,a')=(0,d'a')$ with

$$(\delta'a')(i_0,\dots,i_{l-1})=\sum_{j=0}^{l-1}(-1)^j\,a'\bigl(i_0,\dots,\widehat{i_j},\dots,i_{l-1}\bigr)+(-1)^l\,a\bigl(i_0,\dots,i_{l-1}\bigr)\;, \tag{2}$$

$$(d'a')(i_0,\dots,i_{l-1})=(-1)^ld\bigl(a'(i_0,\dots,i_{l-1})\bigr)\;. \tag{3}$$

We also define a multiplication on $M_P^{\mathrm{mix}}(\Delta_k;R)$ by $(a,a')*(b,b')=(a*b,a'*'b')$, with

$$\begin{aligned}(a'*'b')_{\alpha\gamma}(i_0,\dots,i_{l-1})&\\ =\sum_{j=0}^{l-1}\sum_{\beta\in P}(-1)^{(l-j)(\operatorname{ind}\alpha-\operatorname{ind}\beta)}\,&a_{\alpha\beta}(i_0,\dots,i_j)\,b'_{\beta\gamma}(i_j,\dots,i_{l-1})\\ &+\bigl(a'(i_0,\dots,i_{l-1})\,b'(\emptyset)\bigr)_{\alpha\gamma}\;.\end{aligned} \tag{4}$$

In (2) and (4), $a(i_0,\dots,i_j)$ is regarded as a constant $T_P(R)$-valued function on $\Delta_{k\setminus i_j}$.

6.24. Remark. We collect some properties of the definition above.

(1) Note that (2), (3) and (4) are compatible with (1). Note also that by (1), it would have been enough to specify a' instead of (a,a'), this is why we may write $\delta'a'$ instead of $\delta'(a,a')$

and $a' *' b'$ instead of $(a,a') *' (b,b')$. We have chosen the notation (a,a') mainly for compatibility with Section 1.e, see also Remark 1.52.

(2) We remark in particular that $a'(\emptyset) \in \Omega^*(\Delta_k; \operatorname{End} V^P)$, so that we can define a superconnection $A' = d + a'(\emptyset)$ on the trivial bundle with fibre V^P. Let $a'_j = a'(\emptyset)^{[j]} \in \Omega^j(\Delta_k; \operatorname{End} V^P)$ denote its component of degree j.

(3) The sign factors indicate that we have again two gradings: first, the simplicial degree, which is l for $a'(i_0,\dots,i_{l-1})$ and $a(i_0,\dots,i_l)$; second, we have the sum of the exterior degree and the degree given by P on $\Omega^*(\Delta_{k\setminus i_{l-1}}; T_P(R))$. Thus, $(a,a') \in \mathcal{M}_P^{\mathrm{mix}}(\Delta_k; R)$ is of total degree m iff

$$a'_{\alpha\beta}(i_0,\dots,i_{l-1}) \in \Omega^{m-l-\operatorname{ind}\alpha+\operatorname{ind}\beta}\big(\Delta_{k\setminus i_{l-1}}\big) .$$

Note that "d" and "δ" are super-commuting odd derivations with respect to the total degree, and that "$*$" is associative.

6.25. Definition. The *mixed pre-Whitehead category* $\mathcal{W}h'^{\mathrm{mix}}(R,G)$ is defined as follows.

(1) The objects $\mathcal{W}h_k'^{\mathrm{simp}}(R,G)$ of degree k are triples (P,a,a'), where P is a graded poset, and $(a,a') \in \mathcal{M}_P^{\mathrm{mix}}(\Delta_k; R)$ is of total degree 1 and satisfies

$$a(i_0,\dots,i_l) = a'(i_0,\dots,i_{l-1}) = 0$$

unless $i_0 < \dots < i_l$, and

$$d(a,a') + \delta(a,a') + (a,a') * (a,a') = 0 .$$

(2) For (P,a) and $(Q,a') \in \mathcal{W}h_k'^{\mathrm{mix}}(R,G)$, the morphisms from (P,a) to (Q,a') are pairs (f,g) of a graded closed bijection $f\colon P \to Q$ and a map $g\colon P \to G$, such that

$$a'_{f(\alpha),f(\beta)}(i_0,\dots,i_{l-1}) = g_\alpha\, a_{\alpha\beta}(i_0,\dots,i_{l-1})\, g_\beta^{-1} \qquad \in \Omega^*\big(\Delta_{k\setminus i_{l-1}}, R\big)$$

for all α, $\beta \in P$ and all $0 \le i_0 \le \dots \le i_{l-1} \le k$.

(3) For a sequence $0 \le j_0 \le \dots \le j_m \le k$, the face operator $F_{j_0\dots j_m}\colon \mathcal{W}h_k'^{\mathrm{mix}}(R,G) \to \mathcal{W}h_m'^{\mathrm{mix}}(R,G)$ is given by $F_{j_0\dots j_m}(P,a,a') = (P,b,b')$, with

$$b(i_0,\dots,i_l) = a\big(j_{i_0},\dots,j_{i_l}\big) \qquad \text{and} \qquad b'(i_0,\dots,i_{l-1}) = f^*_{j_0\dots j_m}\big(a(j_{i_0},\dots,j_{i_{l-1}})\big)$$

for all $0 \le i_0 \le \dots \le i_{l-1} \le m$.

The *mixed Whitehead space* $Wh'^{\mathrm{mix}}(R,G)$ is the geometric realisation of the mixed Whitehead category.

6.26. Remark. As in Remark 6.11 and Remark 6.22, the complex representation ρ gives rise to flat bundles $\mathcal{V} \to Wh'^{\mathrm{mix}}(R,G)$ and $\mathcal{H} \to Wh'^{\mathrm{mix}}(R,G)$. Here, $\mathcal{V}$ is now equipped with a mixed superconnection.

6.27. Example. The coefficients $a'(\sigma,\,\cdot\,)$ constructed in Proposition 1.51 clearly give rise to an object $(P_\sigma, a_\sigma, a'_\sigma) \in \mathcal{W}h_k^{\mathrm{mix}}(R;G)$ if $\sigma \in S_k$, by

$$a_\sigma(i_0,\dots,i_l) = a(\sigma_{i_0\dots i_l}) \qquad \text{and} \qquad a'_\sigma(i_0,\dots,i_{l-1}) = \sigma^* a'(\sigma, \sigma_{i_0\dots i_{l-1}}) .$$

Mimicking the proof of Proposition 6.18, we can show that this defines a functor $\Xi^{\mathrm{mix}}_{p,h,F}\colon \mathcal{S} \to \mathcal{W}h'^{\mathrm{mix}}(R;G)$.

To pass to the full Whitehead category, we define elementary extensions of differential and mixed superconnections. Let $P^{[j]}$ and $a^{[j]}$ be given as in (6.12). Similarly, we define a' as in Definition 6.23 with only one non-zero component

$$a_0'^{[j]} = a'^{[j]}(\emptyset) = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} .$$

We also use the symbol "$\oplus$" as in Definition 6.13.

6.28. Definition. The *elementary expansions* of $(P,a') \in \mathcal{W}h_k^{\prime\mathrm{diff}}$ and $(P,a,a') \in \mathcal{W}h_k^{\prime\mathrm{mix}}$ are given by $(P,a') \oplus (P^{[j]}, a'^{[j]})$ and $(P,a,a') \oplus (P^{[j]}, a^{[j]}, a'^{[j]})$, respectively.

6.29. Definition. The *(stabilised) differential Whitehead category* $\mathcal{W}h^{\mathrm{diff}}(R,G)$ has the same objects and face operators as $\mathcal{W}h^{\prime\mathrm{diff}}(R,G)$. Its morphisms are iterated formal compositions of $\mathcal{W}h^{\prime\mathrm{diff}}(R,G)$-morphisms and elementary expansions. The *(stabilised) differential Whitehead space* $Wh^{\mathrm{diff}}(R,G)$ is the geometric realisation of $\mathcal{W}h^{\mathrm{diff}}(R,G)$.

The *(stabilised) mixed Whitehead category* $\mathcal{W}h^{\mathrm{mix}}(R,G)$ and the *(stabilised) mixed Whitehead space* $Wh^{\mathrm{mix}}(R,G)$ are defined analogously.

As before, we define the full subcategories $\mathcal{W}h^{\mathrm{diff},h}(R,G) \subset \mathcal{W}h^{\mathrm{diff}}(R,G)$ and $\mathcal{W}h^{\mathrm{mix},h}(R,G) \subset \mathcal{W}h^{\mathrm{mix}}(R,G)$ consisting of all fibre-wise acyclic objects.

6.30. Remark. The category $\mathcal{W}h^{\mathrm{mix}}(R;G)$ encompasses both $\mathcal{W}h^{\mathrm{simp}}(R;G)$ and $\mathcal{W}h^{\mathrm{diff}}(R;G)$. In fact, there are forgetful functors

$$\begin{aligned} &\Phi_1\colon \mathcal{W}h^{\mathrm{mix}}(R;G) \longrightarrow \mathcal{W}h^{\mathrm{simp}}(R;G) \qquad \text{with} \qquad (P,a,a') \longmapsto (P,a)\,, \\ \text{and} \qquad &\Phi_2\colon \mathcal{W}h^{\mathrm{mix}}(R;G) \longrightarrow \mathcal{W}h^{\mathrm{diff}}(R;G) \qquad \text{with} \qquad (P,a,a') \longmapsto \big(P,a'(\emptyset)\big)\,. \end{aligned}$$

If (f,g) is a morphism in $\mathcal{W}h^{\mathrm{mix}}(R;G)$, then $\Phi_1(f,g)$ and $\Phi_2(f,g)$ are still given by (f,g) in the corresponding category.

It follows from the proof of Proposition 1.51 that Φ_1 has a right-inverse on objects. We will now show that in fact, both functors have right-inverses as functors, which make them homotopy equivalences on the level of realisations. This implies in particular that during the passage from a simplicial superconnection on $V \to B$ to a smooth one in Section 1, up to homotopy, no information was lost. It also implies that the simplicial, mixed, and smooth superconnections of Section 1 are all “classified” by maps into the same pre-Whitehead space $Wh'(R;G)$ that was used by Igusa to construct his torsion classes $\tau_*(M/B;F)$.

6.31. Theorem. *The forgetful functors*

$$\Phi_1\colon \mathcal{W}h^{\mathrm{mix}}(R;G) \longrightarrow \mathcal{W}h^{\mathrm{simp}}(R;G) \qquad \textit{and} \qquad \Phi_2\colon \mathcal{W}h^{\mathrm{mix}}(R;G) \longrightarrow \mathcal{W}h^{\mathrm{diff}}(R;G)$$

have homotopy inverses Ψ_1 *and* Ψ_2 *on realisations.*

6.32. Remark. For each complex representation ρ of R, we clearly have

$$\begin{aligned} \mathcal{V} &\cong |\Phi_1|^*\,\mathcal{V} \cong |\Phi_2|^*\,\mathcal{V} \longrightarrow Wh^{\prime\mathrm{mix}}(R,G)\,, \\ \text{and} \qquad \mathcal{H} &\cong |\Phi_1|^*\,\mathcal{H} \cong |\Phi_2|^*\,\mathcal{H} \longrightarrow Wh^{\mathrm{mix}}(R,G) \end{aligned}$$

as flat bundles. Also by construction, the flat simplicial superconnection on $\mathcal{V}$ and the flat differential superconnection on $\mathcal{V}$ arise by forgetting parts of the mixed superconnection on $\mathcal{V} \to Wh^{\prime\mathrm{mix}}(R,G)$.

Proof of Theorem 6.31. We start with the functor Φ_1, and we work on the level of pre-Whitehead categories, taking care that all our constructions are compatible with stabilisation. The construction in the proof of Proposition 1.51 assigns to each object (P,a) of $\mathcal{W}h^{\prime\mathrm{simp}}(R;G)$ an object (P,a,a') of $\mathcal{W}h^{\prime\mathrm{mix}}(R;G)$. However, this construction is not natural, so it is not clear that we obtain a functor from $\mathcal{W}h^{\prime\mathrm{simp}}(R;G)$ to $\mathcal{W}h^{\prime\mathrm{mix}}(R;G)$. Since morphisms are unaffected by generalised face operators and by the functor Φ_1, it is clear that a right-inverse Ψ_1 of Φ_1 must also map each $\mathcal{W}h^{\prime\mathrm{simp}}(R;G)$-morphism (f,g) to the $\mathcal{W}h^{\prime\mathrm{mix}}(R;G)$-morphism (f,g).

We proceed by induction over the simplicial degree. In degree 0, the functor Φ_1 restricts to an identity of categories

$$\mathcal{W}h_0^{\mathrm{mix}}(R;G) = \mathcal{W}h_0^{\mathrm{simp}}(R;G) ,$$

so $\Psi_{1,0} = \Phi_{1,0}^{-1}$ is simply the inverse of this identity. We assume now that we have constructed functors $\Psi_{1,k'}\colon \mathcal{W}h_{k'}^{\mathrm{simp}}(R;G) \to \mathcal{W}h_{k'}^{\mathrm{mix}}(R;G)$ for all $k' < k$, such that $\Phi_1 \circ \Psi_{1,k'}$ is the identity on $\mathcal{W}h_{k'}^{\mathrm{simp}}(R;G)$.

Given any object $(P,a) \in \mathcal{W}h_k^{\mathrm{simp}}(R;G)$, we want to construct $\Psi_{1,k}(P,a) \in \mathcal{W}h^{\mathrm{mix}}(R;G)$. If (P,a) is degenerated, then $\Psi_{1,k}(P,a)$ is already determined by compatibility with generalised face operators. Otherwise, we first replace P by the graded poset P' which is identical with P as a graded set, but carries the minimal partial ordering that supports a. In other words, we have $\alpha \prec' \beta$ iff there exists $0 \le i_0 < \dots < i_l \le k$ such that $a_{\alpha\beta}(i_0,\dots,i_l) \ne 0$. Then it follows from Definition 6.10 (2) that any morphism $(f,g)\colon (P',a) \to (Q,b)$ is invertible. Also, assume that (Q,b) lies in the same connected component of the nerve $N\,\mathcal{W}h_k^{\prime\mathrm{simp}}(R;G)$, then there exists a morphism $(Q,b) \to (P',a)$.

Assume that we can construct an $\mathrm{Aut}(P',a)$-invariant object $(P',a,a') \in \mathcal{W}h_k^{\mathrm{simp}}(R;G)$, such that

$$F_{j_0\dots j_{k'}}\,(P',a,a') = \Psi_{1,k'}\,F_{j_0\dots j_{k'}}\,(P',a)$$

for all proper face operators $F_{j_0\dots j_{k'}}$ with $k' < k$. Then for any morphism $(f,g)\colon (Q,b) \to (P',a)$, we define

$$\Psi_{1,k}(Q,b) = (Q,b,b') \qquad \text{with} \qquad b'_{\alpha\beta}(i_0,\dots,i_{l-1}) = g_\alpha^{-1}\,\bar a'_{f^{-1}(\alpha),f^{-1}(\beta)}(i_0,\dots,i_{l-1})\,g_\beta\ .$$

By minimality of P', the coefficients of b' are Q-triangular. Also by minimality and because all morphisms are bijective, any two morphisms (f,g), $(f',g')\colon (Q,b) \to (P',a)$ are related by an automorphism of (P',a). Since (P',a,a') is $\mathrm{Aut}(P',a)$-invariant, it does not matter which morphism $(f,g)\colon (Q,b) \to (P',a)$ we choose to define b'.

We first do all this only for objects (P',a) that are not elementary expansions of other objects. Compatibility with stabilisation then forces us to extend $\Psi_{1,k}$ in a prescribed way to all remaining objects in $\mathcal{W}h_k^{\mathrm{simp}}(R,G)$. It is now easy to check that our definition of $\Psi_{1,k}$ is compatible with face operators, morphisms and elementary expansions.

In order to construct a', we proceed as in the proof of Proposition 1.51, keeping in mind that a' should be invariant under all automorphisms of a. Assume that $0 \le i_0 < \dots i_{l-1} \le k$ with $i_{l-1} > l-1$, then $a'(i_0,\dots,i_{l-1})$ is already completely determined by $\Psi_{1,k'}\,F_{i_0,\dots,i_{l-1},i_{l-1}+1,\dots,k}(P',a)$. Since

$$\mathrm{Aut}(P',a) \subset \mathrm{Aut}\bigl(F_{i_0,\dots,i_{l-1},i_{l-1}+1,\dots,k}(P',a)\bigr) ,$$

the coefficient $a'(i_0,\dots,i_{l-1})$ is $\mathrm{Aut}(P',a)$-invariant. Let us assume that we already inductively constructed all $a'(0,\dots,l'-1)$ with $l < l' \le k+1$. Then the restriction of $a'(0,\dots,l-1)$ to $\partial\Delta_{k\setminus l-1}$ is determined by induction and by (1.55) (for $l \ge 1$) or (1.58) (for $l = 0$), and is also $\mathrm{Aut}(P',a)$-invariant. Because $\mathrm{Aut}(P',a)$ acts linearly on $T_P(R)$, the fixpoint set is an $\mathbb{R}$-subalgebra of $T_P(R)$. We can thus find an invariant continuation of $a'(0,\dots,l-1)|_{\partial\Delta_{k\setminus l-1}}$ to $\Delta_{k\setminus l-1}$. This completes the construction of $(P',a,a') \in \mathcal{W}h_k^{\mathrm{mix}}(R;G)$.

This completes the construction of Ψ_1. It remains to show that $\Psi_1 \circ \Phi_1$ is simplicially homotopic to the identity on $\mathcal{W}h_k^{\mathrm{mix}}(R;G)$. Such a homotopy can be constructed by the same procedure as above, see e.g. the proofs of Proposition 1.70 and Proposition 6.18. Details are left to the reader.

Now, consider the functor $\Phi_2\colon \mathcal{W}h^{\mathrm{mix}}(R;G) \to \mathcal{W}h^{\mathrm{simp}}(R;G)$. The general arguments are the same as in the construction of Ψ_1, so it is enough to give an inductive $\mathrm{Aut}(P,a')$-invariant construction of $(P,\bar a,\bar a') = \Psi_1(P,a')$, where $(P,a') \in \mathcal{W}h_k^{\mathrm{diff}}(R,G)$ is neither degenerated nor an elementary expansion, and P carries the minimal ordering that supports a'.

Note that $\bar{a}'$ determines $\bar{a}$ by Definition 6.23 (1), so we only construct $\bar{a}'$. As before, all coefficients $\bar{a}'(i_0, \dots, i_{l-1})$ with $0 \leq i_0 < \cdots < i_{l-1} < k$ and $i_{l-1} > l-1$ are inductively fixed by compatibility with face operators, and of course, they are $\mathrm{Aut}(P, a')$-invariant. We will construct the coefficients $\bar{a}'(0, \dots, l-1)$ by induction over l. For $l = 0$, we clearly have $\bar{a}'(\emptyset) = a'$. For $l > 0$, we already know the restriction of $\bar{a}'(0, \dots, l-1)$ to the set

$$\bigcup_{j=l}^{k} f_{l-1,\dots,\widehat{j},\dots,k}(\Delta_{k-l}) = \partial\Delta_{k\setminus l-1} \setminus \mathring{\Delta}_{k\setminus l} \; .$$

Note that this set is a deformation retract of $\Delta_{k\setminus l-1}$. By Definition 6.25 (1), we need an $\mathrm{Aut}(P, a')$-invariant solution $\bar{a}'(0, \dots, l-1)$ of the system of linear partial first-order differential equations

$$(-1)^l \, d\big(\bar{a}'_{\alpha\beta}(0, \dots, l-1)\big) = -(\partial'\bar{a}' + a' *' a')_{\alpha\beta}(0, \dots, l-1) \; , \tag{6.33}$$

such that $\bar{a}'(0, \dots, l-1)$ has degree $1 - l$ in $\Omega^*(\Delta_{k\setminus l-1}; T_P(R))$ and restricts to the given initial data on $\partial\Delta_{k\setminus l-1} \setminus \mathring{\Delta}_{k\setminus l}$. By triangularity of $\bar{a}'$, this system can be solved inductively in the distance of α and β in the partial ordering of P. The integrability condition is inductively given by

$$0 = \big(d'(\partial'\bar{a}' + \bar{a}' *' \bar{a}')\big)_{\alpha\beta}(0, \dots, l-1) \; ,$$

which follows from the equations

$$0 = \big(d'\bar{a}' + \delta'\bar{a}' + \bar{a}' *' \bar{a}'\big)_{\gamma\delta}(i_0, \dots, i_m)$$

for those indices γ, δ and those sequences $0 \leq i_0 < \cdots < i_m \leq k$ where all coefficients are already known, because

$$\begin{aligned}
&\big(d'(\partial'\bar{a}' + \bar{a}' *' \bar{a}')\big)_{\alpha\beta}(0, \dots, l-1) \\
&\quad = \big(-\partial' d'\bar{a}' + (d'\bar{a}') *' \bar{a}' - \bar{a}' *' d'\bar{a}'\big)_{\alpha\beta}(0, \dots, l-1) \\
&\quad = \big(\partial'(\partial'\bar{a}' + \bar{a}' *' \bar{a}') - (\partial'\bar{a}' + \bar{a}' *' \bar{a}') *' \bar{a}' + \bar{a}' *' (\partial'\bar{a}' + \bar{a}' *' \bar{a}')\big)_{\alpha\beta}(0, \dots, l-1) \\
&\quad = 0 \; .
\end{aligned}$$

In order to get initial data for a Cauchy problem, we have to specify all coefficients of $\bar{a}'(0, \dots, l-1)$ along $\partial\Delta_{k\setminus l-1} \setminus \mathring{\Delta}_{k\setminus l}$, including those that involve directions normal to this set. It is clear that this is possible. Taking for $\bar{a}'(0, \dots, l-1)$ any $\mathrm{Aut}(P, a')$-invariant solution of (6.33), we have finished the inductive construction of $(P, \bar{a}, \bar{a}') = \Psi_2(P, a')$.

This shows that Φ_2 has a right-inverse Ψ_2. By the same argument as above, the simplicial functor $\Psi_2 \circ \Phi_2$ is simplicially homotopic to the identity on $\mathcal{W}h^{\mathrm{mix}}(R; G)$. $\square$

6.34. Remark. In [I3], Igusa constructs a simplicial functor $\mathcal{W}h^{\mathrm{diff}}(R, G) \to \mathcal{W}h^{\mathrm{simp}}(R, G)$ using Chen's iterated integrals. It seems that on realisations, Igusa's functor is homotopic to $\Phi_1 \circ \Psi_2$.

7. Properties of the finite-dimensional higher torsion

In this chapter, we collect some elementary properties of the Chern torsion classes $T(A', g^V, h^V)$ introduced in Section 2.

We start by giving a description of $T(A', g^V, h^V) = T(\nabla^V, A', g^V)$ in Subsections 7.a and 7.b, which does not make use of the function h^V. This is important for two reasons. First, it allows us to define a torsion class $T(R, G)$ on the acyclic Whitehead space $Wh^h(R, G)$ in Definition 7.29, and as well on the unitary Whitehead space $Wh^u(R, G)$ in Section 9. Second, the new representation of $T(\nabla^V, A'g^V)$ will allow us to compare Igusa's higher Franz-Reidemeister torsion and T on a suitable Whitehead space in Section 8.

We then analyse the behaviour of $T(\nabla^V, A', g^V)$ on globally filtered complexes. Applications include a gluing formula for the combinatorial torsion, an extension of the examples in Section 5, and a reduction of the unitary higher torsion to an acyclic situation in Section 9.

7.a. The Chern normalisation of the torsion forms. We give a formula for finite-dimensional the Chern torsion form of Section 2 as an integral of the Chern form of a superconnection over a two-dimensional domain. This is the first step in finding a formula for $T(A', g^V, h^V)$ that does not involve the endomorphism h^V.

Let $f(x) = x\,e^{x^2}$, and recall that by Definition 2.46 and Definition 2.91,

(7.1)
$$\begin{aligned}
T\big(A', g^V, h^V\big) &= \int_0^1 \big(4s(1-s)\big)^{\frac{N^B}{2}} T_f\big(A', g^V, h^V\big)\,ds \\
&= -\int_0^1 \left(\frac{4s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \int_1^\infty \Big(\mathrm{str}_V\big(N^V f'(X_{t,1})\big) - \chi'(H) f'(0) \\
&\qquad\qquad - \big(\chi'(V) - \chi'(H)\big)\, f'\big(\sqrt{-t}/2\big)\Big)\,\frac{dt}{2t}\,ds \\
&\qquad - \int_0^1 \left(\frac{4s(1-s)}{2\pi i}\right)^{\frac{N^B}{2}} \int_0^1 \Big(\mathrm{str}_V\big(h^V f'(X_{1,T})\big) - \mathrm{str}_V\big(h^V\big)\, f'(0) \\
&\qquad\qquad - \big(\chi'(V) - \chi'(H)\big)\,\big(f'(\sqrt{-T}/2) - f'(0)\big)\Big)\,\frac{dT}{2T}\,ds\,.
\end{aligned}$$

Let $\overline{A}' = A' + \frac{\partial}{\partial s}\,ds + \frac{\partial}{\partial t}\,dt + \frac{\partial}{\partial T}\,dT$ be the extension of the flat superconnection $A' = \nabla^V + a_0 + \dots$ to the pullback ${}^1\overline{B} = B \times (0,1) \times \mathbb{R}^2_>$. We define a metric

$${}^1\overline{g^V} = g^V\left(\left(\frac{t}{4s(1-s)}\right)^{N^V} T^{h^V}\,\cdot\,,\,\cdot\,\right),$$

which coincides with the metric of (2.39) at $s = \frac{1}{2}$. Let ${}^1\overline{A}''$ denote the adjoint of $\overline{A}'$ with respect to ${}^1\overline{g^V}$, then

$$\begin{aligned}
{}^1\overline{A}'' &= \nabla^{V,*} + \sum_j \left(\frac{t}{4s(1-s)}\right)^{1-j} T^{-h^V} a_j''\, T^{h^V} \\
&\quad + \log T\,dh^V + \left(\frac{\partial}{\partial s} + \frac{N^V}{1-s} - \frac{N^V}{s}\right) ds + \left(\frac{\partial}{\partial t} + \frac{N^V}{t}\right) dt + \left(\frac{\partial}{\partial T} + \frac{h^V}{T}\right) dT\,.
\end{aligned}$$

Conjugating everything by $\big(\frac{t}{4(1-s)^2}\big)^{\frac{N^V}{2}}\, T^{\frac{h^V}{2}}$ gives two flat superconnections

(7.2)
$$\begin{aligned}
{}^1\widetilde{A}' &= \nabla^V + \sum_j \left(\frac{t}{4(1-s)^2}\right)^{\frac{1-j}{2}} T^{\frac{h^V}{2}}\, a_j'\, T^{-\frac{h^V}{2}} - \frac{\log T}{2}\,dh^V \\
&\quad + \left(\frac{\partial}{\partial s} - \frac{N^V}{1-s}\right) ds + \left(\frac{\partial}{\partial t} - \frac{N^V}{2t}\right) dt + \left(\frac{\partial}{\partial T} - \frac{h^V}{2T}\right) dT\,,
\end{aligned}$$

(7.3)
$$\begin{aligned}
{}^1\widetilde{A}'' &= \nabla^{V,*} + \sum_j \left(\frac{t}{4s^2}\right)^{\frac{1-j}{2}} T^{-\frac{h^V}{2}}\, a_j''\, T^{\frac{h^V}{2}} + \frac{\log T}{2}\,dh^V \\
&\quad + \left(\frac{\partial}{\partial s} - \frac{N^V}{s}\right) ds + \left(\frac{\partial}{\partial t} + \frac{N^V}{2t}\right) dt + \left(\frac{\partial}{\partial T} + \frac{h^V}{2T}\right) dT\,.
\end{aligned}$$

As in [BG], we consider the superconnection

$$\begin{aligned} {}^1\widetilde{A} &= (1-s)\ {}^1\widetilde{A}' + s\ {}^1\widetilde{A}'' \\ &= \nabla^{\overline{V},u} + \sum_j \left(\frac{t}{4}\right)^{\frac{1-j}{2}} \left((1-s)^j\, T^{\frac{h^V}{2}}\, a'_j\, T^{-\frac{h^V}{2}} + s^j\, T^{-\frac{h^V}{2}}\, a''_j\, T^{\frac{h^V}{2}}\right) \\ &\quad + (2s-1)\left(\frac{\omega^V}{2} + \frac{\log T}{2}\, dh^V + \frac{N^V}{2t}\, dt + \frac{h^V}{2T}\, dT\right) - 2N^V\, ds\ . \end{aligned} \tag{7.4}$$

Note that ${}^1\widetilde{A}$ contains no negative powers of $1-s$ or s, so we can extend it to $B \times [0,1] \times \mathbb{R}^2_>$. Because both ${}^1\widetilde{A}'$ and ${}^1\widetilde{A}''$ are flat, we find that

$$ {}^1\widetilde{A}^2 = -s(1-s)\left({}^1\widetilde{A}'' - {}^1\widetilde{A}'\right)^2 - \left({}^1\widetilde{A}'' - {}^1\widetilde{A}'\right) ds\ . \tag{7.5}$$

7.6. Remark. Formula(7.5) is well suited for describing ${}^1\widetilde{A}^2$ in $B \times (0,1) \times [1,\infty) \times \mathbb{R}$. For the restriction to $s=0$, we obtain instead

$$\begin{aligned} {}^1\widetilde{A}\Big|_{s=0} &= {}^1\widetilde{A}'\Big|_{s=0} + \frac{\sqrt{t}}{2}\, T^{-\frac{h^V}{2}}\, a''_0\, T^{\frac{h^V}{2}} \\ \text{and} \qquad {}^1\widetilde{A}^2\Big|_{s=0} &= \left[{}^1\widetilde{A}'\Big|_{s=0},\ \frac{\sqrt{t}}{2}\, T^{-\frac{h^V}{2}}\, a''_0\, T^{\frac{h^V}{2}}\right]\ . \end{aligned} \tag{7.7}$$

Note that ${}^1\widetilde{A}^2|_{s=0}$ is in triangular shape with respect to the $\mathbb{Z}$-grading of V, with the diagonal term given by $\frac{t}{4}\left[T^{\frac{h^V}{2}}\, a'_j\, T^{-\frac{h^V}{2}}, T^{-\frac{h^V}{2}}\, a''_0\, T^{\frac{h^V}{2}}\right]$. This has two consequences.

(1) The kernel of $({}^1\widetilde{A}^2)^{[0]}$ does not jump at $s=0$, and
(2) The restriction of $\operatorname{str}({}^1\widetilde{A}^2)^{[0]}$ to $s=0$ is of degree 0.

Similar observations are of course true at $s=1$.

7.8. Proposition. *Up to exact forms on B, the Chern analytic torsion form is given by*

$$\begin{aligned} T\big(A', g^V, h^V\big) = -(2\pi i)^{-\frac{N^B}{2}} \int_{(s,t)\in[0,1]\times[1,\infty)} \Bigg(&\frac{1}{2}\operatorname{str}_V\left(e^{-{}^1\widetilde{A}^2}\right) - \chi'(H)\,\frac{ds\,dt}{2t} \\ &- \big(\chi'(V) - \chi'(H)\big)\, f'\big(\sqrt{-t}/2\big)\,\frac{ds\,dt}{2t}\Bigg) \\ - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,T)\in[0,1]^2} \Bigg(&\frac{1}{2}\operatorname{str}_V\left(e^{-{}^1\widetilde{A}^2}\right) - \operatorname{str}_V\big(h^V\big)\,\frac{ds\,dT}{2T} \\ &- \big(\chi'(V) - \chi'(H)\big)\,\Big(\big(f'\big(\sqrt{-T}/2\big) - 1\Big)\,\frac{ds\,dT}{2T}\Bigg)\ . \end{aligned}$$

Proof. Comparing Definition 2.46 and Definition 2.91 with the formula above, we easily see that the proposition holds in degree 0. In particular, we do not need to investigate any correction terms.

Let ${}^1\widetilde{A}'$ and ${}^1\widetilde{A}''$ be defined by (7.2), (7.3), and set

$$ {}^1\widetilde{X} = \frac{1}{2}\left({}^1\widetilde{A}'' - {}^1\widetilde{A}'\right) = {}^1\widetilde{X}_{s,t,T} + \left(\frac{N^V}{2(1-s)} - \frac{N^V}{2s}\right) ds + \frac{N^V}{2t}\, dt + \frac{h^V}{2T}\, dT\ ,$$

then

$$ {}^1\widetilde{X}^2 = {}^1\widetilde{X}^2_{s,t,T} + \left[\left(\frac{N^V}{2(1-s)} - \frac{N^V}{2s}\right) ds + \frac{N^V}{2t}\,dt + \frac{h^V}{2T}\,dT\ ,\ {}^1\widetilde{X}_{s,t,T}\right] . \tag{7.9} $$

We calculate the following contribution to the integrand in formula (7.1) for $T(A', g^V, h^V)$:

$$ \begin{aligned} &\int_0^1 (4s(1-s))^{\frac{N^B}{2}} \operatorname{str}\left(\left(\frac{N^V}{2t}\,dt + \frac{h^V}{2T}\,dT\right) f'\left(\widetilde{X}_t\right)\right) ds \\ &= \int_0^1 \operatorname{str}\left(\left(\frac{N^V}{2t}\,dt + \frac{h^V}{2T}\,dT\right) f'\left(\sqrt{s(1-s)}\left(\widetilde{A}''_{\frac{t}{4s(1-s)},T} - \widetilde{A}'_{\frac{t}{4s(1-s)},T}\right)\right)\right) ds \\ &= \int_0^1 \operatorname{str}\left(\left(\frac{N^V}{2t}\,dt + \frac{h^V}{2T}\,dT\right) f'\left(\sqrt{s(1-s)}\left(\widetilde{A}''_{\frac{t}{4s^2},T} - \widetilde{A}'_{\frac{t}{4(1-s)^2},T}\right)\right)\right) ds \\ &= \int_0^1 \operatorname{str}\left(\left(\frac{N^V}{2t}\,dt + \frac{h^V}{2T}\,dT\right) f'\left(\sqrt{4s(1-s)}\ {}^1\widetilde{X}_{s,t,T}\right)\right) ds\ . \end{aligned} \tag{7.10} $$

For the second equation, we have conjugated by $\left(\frac{s}{1-s}\right)^{\frac{N^V}{2}}$ within the trace.

On the other hand, by (7.5) and (7.9), and because the supertrace of an even positive power of an odd operators vanishes, we have

$$ \begin{aligned} \frac{1}{2}\operatorname{str}\left(e^{-{}^1\widetilde{A}^2}\right) &= \frac{1}{2}\operatorname{str}\left(e^{4s(1-s)\ {}^1\widetilde{X}^2 + 2\ {}^1\widetilde{X}\,ds}\right) \\ &= \chi(V) + \frac{1}{\sqrt{4s(1-s)}}\operatorname{str}\left(f\left(\sqrt{4s(1-s)}\ {}^1\widetilde{X}\right)\right) ds \\ &= \chi(V) + \frac{1}{\sqrt{4s(1-s)}}\operatorname{str}\left(f\left(\sqrt{4s(1-s)}\ {}^1\widetilde{X}_{s,t,T}\right)\right) ds \\ &\qquad + \operatorname{str}\left(\left(\frac{N^V}{2t}\,dt + \frac{h^V}{2T}\,dT\right) f'\left(\sqrt{4s(1-s)}\ {}^1\widetilde{X}_{s,t,T}\right)\right) ds + \dots\ , \end{aligned} \tag{7.11} $$

where the remaining term involves ds, dt and dT. The previous equation and (7.10) together with Definition 2.46 and Definition 2.91 prove the proposition. □

7.b. A new formula for the Chern torsion classes. We now want to get rid of the endomorphism h^V in our formula for $T(A', g^V, h^V)$. This implies that we have to replace the last integral in Proposition 7.8 by something different. This is done in two steps, Proposition 7.18 and Theorem 7.23.

First, let

$$ {}^2V \longrightarrow {}^2B = B \times (0,1) \times \mathbb{R}_> \times [0,1] $$

be the pullback of $V \to B$, and let (s, T, u) be the coordinates on $(0,1) \times \mathbb{R}_> \times [0,1]$. We extend the superconnections $\widetilde{A}'|_{t=1/4}$ and $\widetilde{A}''|_{t=1/4}$ of (2.40) and ${}^1\widetilde{A}|_{t=1}$ to ${}^2B$, and set

$$ \widetilde{A} = (1-s)\,\widetilde{A}' + s\,\widetilde{A}''\ . \tag{7.12} $$

We then consider the superconnection

$$\begin{aligned}
{}^2\widetilde{A} &= u\ {}^1\widetilde{A}|_{t=1} + (1-u)\ \widetilde{A}|_{t=1/4} \\
&= \nabla^{\overline{V},u} + \sum_j 2^{j-1} \Bigl(\bigl((1-s)^j u + (1-s)(1-u)\bigr)\, T^{\frac{h^V}{2}}\, a_j'\, T^{-\frac{h^V}{2}} \\
&\qquad\qquad + \bigl(s^j u + s(1-u)\bigr)\, T^{-\frac{h^V}{2}}\, a_j''\, T^{\frac{h^V}{2}} \Bigr) \\
&\quad + (2s-1)\left(\frac{\omega^V}{2} + \frac{\log T}{2}\, dh^V + \frac{h^V}{2T}\, dT\right) - 2u\, N^V\, ds\ .
\end{aligned} \tag{7.13}$$

As $T \to 0$, this superconnection becomes singular. Note that by Definition 2.35, there exists $\varepsilon > 0$ with

$$T^{\frac{h^V}{2}}\, a_j'\, T^{-\frac{h^V}{2}} = O\Bigl(T^{\frac{\varepsilon}{2}}\Bigr) \qquad \text{and} \qquad T^{-\frac{h^V}{2}}\, a_j''\, T^{\frac{h^V}{2}} = O\Bigl(T^{\frac{\varepsilon}{2}}\Bigr)\ . \tag{7.14}$$

The remaining terms in ${}^2\widetilde{A}$ define a new superconnection

$${}^2\widetilde{A}_0 = \nabla^{\overline{V},u} + (2s-1)\ {}^2\widetilde{X}_0 - 2u\, N^V\, ds\ ,$$

where

$${}^2\widetilde{X}_0 = \frac{\omega^V}{2} + \frac{\log T}{2}\, dh^V + \frac{h^V}{2T}\, dT\ .$$

Note that ${}^2\widetilde{X}_0^2 = \omega^V/4$. Because N^V commutes with ${}^2\widetilde{A}_0$, the curvature of ${}^2\widetilde{A}_0$ is given by

$${}^2\widetilde{A}_0^2 = -s(1-s)\, \bigl(\omega^V\bigr)^2 - 2\ {}^2\widetilde{X}_0\, ds - 2\, N^V\, du\, ds$$

Let us now compute the (unnormalised) Chern character form of ${}^2\widetilde{A}_0$. We find that

$$\begin{aligned}
\operatorname{str}\Bigl(e^{-{}^2\widetilde{A}_0^2}\Bigr) &= \operatorname{str}\Bigl(e^{s(1-s)\,(\omega^V)^2}\Bigr) - 2\operatorname{str}\Bigl(N^V\, e^{s(1-s)\,(\omega^V)^2}\Bigr)\, du\, ds \\
&\quad + \frac{1}{\sqrt{s(1-s)}}\, \operatorname{str}\Bigl(f\Bigl(\sqrt{4s(1-s)}\ {}^2\widetilde{X}_0\Bigr)\Bigr)\, ds \\
&= \chi(V) - 2\chi'(V)\, du\, ds + d\operatorname{str}\bigl(h^V\ \log T\bigr)\, ds \\
&\quad + \operatorname{str}\Bigl(\omega^V\ e^{s(1-s)\,(\omega^V)^2}\Bigr)\, ds\ .
\end{aligned} \tag{7.15}$$

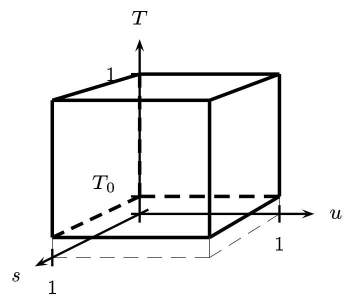
Here we have used that positive even powers of odd operators vanish, that $\bigl[\nabla^{V,u}, \omega^V\bigr] = 0$, and that ω^V, h^V and N^V supercommute pairwise.

As in (2.45), it follows that there exists an $\varepsilon' > 0$ with

$$\begin{aligned}
&\operatorname{str}\Bigl(e^{-{}^2\widetilde{A}^2}\Bigr) - d\operatorname{str}\bigl(h^V\ \log T\bigr)\, ds \\
&\qquad = \chi(V) - 2\chi'(V)\, du\, ds + \operatorname{str}\Bigl(\omega^V\ e^{s(1-s)\,(\omega^V)^2}\Bigr)\, ds + O\bigl(T^{\varepsilon'}\bigr) + O\bigl(T^{\varepsilon'}\bigr)\, \frac{dT}{T}\ ,
\end{aligned} \tag{7.16}$$

where the terms $O(T^{\varepsilon'})$ do not contain the exterior variable dT.

For $T_0 > 0$, consider the polyhedron ${}^2D = [0,1] \times [T_0, 1] \times [0,1]$.



By Stokes' theorem, the form

$$-(2\pi i)^{-\frac{N^B}{2}} \int_{(B\times\partial\,{}^2D)/B} \frac{1}{2}\left(\operatorname{str}\left(e^{-{}^2\widetilde{A}^2}\right) - \operatorname{str}\left(h^V\right)\frac{dT\,ds}{T} + 2\chi'(V)\,du\,ds\right)$$

is exact. We study the behaviour of the integrals over the six faces as $T_0 \to 0$. This will allow us to replace one contribution to the Chern torsion class by two contributions which are easier to handle.

α) The faces $s = 0$ and $s = 1$. Let us consider the restriction to $s = 0$, where by (7.7), (7.13) and (7.12),

$$\begin{aligned} {}^2\widetilde{A}|_{s=0} &= \widetilde{A}'|_{t=1/4} + \frac{u}{2}\, T^{-\frac{h^V}{2}}\, a_0''\, T^{\frac{h^V}{2}} \\ \text{and} \qquad {}^2\widetilde{A}^2|_{s=0} &= \frac{u}{2}\left[\widetilde{A}'|_{t=1/4}\,,\ T^{-\frac{h^V}{2}}\, a_0''\, T^{\frac{h^V}{2}}\right] - \frac{1}{2}\, T^{-\frac{h^V}{2}}\, a_0''\, T^{\frac{h^V}{2}}\, du \end{aligned}$$

because $\left(\widetilde{A}'\right)^2 = (a_0'')^2 = 0$. Note that ${}^2\widetilde{A}^2|_{s=0}$ is in triangular shape with respect to N^V, with diagonal term

$$\frac{u}{4}\left[T^{\frac{h^V}{2}}\, a_0'\, T^{-\frac{h^V}{2}}\,,\ T^{-\frac{h^V}{2}}\, a_0''\, T^{\frac{h^V}{2}}\right].$$

This implies that the supertrace of $e^{-{}^2\widetilde{A}^2}$ contains neither dT nor du. A similar statement holds at $s = 1$. In particular

$$0 = \int_{(s,T)\in[T_0,1]\times[0,1]} \frac{1}{2}\operatorname{str}\left(e^{-{}^2\widetilde{A}^2|_{s=0}}\right) = \int_{(s,T)\in[T_0,1]\times[0,1]} \frac{1}{2}\operatorname{str}\left(e^{-{}^2\widetilde{A}^2|_{s=1}}\right). \tag{7.17}$$

β) The face $u = 0$. With (2.40), (2.41) and (7.5), we have

$$\begin{aligned} {}^2\widetilde{A}|_{u=0} &= (1-s)\,\widetilde{A}'|_{t=1/4} + s\,\widetilde{A}''|_{t=1/4} \\ \text{and} \qquad {}^2\widetilde{A}^2|_{u=0} &= -4s(1-s)\,\widetilde{X}|_{t=1/4}^2 - 2\,\widetilde{X}|_{t=1/4}\,ds\,. \end{aligned}$$

Again by (7.16), we know that the following integral exists:

$$\int_{(s,T)\in[0,1]^2} \left(\frac{1}{2}\operatorname{str}\left(e^{-{}^2\widetilde{A}^2|_{u=0}}\right) - \operatorname{str}\left(h^V\right)\frac{dT\,ds}{2T}\right).$$

γ) The face $u=1$. By (7.4) and (7.13),

$$ {}^2\widetilde{A}|_{u=1} = {}^1\widetilde{A}|_{t=1} \; . $$

In particular, (7.10) and (7.11) imply that

$$ \int_{(s,T)\in[0,1]\times[T_0,1]} \frac{1}{2}\operatorname{str}\Bigl(e^{-{}^2\widetilde{A}^2|_{u=1}}\Bigr) = -\int_{T_0}^{T}\int_0^1 \bigl(4s(1-s)\bigr)^{\frac{N^B}{2}} \operatorname{str}\Bigl(h^V f'\bigl(\widetilde{X}_t\bigr)\Bigr)\frac{dT}{2T} \; . $$

By (7.16),, we know that the following integral exists:

$$ \int_{(s,T)\in[0,1]\times[T_0,1]} \biggl(\frac{1}{2}\operatorname{str}\Bigl(e^{-{}^2\widetilde{A}^2|_{u=1}}\Bigr) - \operatorname{str}\bigl(h^V\bigr)\frac{dT}{2T}\,ds\biggr) \; . $$

δ) The face $T=1$. The integral

$$ \int_{(s,u)\in[0,1]^2} \biggl(\frac{1}{2}\operatorname{str}\Bigl(e^{-{}^2\widetilde{A}^2|_{T=1}}\Bigr) + \chi'(V)\,du\,ds\biggr) $$

exists and can be nonzero. It will become a contribution to $T(A',g^V,h^V)$ in the next intermediate step.

ε) The face $T=T_0$. From (7.16), we conclude that

$$ \lim_{T_0\to 0}\int_{(s,u)\in[0,1]^2} \biggl(\frac{1}{2}\operatorname{str}\Bigl(e^{-{}^2\widetilde{A}^2|_{T=T_0}}\Bigr) + \chi'(V)\,du\,ds\biggr) = 0 \; . $$

Stokes' theorem and α–ε) imply the following intermediate result.

7.18. Proposition. *Up to exact forms on B, the Chern analytic torsion $T(A',g^V,h^V)$ can be expressed as*

$$ \begin{aligned} T\bigl(A',g^V,h^V\bigr) = -(2\pi i)^{-\frac{N^B}{2}} \int_{(s,t)\in[0,1]\times[1,\infty)} &\biggl(\frac{1}{2}\operatorname{str}_V\Bigl(e^{-{}^1\widetilde{A}^2}\Bigr) - \chi'(H)\,\frac{dt}{2t}\,ds \\ &\qquad - \bigl(\chi'(V)-\chi'(H)\bigr)\,f'\bigl(\sqrt{-t}/2\bigr)\,\frac{dt}{2t}\,ds\biggr) \\ - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,u)\in[0,1]^2} &\biggl(\frac{1}{2}\operatorname{str}_V\Bigl(e^{-{}^2\widetilde{A}^2|_{T=1}}\Bigr) + \chi'(V)\,du\,ds\biggr) \\ - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,T)\in[0,1]^2} &\biggl(\frac{1}{2}\operatorname{str}_V\Bigl(e^{-{}^2\widetilde{A}^2|_{u=0}}\Bigr) - \operatorname{str}_V\bigl(h^V\bigr)\,\frac{dT\,ds}{2T} \\ &\qquad - \bigl(\chi'(V)-\chi'(H)\bigr)\Bigl(f'\bigl(\sqrt{-t}/2\bigr)-1\Bigr)\,\frac{dT\,ds}{2T}\biggr) \; . \end{aligned} $$

We are now in a position to finally eliminate h^V from our representation of $T(A',g^V,h^V)$. For the final step, we let

$$ {}^3\overline{V} \longrightarrow {}^3\overline{B} = B\times[0,1]\times\mathbb{R}_>\times[0,1] $$

be the pullback of $V \to B$, and let (s,T,v) be the coordinates on $[0,1] \times \mathbb{R}_> \times [0,1]$. We define a superconnection

$$\begin{aligned}
{}^3\widetilde{A} &= v\,\Big((1-s)\,\widetilde{A}'|_{t=1/4} + s\,\widetilde{A}''|_{t=1/4}\Big) + (1-v)\,\Big((1-s)\,\nabla^{\overline{V}} + s\,\nabla^{\overline{V},*}\Big) \\
&= \nabla^{\overline{V},u} + v\,\sum_j 2^{j-1}\,\Big((1-s)\,T^{\frac{h^V}{2}}\,a_j'\,T^{-\frac{h^V}{2}} + s\,T^{-\frac{h^V}{2}}\,a_j''\,T^{\frac{h^V}{2}}\Big) \\
&\qquad + (2s-1)\,\Big(\frac{\omega^V}{2} + v\,\frac{\log T}{2}\,dh^V + v\,\frac{h^V}{2T}\,dT\Big)\;.
\end{aligned} \tag{7.19}$$

As above, we use those terms that do not vanish as $T \to 0$ to define a new superconnection

$$ {}^3\widetilde{A}_0 = \nabla^{\overline{V},u} + (2s-1)\,\Big(\frac{\omega^V}{2} + v\,\frac{\log T}{2}\,dh^V + v\,\frac{h^V}{2T}\,dT\Big)\;. $$

Similar as above, we calculate

$$ {}^3\widetilde{A}_0^2 = -s(1-s)\big(\omega^V\big)^2 - \omega^V\,ds - \frac{1}{2}\,d\big(h^V\;\log T\big)\,d\big((2s-1)v\big)\;. $$

As above, this implies that there exists an $\varepsilon' > 0$ such that

$$\begin{aligned}
&\operatorname{str}\Big(e^{-{}^3\widetilde{A}^2}\Big) - \frac{1}{2}\,d\operatorname{str}\big(h^V\;\log T\big)\,d\big((2s-1)v\big) \\
&\qquad = \chi(V) + \operatorname{str}\Big(\omega^V\,e^{s(1-s)\,(\omega^V)^2}\Big)\,ds + O\big(T^{\varepsilon'}\big) + O\big(T^{\varepsilon'}\big)\,\frac{dT}{T}\;,
\end{aligned} \tag{7.20}$$

Let ${}^3D = [0,1]^3$, with coordinates s, T and v. By Stokes' theorem, the form

$$ -(2\pi i)^{-\frac{N^B}{2}} \int_{(B\times\partial\,{}^3D)/B} \frac{1}{2}\,\Big(\operatorname{str}\Big(e^{-{}^3\widetilde{A}^2}\Big) - \operatorname{str}\big(h^V\big)\,\frac{dT}{2T}\,d\big((2s-1)v\big)\Big) $$

is exact. We study the behaviour of the integrals over the six faces of ${}^3D$ as $T_0 \to 0$. This will allow us to replace one contribution to the Chern torsion class by another one that is easier to handle.

α) The faces $s=0$ and $s=1$. Let us consider the restriction to $s=0$, where by (7.19),

$$\begin{aligned}
{}^3\widetilde{A}|_{s=0} &= v\,\widetilde{A}'|_{t=1/4} + (1-v)\,\nabla^{\overline{V}} \\
\text{and}\qquad {}^3\widetilde{A}^2|_{s=0} &= -4v(1-v)\,\Big(\widetilde{A}'|_{t=1/4} - \nabla^{\overline{V}}\Big)^2 - 2\,\Big(\widetilde{A}'|_{t=1/4} - \nabla^{\overline{V}}\Big)\,dv\;.
\end{aligned}$$

Note that $\big(\widetilde{A}'|_{t=1/4} - \nabla^{\overline{V}}\big)$ is in triangular shape with respect to the local partial ordering induced by h^V, and the diagonal term is just $\frac{1}{2}\,d(h^V\;\log T)$. A similar statement holds at $s=1$. In particular

$$\begin{aligned}
0 &= \int_{(T,v)\in[T_0,1]\times[0,1]} \frac{1}{2}\,\Big(\operatorname{str}\Big(e^{-{}^3\widetilde{A}^2|_{s=0}}\Big) + \operatorname{str}\big(h^V\big)\,\frac{dT}{2T}\,dv\big)\Big) \\
&= \int_{(T,v)\in[T_0,1]\times[0,1]} \frac{1}{2}\,\Big(\operatorname{str}\Big(e^{-{}^3\widetilde{A}^2|_{s=1}}\Big) - \operatorname{str}\big(h^V\big)\,\frac{dT}{2T}\,dv\big)\Big)\;.
\end{aligned}$$

β) The face $v=0$. With (2.40), (2.41) and (7.5), we have

$$\text{(7.21)}\qquad {}^3\widetilde{A}|_{v=0} = (1-s)\,\nabla^{\overline{V}} + s\,\nabla^{\overline{V},*} \qquad\text{and}\qquad {}^3\widetilde{A}^2|_{v=0} = -4s(1-s)\,\big(\omega^V\big)^2 - 2\,\omega^V\,ds\;.$$

Clearly, ${}^3\widetilde{A}^2|_{v=0}$ does not contain the exterior variable dT, so

$$\int_{(s,T)\in[0,1]^2} \frac{1}{2}\,\operatorname{str}\Big(e^{-\,{}^3\widetilde{A}^2|_{v=0}}\Big) = 0\;.$$

γ) The face $v=1$. By (7.13) and (7.19),

$${}^3\widetilde{A}|_{v=1} = {}^2\widetilde{A}|_{u=1}\;.$$

By (7.16), we know that the following integrals exist and are equal:

$$\begin{aligned}\int_{(s,T)\in[0,1]^2} &\frac{1}{2}\left(\operatorname{str}\Big(e^{-\,{}^2\widetilde{A}^2|_{u=0}}\Big) - 2\operatorname{str}\big(h^V\big)\,\frac{dT}{2T}\,ds\right)\\ &= \int_{(s,T)\in[0,1]^2} \frac{1}{2}\left(\operatorname{str}\Big(e^{-\,{}^3\widetilde{A}^2|_{v=1}}\Big) - 2\operatorname{str}\big(h^V\big)\,\frac{dT}{2T}\,ds\right)\;.\end{aligned}$$

δ) The face $T=1$. The integral

$$\int_{(s,v)\in[0,1]^2} \frac{1}{2}\,\operatorname{str}\Big(e^{-\,{}^3\widetilde{A}^2|_{T=1}}\Big)$$

exists and can be nonzero. It will become a contribution to $T(A',g^V,h^V)$.

ε) The face $T=T_0$. From (7.20), we conclude that

$$\lim_{T_0\to 0}\int_{(s,v)\in[0,1]^2} \frac{1}{2}\,\operatorname{str}\Big(e^{-\,{}^3\widetilde{A}^2|_{T=T_0}}\Big) = 0\;.$$

We may thus define a Chern torsion form without using h^V.

7.22. Definition. The finite dimensional *Chern torsion class* is modulo exact forms defined as

$$\begin{aligned}T\big(\nabla^V,A',g^V\big) = -(2\pi i)^{-\frac{N^B}{2}} &\int_{(s,t)\in[0,1]\times[1,\infty)} \left(\frac{1}{2}\,\operatorname{str}_V\Big(e^{-\,{}^1\widetilde{A}^2}\Big) - \chi'(H)\,\frac{dt\,ds}{2t}\right.\\ &\qquad\qquad \left. - \big(\chi'(V)-\chi'(H)\big)\,f'\big(\sqrt{-t}/2\big)\,\frac{dt\,ds}{2t}\right)\\ &- (2\pi i)^{-\frac{N^B}{2}} \int_{(s,u)\in[0,1]^2} \left(\frac{1}{2}\,\operatorname{str}_V\Big(e^{-\,{}^2\widetilde{A}^2|_{T=1}}\Big) - \chi'(V)\,du\,ds\right)\\ &- (2\pi i)^{-\frac{N^B}{2}} \int_{(s,v)\in[0,1]^2} \left(\frac{1}{2}\,\operatorname{str}_V\Big(e^{-\,{}^3\widetilde{A}^2|_{T=1}}\Big)\right.\\ &\qquad\qquad \left. - \big(\chi'(V)-\chi'(H)\big)\,\Big(f'\big(\sqrt{-t}/2\big)-1\Big)\,\frac{dv\,ds}{2v}\right)\;.\end{aligned}$$

Then the considerations above immediately prove the following result.

7.23. Theorem. *The two definitions of a finite-dimensional Chern torsion form agree, so up to exact forms on B,*

$$T\bigl(A',g^V,h^V\bigr)=T\bigl(\nabla^V,A',g^V\bigr)\;.$$

If $A'=\nabla^V+a_0'$ with $a_0'\in\Gamma(\operatorname{End}^1V)$, we let $T^{\mathrm{BL}}(A',g^V)$ denote the Bismut-Lott torsion form of [BL] in the Chern normalisation of [BG], cf. Section 2.i. Together with Remark 2.47, Theorem 7.23 has the following consequence.

7.24. Corollary. *Let $A'=\nabla^V+a_0'$ with $a_0'\in\Gamma(\operatorname{End}^1V)$, then*

$$T\bigl(\nabla^V,A',g^V\bigr)=T^{\mathrm{BL}}(A',g^V)\;.$$

7.25. Theorem. *Let $(V,\nabla^V)\to B$ be a flat $\mathbb{Z}$-graded vector bundle with a flat superconnection A' such that $A'-\nabla^V\in\Omega^*(B;\operatorname{End}V)$ takes values in a nilpotent subalgebra of $\operatorname{End}V$, which may vary over B. Then for any metric g^V on V, we have*

$$dT\bigl(\nabla^V,A',g^V\bigr)=\mathrm{ch}^{\mathrm{o}}\bigl(\nabla^V,g^V\bigr)-\mathrm{ch}^{\mathrm{o}}\bigl(\nabla^H,g^H\bigr)\;.$$

Proof. By Definition 7.22 and the preceeding arguments, we find that

$$\begin{aligned}(7.26)\quad dT\bigl(\nabla^V,A',g^V\bigr)=\frac12\,(2\pi i)^{\frac{1-N^B}{2}}\biggl(&-\lim_{t\to\infty}\int_{s\in[0,1]}\mathrm{str}_V\Bigl(e^{-{}^1\widetilde A^2|_{T=1}}\Bigr)\\&+\int_{s\in[0,1]}\mathrm{str}_V\Bigl(e^{-{}^3\widetilde A^2|_{\substack{v=0\\T=1}}}\Bigr)+\int_{t\in[1,\infty)}\mathrm{str}_V\Bigl(e^{-{}^1\widetilde A^2|_{T=1}}\Bigr)\Big|_{s=0}^1\\&+\int_{u\in[0,1]}\mathrm{str}_V\Bigl(e^{-{}^2\widetilde A^2|_{T=1}}\Bigr)\Big|_{s=0}^1+\int_{v\in[0,1]}\mathrm{str}_V\Bigl(e^{-{}^3\widetilde A^2|_{T=1}}\Bigr)\Big|_{s=0}^1\biggr)\;.\end{aligned}$$

Arguing as in (7.10) and (7.11), we compute

$$\begin{aligned}&(2\pi i)^{\frac{1-N^B}{2}}\lim_{t\to\infty}\int_{s\in[0,1]}\frac12\,\mathrm{str}_V\Bigl(e^{-{}^1\widetilde A^2|_{T=1}}\Bigr)\\&\qquad=(2\pi i)^{\frac{1-N^B}{2}}\lim_{t\to\infty}\int_{s\in[0,1]}\frac{1}{\sqrt{4s(1-s)}}\;\mathrm{str}\Bigl(f\bigl(\sqrt{4s(1-s)}\;{}^1\widetilde X_{s,t,1}\bigr)\Bigr)\,ds\\&\qquad=(2\pi i)^{\frac{1-N^B}{2}}\lim_{t\to\infty}\int_{s\in[0,1]}\frac{1}{\sqrt{4s(1-s)}}\;\mathrm{str}\Bigl(f\Bigl(\sqrt{4s(1-s)}\,\widetilde X_{\frac{t}{4s(1-s)}}\Bigr)\Bigr)\,ds\\&\qquad=\lim_{t\to\infty}\int_{s\in[0,1]}\Bigl(\frac{4s(1-s)}{2\pi i}\Bigr)^{\frac{N^B-1}{2}}\;\mathrm{str}\Bigl(f\bigl(\widetilde X_t\bigr)\Bigr)\,ds=\mathrm{ch}^{\mathrm{o}}\bigl(\nabla^H,g^H_V\bigr)\end{aligned}$$

as in the proof of Theorem 2.48. On the other hand, we have

$$\begin{aligned}&(2\pi i)^{\frac{1-N^B}{2}}\int_{s\in[0,1]}\frac12\,\mathrm{str}_V\Bigl(e^{-{}^3\widetilde A^2|_{\substack{v=0\\T=1}}}\Bigr)\\&\qquad=(2\pi i)^{\frac{1-N^B}{2}}\lim_{v\to0}\int_{s\in[0,1]}\frac{1}{\sqrt{4s(1-s)}}\;\mathrm{str}_V\Bigl(f\bigl(\sqrt{s(1-s)}\,\omega^V\bigr)\Bigr)\,ds=\mathrm{ch}^{\mathrm{o}}\bigl(\nabla^V,g^V\bigr)\;.\end{aligned}$$

We still have to show that the remaining terms on the right hand side of (7.26) vanish. By Remark 7.6, the restriction of $\operatorname{str}_V\bigl(e^{-{}^1\widetilde{A}^2}\bigr)$ to $s=0$ and $s=1$ is of degree 0 and hence does not contain dt, so

$$\int_{t\in[1,\infty)} \frac{1}{2}\operatorname{str}_V\Bigl(e^{-{}^1\widetilde{A}^2|_{T=1}}\Bigr)\Big|_{s=0}^{1} = 0\;.$$

By (7.15), the integrand $\operatorname{str}_V\bigl(e^{-{}^2\widetilde{A}^2}\bigr)$ involves no term that contains du but not ds, so

$$\int_{u\in[0,1]} \frac{1}{2}\operatorname{str}_V\Bigl(e^{-{}^2\widetilde{A}^2|_{T=1}}\Bigr)\Big|_{s=0}^{1} = 0\;.$$

Finally, we have

$$\begin{aligned}\int_{v\in[0,1]} &\frac{1}{2}\operatorname{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{\substack{s=0\\T=1}}}\Bigr)\\ &= \int_{v\in[0,1]} \frac{1}{\sqrt{4v(1-v)}}\operatorname{str}_V\Bigl(f\Bigl(\sqrt{4v(1-v)}\,\bigl(\widetilde{A}'|_{t=1/4}-\nabla^V\bigr)\Bigr)\Bigr)\,dv = 0\;,\end{aligned}$$

because $\widetilde{A}'-\nabla^V$ takes values in a nilpotent subalgebra of $\operatorname{End} V$. Our theorem follows from the equations above. $\square$

Theorem 7.25 shows that $T(\nabla^V,A',g^V)$ behaves as expected even without the existence of a Morse endomorphism h^V. Proceeding as in the proof of Corollary 2.63, we can therefore draw the following conclusion.

7.27. Corollary. *Let $(V,\nabla^V)\to B$ be a flat $\mathbb{Z}$-graded vector bundle with a flat superconnection A' such that $A'-\nabla^V\in\Omega^*(B;\operatorname{End}V)$ takes values in a nilpotent subalgebra of $\operatorname{End}V$, which may vary over B. Assume that both V and $H=H^*(V,a_0')$ carry parallel metrics, then the form*

$$T\bigl(\nabla^V,A',g^V\bigr)+\widetilde{\operatorname{ch}}^{\circ}\bigl(\nabla^H,g^H,g^H_V\bigr)$$

is even, closed and real, and its cohomology class is independent of the choice of parallel metrics in degree ≥ 2.

7.28. Remark. Let us state some properties of Definition 7.22 above.

(1) The formula for $T(\nabla^V,A',g^V)$ makes essential use of the Chern normalisation. Integrating over two-dimensional domains instead of one-dimensional contours as in Definition 2.46 allows a much larger choice of superconnections to use in the definition. In fact, the integrals in Definition 7.22 do not arise from applying the Chern normalisation operator $\int_0^1(s(1-s))^{N^B}\,ds$ to a one-dimensional contour integral.
(2) We have to include the flat reference connection ∇^V in the notation $T(\nabla^V,A',g^V)$ because ∇^V is used for the construction of the superconnection ${}^3\widetilde{A}$ in (7.19) above.
(3) We had to assume that $A'-\nabla^V$ is an inhomogeneous differential form with values in a nilpotent subalgebra of $\operatorname{End}V$, which may vary over B. In fact, we could define $T(\nabla^V,A',g^V)$ without this assumption. However, we would then be unable to prove that

$$dT\bigl(\nabla^V,A',g^V\bigr)=\operatorname{ch}^{\circ}\bigl(\nabla^V,g^V\bigr)-\operatorname{ch}^{\circ}\bigl(\nabla^H,g^H_V\bigr)\;.$$

By the way, we do not have to demand that the nilpotent subalgebra mentioned above can locally be chosen parallel. This is because by flatness of A',

$$\bigl[\nabla^V,a_k'\bigr]=-a_0'a_{k+1}'-\cdots-a_{k+1}'a_0'$$

is again an element of the subalgebra.

Let from now on $T(\nabla^V, \mathcal{A}', g^{\mathcal{V}})$ denote the form constructed in Theorem 7.23, which can be used to define a universal torsion class on the acyclic Whitehead space. We assume that G is a subgroup of a unitary group, so that the bundle $\mathcal{V} \to Wh^h(R,G)$ carries a canonical parallel metric $g^{\mathcal{V}}$.

7.29. Definition. Define $T(R,G) \in H^{\text{even}}\big(Wh^h(R,G), \mathbb{R}\big)$ to be the class represented by the cocycle which assigns to every simplex Δ in $\mathcal{W}h^{\text{diff}}(R,G)$ the integral

$$\int_\Delta T\big(\nabla^{\mathcal{V}}, \mathcal{A}', g^{\mathcal{V}}\big) \,.$$

Then $T(R,G)^{[0]}$ equals the fibre-wise Franz-Reidemeister torsion of the acyclic complex $(\mathcal{V}, a_0)$. Another possibility would have been to find an adapted endomorphism $h^{\mathcal{V}}$ over $\big|\mathcal{W}h^{\text{diff}}(R,G)\big|$. Here the difficulty is that one has to make sure that Definition 2.35 (3) is satisfied. This is possible if $h^{\mathcal{V}}$ is chosen suitably and the tautological superconnection $\mathcal{A}'$ on $\mathcal{V} \to \big|\mathcal{W}h^{\text{diff}}(R,G)\big|$ is pulled back to the interior of each simplex using cut-off functions as in Definition 1.65. Of course, by Theorem 7.23, the class obtained this way would be cohomologous to the class $T(R,G)$ defined in Definition 7.29 above.

Let (V, A', h^V) be a family Thom-Smale complex over B such that ∇^V is unitarily flat and fibre-wise acyclic. By Proposition 6.18 and Remark 6.19, there exists a "classifying functor" $\Xi = \Xi^{\text{diff}}_{p,h,F}\colon S \to \mathcal{W}h^{\text{diff},h}(R,G)$, inducing a classifying map $\xi\colon B \to Wh^h(R,G)$ up to homotopy, such that

$$(V, \nabla^V, A', g^V) \cong \xi^*\big(\mathcal{V}, \nabla^{\mathcal{V}}, \mathcal{A}', g^{\mathcal{V}}\big) \,.$$

7.30. Corollary. *Let S be a simplicial complex on B and let $\Xi\colon S \to \mathcal{W}h^{\text{diff},h}(R,G)$ be a simplicial map. Let $(V, \nabla^V, A', g^V) = \Xi^*(\mathcal{V}, \nabla^{\mathcal{V}}, \mathcal{A}', g^{\mathcal{V}})$, then*

$$T\big(\nabla^V, A', g^V\big) = |\Xi|^*\, T(R,G) \qquad \in H^{\text{even}}(|S|, \mathbb{R}) \,. \tag{1}$$

In particular, if $p\colon M \to B$ is a proper submersion with a fibrewise Morse-function $h\colon M \to \mathbb{R}$, if $(F, \nabla^F, g^F) \to M$ is a unitarily flat and fibrewise acyclic vector bundle, and if $\xi\colon B \to Wh(R,G)$ is constructed as Chapter 4 of [I2], cf. Remark 6.19, then

$$T(M/B, F, h) = \xi^* T(R,G)^{[\geq 2]} \,. \tag{2}$$

Proof. Claim (1) is an immediate consequence of Theorem 7.23 and Definition 7.29. By Theorem 6.20, Igusa's functor Ξ is homotopic to the functor constructed in Section 1. Thus, (2) follows from Definition 2.64 and (1). □

7.c. Higher torsion and filtrations. If there is a filtration of the vector bundle V that is compatible with ∇^V and A', and preserves the local splitting of V of Definition 2.35, then we show that the higher torsion $T(\nabla^V, A', g^V)$ can be computed using higher torsions associated to a spectral sequence. Such filtered complexes arise in the following situations.

(1) In the case that the fibre X of $p\colon N \to B$ is itself fibred by $p = p_2 \circ p_1$, with $p_1\colon N \to M$ and $p_2\colon M \to B$, the filtration of $\Omega^*(M/B; F)$ by relative horizontal degree gives a family of Leray spectral sequences over B. The analytic torsion of such a multifibration was treated by Ma in [Ma]. Here, we recover Ma's result in the case that both fibrations admit fibre-wise Morse functions, if the fibre-wise Morse function on M with respect to h_2 admits a Smale gradient field.

(2) If the bundle $p\colon M \to B$ admits a fibre-wise function such that 0 is a regular value on each fibre of p, one has a decomposition $M = M^+ \cup M^-$ with $M^- = h^{-1}(-\infty, 0]$ and $M^+ = h^{-1}[0,\infty)$. Under certain assumptions on H, Igusa proves in [I2], Chapter 5 that the higher Franz-Reidemeister torsion of $M \to B$ equals the sum of the torsions of $M^+ \to B$ and $M^- \to B$ already on the level of maps from B to $Wh(R,G)$. If h can be chosen to be Morse on every fibre of p, we obtain a gluing formula for $T_{\mathrm{ch}}(M/B; F, h)$. A gluing formula for the infinitesimal equivariant analytic torsion in a similar situation was proved by Bunke in [Bu].

(3) Suppose that $F \to M$ and $H = H^*(M/B; F)$ admit parallel metrics, then we get a filtered complex on $\widehat{V} = V \oplus H \to B$ which is fibre-wise acyclic. Here, the spectral sequence degenerates, and $T_{\mathrm{ch}}(M/B; F, h) = T_{\mathrm{ch}}\big(\nabla^{\hat V}, \widehat{A}', g^{\hat V}\big)$. In other words, the torsion class on the acyclic Whitehead space determines the torsion class on the unitary Whitehead spaces that will be defined in Section 9.

We will prove a finite dimensional analogue of Ma's results in [Ma], Section 3.2, where the fibre-wise Leray spectral sequence of a family of submersions is considered.

Suppose that the vector bundle (V, ∇^V, A') admits a metric g^V and a decreasing graded filtration

$$F^0V^j = V \supset \cdots \supset F^{k_0}V^j = 0 \tag{7.31}$$

that is compatible with ∇^V and $A' = \nabla^V + a'$, and put $F^pV = V$ for $k \le 0$ and $F^pV = 0$ for $p \ge k_0$. Then let $(E_k, d'_k) \to B$ be the spectral sequence associated to the filtered complex (V, a'_0). There is an associated filtration

$$F^0H = H \supset \cdots \supset F^{k_0}H = 0 \qquad \text{with} \qquad F^pH = \big(\ker(a'_0) \cap F^pV\big) \big/ \big(\mathrm{im}(a'_0) \cap F^pV\big) . \tag{7.32}$$

The following result has been proved by Ma in [Ma] in an infinite-dimensional setting.

7.33. Proposition. *For $k = 0$, let the operator d'_0 on $E_0^p = F^pV/F^{p+1}V$ be induced by a'_0. There are a flat connection ∇^{E_0} and a flat superconnection D'_0 of total degree 1 on E_0, induced by ∇^V and A', such that D'_0 starts with d'_0 in degree 0, and $D'_0 - \nabla^{E_0}$ is a differential form with values in some nilpotent subalgebra of* $\operatorname{End} E_0$ *which may vary over B.*

For $k \ge 0$, we have a vector bundle $E_{k+1} = (H^*(E_k, d'_k)) \to B$. If $\nabla^{E_{k+1}}$ denotes the Gauß-Manin connection induced by D'_k then $\nabla^{E_{k+1}}$ is flat, and the fibre-wise differential d'_k is parallel with respect to $\nabla^{E_{k+1}}$. In particular, the superconnection D'_{k+1} defined by $D'_{k+1} = \nabla^{E_{k+1}} + d'_{k+1}$ is flat and of total degree 1.

The spectral sequence E_k converges to $E_\infty = E_{k_0}$ with $E_\infty^{p,q} = F^pH^q/F^{p+1}H^q$ and ∇^{E_∞} is the connection induced by ∇^H. $\square$

Let g^V be a metric on V and identify E_0^p with $(F^{p+1}V)^\perp \cap F_pV$. This allows us to identify

$$V \cong E_0 = \bigoplus_p E_0^p , \tag{7.34}$$

and g^V induces a metric g^{E_0} on E_0. We inductively define metrics on E_k as follows. Assume that g^{E_k} is already defined, and let d'^*_k denote the adjoint of d'_k with respect to g^{E_k}. Then we can identify $E_{k+1} = H(E_k, d'_k)$ with the $(d'_k + d'^*_k)$-harmonic elements of E_k. This induces a metric $g^{E_{k+1}}$ on E_{k+1}.

By assumption, the spectral sequence converges to $E_{k_0+1} = E_\infty \to B$, and we obtain a metric $g^{E_\infty} = g^{E_{k_0+1}}$. A metric g^H on H allows us to identify E_∞^{p+q} with $F^pH^{p+q} \cap (F^{p+1}H^{p+q})^\perp$ similar as above.

7.35. Definition. Let (V, ∇^V) be a flat vector bundle with a ∇^V-parallel filtration F^*V. A metric g^V is called *adapted to* F^*V if the subbundles $(F^{p+1}V)^\perp \cap F_pV$ are also ∇^V-parallel.

7.36. Remark. We collect some examples and properties of adapted metrics.

(1) Any ∇^V-parallel metric is adapted to F^*V.
(2) Suppose that (V, A', h^V) is a family Thom-Smale complex. If each subbundle F^p can locally be written as the direct sum of some of the bundles V^j_α of Definition 2.35, then any metric g^V adapted to (V, A', h^V) in the sense of Definition 2.37 is in particular adapted to F^*V. This will be the case in all the applications we have in mind.
(3) Suppose g^V is adapted to F^*V. Then the adjoint $\nabla^{V,*}$ of ∇^V with respect to g^V also respects the filtration F^*V and the decomposition (7.34), and we have
$$\mathrm{ch}^o\bigl(\nabla^V, g^V\bigr) = \sum_{p,q} (-1)^{p+q}\, \mathrm{ch}^o\Bigl(\nabla^{E_0^p}, g^{E_0^p}\Bigr) = \mathrm{ch}^o\bigl(\nabla^{E_0}, g^{E_0}\bigr)\;.$$
Similarly, if g_0^V and g_1^V are two adapted metrics that induce the same splitting (7.34), then $g_t^V = (1-t)\, g_0^V + t\, g_1^V$ is adapted for all t, so modulo exact forms,
$$\widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^V, g_0^V, g_1^V\bigr) = \widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^{E_0}, g_0^{E_0}, g_1^{E_0}\bigr)\;.$$
(4) The bundle $H \to B$ is filtered by (7.32), but even if g^V is adapted to F^*V, it is not clear that there is a metric g^H adapted to F^*H. However, if we assume that g^H is adapted, we can use g^H to identify
$$H^{p,q} := F^pH^{p+q} \cap \bigl(F^{p+1}H^{p+q}\bigr)^\perp \cong F^pH^{p+q} \bigm/ F^{p+1}H^{p+q} = E_\infty^{p,q}$$
as flat vector bundles with respect to ∇^H and ∇^{E_∞}, and to define
$$\widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^H, g^H, g^{E_\infty}\bigr) = \sum_{p,q} (-1)^{p+q}\, \widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^{E_\infty^{p,q}}, g^{H^{p,q}}, g^{E_\infty^{p,q}}\bigr)\;.$$

We can now state the main result of this subsection, which we prove below.

7.37. Theorem. *Suppose that g^V is adapted to the filtration F^*V, and that g^H is adapted to the induced filtration on H. Then the torsion of V can be computed as*
$$\begin{aligned} &T\bigl(\nabla^V, A', g^V\bigr) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^H, g^H, g_V^H\bigr) \\ &\qquad = T\bigl(\nabla^{E_0}, D_0', g^{E_0}\bigr) + \sum_{k=1}^{k_0} T^{\mathrm{BL}}\bigl(D_k', g^{E_k}\bigr) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^{E_\infty}, g^H, g^{E_\infty}\bigr)\;. \end{aligned}$$

7.38. Remark. Let $N \xrightarrow{p_1} M \xrightarrow{p_2} B$ be a family of proper submersions as initem (1) at the beginning of this subsection. Assume that N carries a fibre-wise Morse function h_1 with respect to π_1, and that M carries a fibre-wise Morse function h_2. Over each compact subset of B, $h = \varepsilon, h_1 + h_2 \circ \pi_1$ becomes a fibre-wise Morse function with respect to $\pi = \pi_2 \circ \pi_1$ for ε small enough. The family-Thom-Smale complex of h carries a natural filtration by horizontal degree with respect to π_1 if h_2 admits a fibre-wise Morse-Smale gradient field. In this case, the filtration gives rise to the Leray spectral sequence of $N_b \to M_b$ for each $b \in B$.

In this situation, Theorem 7.37 implies that Theorem 0.1 is compatible with Ma's formula for the analytic torsion of iterated fibre bundles in [Ma]. We leave the details to the interested reader. The necessity for the existence of a Morse-Smale gradient field for h_2 is reflected by the fact that already the E_1-term in the Leray spectral of the filtered complex V associated with h carries a superconnection of the form $\nabla^{E_1} + d_1'$.

Note that if h_2 does not admit a fibre-wise Morse-Smale gradient field, we still get local filtrations over subsets of B by the values of h_2. However, the type of the sectral sequence changes as $b \in B$ varies, and we are situation similar to the one describe in Section 1.

7.d. A proof of Theorem 7.37. If g^V is adapted, we may define a ∇^V-parallel number operator N^F on V, which acts on $F^p \cap (F^{p+1})^\perp$ by p. We define a new metric g_r^V on V by

$$g_r^V = g^V\, r^{N^F} = \bigoplus_p r^p\, g^V|_{E_0^p} \;. \tag{7.39}$$

Let $g_r^{E_k}$ denote the metrics on the bundles $E_k \to B$ induced by g_r^V as above. Because we have identified $E_{k+1}^{p,q}$ with a subbundle of $E_k^{p,q}$ in order to define $g^{E_{k+1}}$, it follows by induction that

$$g_r^{E_k^{p,q}} = g^{E_k^{p,q}}\, r^{N^F} = r^p\, g^{E_k^{p,q}}$$

for all $k \geq 0$, including $k = \infty$.

Clearly, the statement of our theorem is compatible with variation of g^V by metrics that preserve the splitting (7.34) of V. We may therefore prove the theorem by replacing g^V with g_r^V for some small $r > 0$, and considering the limit $r \to 0$. Note that modulo exact forms,

$$\widetilde{\mathrm{ch}}^{\,\mathrm{o}}\big(\nabla^V, g^V, g_r^V\big) = \frac{1}{2}\int_1^r \mathrm{str}_V\big(N^F\big)\,\frac{dr}{r} = \frac{\log r}{2}\,\mathrm{str}_V\big(N^F\big)$$

because N^F is parallel with respect to ∇^V and $\nabla^{V,*}$. We still denote the induced number operators on E_k by N^F. A careful count shows that

$$\mathrm{str}_{E_k}\big(N^F\big) = \mathrm{str}_{E_{k+1}}\big(N^F\big) + k\,\big(\chi'(E_{k+1}) - \chi'(E_k)\big)\;,$$

so

$$\widetilde{\mathrm{ch}}^{\,\mathrm{o}}\big(\nabla^V, g^V, g_r^V\big) = \frac{\log r}{2}\left(\mathrm{str}_{E_\infty}(N^F) + \sum_{k=1}^{\infty}\big(\chi'(E_k) - \chi'(E_\infty)\big)\right). \tag{7.40}$$

If A'' denotes the adjoint of A' with respect to g^V, then the adjoint of A' with respect to g_r^V is given by

$$A_r'' = r^{-N^F}\, A''\, r^{N^F}\;.$$

Conjugating both A' and A_r'' with $r^{\frac{N^F}{2}}$ gives new superconnections

$$\widetilde{A}_r' = r^{\frac{N^F}{2}}\, A'\, r^{-\frac{N^F}{2}} \qquad \text{and} \qquad \widetilde{A}_r'' = r^{-\frac{N^F}{2}}\, A''\, r^{\frac{N^F}{2}}\;.$$

Let

$$\widetilde{A}_r = \frac{1}{2}\big(\widetilde{A}_r' + \widetilde{A}_r''\big) \qquad \text{and} \qquad \widetilde{X}_r = \frac{1}{2}\big(\widetilde{A}_r'' - \widetilde{A}_r'\big)\;.$$

We note that under the identification of (7.34),

$$\lim_{r\to 0}\widetilde{A}_r' = D_0' \qquad \text{and} \qquad \lim_{r\to 0}\widetilde{A}_r'' = D_0''\;, \tag{7.41}$$

where D_0'' is the adjoint of D_0' with respect to g^{E_0}, and thus also with respect to all $g_r^{E_0}$ for all $r > 0$. If we define superconnections ${}^1\widetilde{A}_r$, ${}^2\widetilde{A}_r$ and ${}^3\widetilde{A}_r$ on V and ${}^1\widetilde{D}_0$, ${}^2\widetilde{D}_0$ and ${}^3\widetilde{D}_0$ on E_0 as in (7.4), (7.13), (7.19) at $T = 1$, then similarly,

$$\lim_{r\to 0}{}^j\widetilde{A}_r = {}^j\widetilde{D}_0 \tag{7.42}$$

for $j = 1, 2, 3$.

Equation (7.42) immediately clarifies the following behaviour of two of the three integrals in Definition 7.22. We have

$$\lim_{r\to 0}\int_{(s,u)\in[0,1]^2}\frac{1}{2}\,\mathrm{str}_V\Bigl(e^{-{}^2\widetilde{A}_r^2}\Bigr)=\int_{(s,u)\in[0,1]^2}\frac{1}{2}\,\mathrm{str}_V\Bigl(e^{-{}^2\widetilde{D}_0^2}\Bigr)\,, \tag{7.43}$$

and

(7.44)
$$\begin{aligned}\lim_{r\to 0}&\int_{(s,v)\in[0,1]^2}\biggl(\frac{1}{2}\,\mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2}\Bigr)-\bigl(\chi'(V)-\chi'(H)\bigr)\Bigl(f'\bigl(\sqrt{-v}/2\bigr)-1\Bigr)\frac{dv\,ds}{2v}\biggr)\\&=\int_{(s,v)\in[0,1]^2}\biggl(\frac{1}{2}\,\mathrm{str}_V\Bigl(e^{-{}^3\widetilde{D}_0^2}\Bigr)-\bigl(\chi'(E_0)-\chi'(E_1)\bigr)\Bigl(f'\bigl(\sqrt{-v}/2\bigr)-1\Bigr)\frac{dv\,ds}{2v}\biggr)\\&\qquad-\bigl(\chi'(E_1)-\chi'(E_\infty)\bigr)\int_0^1\Bigl(f'\bigl(\sqrt{-v}/2\bigr)-1\Bigr)\frac{dv}{2v}\,.\end{aligned}$$

In particular, in the limit we recover two of the three terms in the definition of $T(\nabla^{E_0},D_0',g^{E_0})$ up to a simple correction term.

If we write

$$\alpha_{r,t}=\Bigl(\mathrm{str}_V\Bigl(N^V\,f'\bigl(\widetilde{X}_{r,t}\bigr)\Bigr)-\chi'(H)\Bigr)\frac{dt}{2t}\,,$$

then by its construction in Section 7.a, the remaining integral takes the form

$$\begin{aligned}\int_{(s,t)\in[0,1]\times[1,\infty)}&\biggl(\frac{1}{2}\,\mathrm{str}_V\Bigl(e^{-{}^1\widetilde{A}_r^2}\Bigr)-\chi'(H)\,\frac{ds\,dt}{2t}-\bigl(\chi'(V)-\chi'(H)\bigr)\,f'\bigl(\sqrt{-t}/2\bigr)\,\frac{ds\,dt}{2t}\biggr)\\&=\int_0^1\bigl(4s(1-s)\bigr)^{\frac{N^B}{2}}\,ds\int_1^\infty\alpha_{r,t}-\bigl(\chi'(E_0)-\chi'(E_\infty)\bigr)\int_1^\infty f'\bigl(\sqrt{-t}/2\bigr)\,\frac{dt}{2t}\,.\end{aligned}\tag{7.45}$$

We will split the integral over $\alpha_{r,t}$ into finitely many subintervals, which we will rescale differently.

Similar as in [MM], [D] or [Ma], we have to control the "small eigenvalues" of $\tilde{a}_{0,r}=\frac{1}{2}\left(\tilde{a}_{0,r}'+\tilde{a}_{0,r}''\right)$ as $r\to 0$. Let us write

$$\begin{aligned}a_j'&=\sum_k a_j'^{(k)}&&\text{with}&&a_j'^{(k)}|_{E_0^{p,q}}\colon\ E_0^{p,q}\longrightarrow E_0^{p+k,q+1-k}\,,\\a_j''&=\sum_k a_j''^{(k)}&&\text{with}&&a_j''^{(k)}|_{E_0^{p,q}}\colon\ E_0^{p,q}\longrightarrow E_0^{p-k,q+k-1}\,,\\\text{and}\qquad x_j&=\sum_k x_j^{(k)}&&\text{with}&&x_j^{(k)}=\bigl(a_j''^{(k)}-a_j'^{(k)}\bigr)\,/\,2\end{aligned}$$

for all p, q. Then the components $\tilde{a}_{j,r}$ and $\tilde{x}_{j,r}$ of $\widetilde{A}_r-\nabla^{E_k,u}$ and $\widetilde{X}_r$ in $\Omega^j(B;\mathrm{End}\,V)$ are given by

$$\tilde{a}_{j,r}=\sum_k r^{\frac{k}{2}}\,a_j^{(k)}\qquad\text{and}\qquad\tilde{x}_{j,r}=\sum_k r^{\frac{k}{2}}\,x_j^{(k)}\,. \tag{7.46}$$

We define a superconnection $D_k=\frac{1}{2}\left(D_k'+D_k''\right)$ on E_k, and let $d_k=\frac{1}{2}\left(d_k'+d_k''\right)$ denote its component of degree 0. Similarly, we construct fibrewise operators

$$Y_k=\frac{1}{2}\left(D_k''-D_k'\right)=y_k+\frac{\omega_k}{2}\quad\in\quad\mathrm{End}\,E_k$$

for $k\geq 1$ with $y_k=\frac{1}{2}\left(d_k''-d_k'\right)$ and $\omega_k=\nabla^{E_k,*}-\nabla^{E_k}$. We put

$$Y_{k,t}=\sqrt{t}\,y_k+\frac{\omega^{E_k}}{2}\,.$$

7.47. Definition. Set
$$U_{\varepsilon,k} = \left\{\, z \in \mathbb{C} \;\middle|\; d\big(z, \operatorname{Spec}(y_k)\big) > \varepsilon \,\right\} .$$

Let $\Pi_k \in \operatorname{End} E_0$ denote the orthogonal projection onto E_k, regarded as a subset of $V \cong E_0$. The evaluation of (7.45) is based on a quantitative version of a result of Bismut-Berthomieu and Ma in our finite-dimensional setting. We will prove the following theorem in Section 7.e.

7.48. Theorem ([BB], Theorem 6.5; [Ma], Theorem 2.2). *There exist $r_0 > 0$, $c > 0$ and $C > 0$ such that for all $r \in (0, r_0)$, $k \geq 1$ and all $\lambda \in U_{\varepsilon,k}$ with $\max(|\lambda|, \varepsilon^{-1}) < c\, r^{-\frac{1}{2}}$, one has*
$$\left\| \left(\lambda - r^{-\frac{k}{2}}\, \tilde{x}_{0,r}\right)^{-1} - \Pi_k\, (\lambda - y_k)^{-1}\, \Pi_k \right\| \leq C \sqrt{r}\, \varepsilon^{-1} \left(1 + \max(|\lambda|, \varepsilon^{-1})\right) ,$$
and for $r \in (0, r_0)$, $k = 0$ and $\lambda \in U_{\varepsilon,0}$ with $\varepsilon^{-1} < c\, r^{-\frac{1}{2}}$,
$$\left\| \left(\lambda - \tilde{x}_{0,r}\right)^{-1} - \left(\lambda - \tilde{x}_{0,0}\right)^{-1} \right\| \leq C \sqrt{r}\, \varepsilon^{-1} \left(1 + \varepsilon^{-1}\right) .$$

Let us fix constants $0 < c_1 < c_2 < \infty$ such that for all k,
$$\operatorname{Spec}(Y_k) \setminus \{0\} \subset i[c_1, c_2] \cup -i[c_1, c_2] .$$

Let $\Gamma_{r,t} = \Gamma^1_{r,t} \cup \Gamma^2_{r,t} \cup \Gamma^3_{r,t} \subset \mathbb{C}$ be the contour depicted in Figure 7.49.

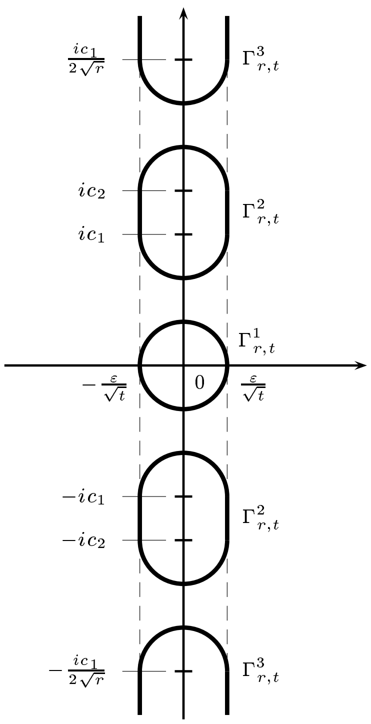


Figure 7.49. The contour $\Gamma_{r,t} = \Gamma^1_{r,t} \cup \Gamma^2_{r,t} \cup \Gamma^3_{r,t} \subset \mathbb{C}$.

We split the integral (7.45) as
$$\begin{aligned} \int_1^\infty \alpha_{r,t} &= \int_1^{r^{-\frac{1}{2}}} \alpha_{r,t} + \sum_{k=1}^{k_0-1} \int_{r^{-k+\frac{1}{2}}}^{r^{-k-\frac{1}{2}}} \alpha_{r,t} + \int_{r^{-k_0+\frac{1}{2}}}^\infty \alpha_{r,t} \\ &= \int_1^{r^{-\frac{1}{2}}} \alpha_{r,t} + \sum_{k=1}^{k_0-1} \int_{r^{\frac{1}{2}}}^{r^{-\frac{1}{2}}} \alpha_{r,r^{-k}t} + \int_{r^{\frac{1}{2}}}^\infty \alpha_{r,r^{-k_0}t} \,. \end{aligned}$$

For $k \geq 0$, $r \in (0, r_0)$ with r_0 sufficiently small and for $t \in \left(t^{\frac{r}{2}}, t^{-\frac{r}{2}}\right)$, we consider

$$\begin{aligned}\alpha'_{k,r,t} &:= \left(\operatorname{str}_V\left(N^V f'\big(\widetilde{X}_{r,r^{-k}t}\big)\right) - \chi'(E_{k+1})\right)\frac{dt}{2t}\\ &= \left(\operatorname{str}_V\left(N^V f'\left(t^{\frac{1}{2}} r^{-\frac{k}{2}} \tilde{x}_{0,r} + \tilde{x}_{1,r} + t^{-\frac{1}{2}} r^{\frac{k}{2}} \tilde{x}_{2,r} + \dots\right)\right) - \chi'(E_{k+1})\right)\frac{dt}{2t}\,,\end{aligned}$$

such that

$$\int_{r^{\frac{1}{2}}}^{r^{-\frac{1}{2}}} \alpha_{r,r^{-k}t} = \int_{r^{\frac{1}{2}}}^{r^{-\frac{1}{2}}} \alpha'_{k,r,t} - \big(\chi'(E_{k+1}) - \chi'(E_\infty)\big)\frac{\log r}{2}\,. \tag{7.50}$$

We will treat the integrals over $t \in \left[r^{\frac{1}{2}}, 1\right]$ and $t \in \left[1, r^{-\frac{1}{2}}\right]$ separately. For $t \in \left[1, r^{-\frac{1}{2}}\right]$ and $k \geq 0$, we write

$$\begin{aligned}f'\big(\widetilde{X}_{r,r^{-k}t}\big) = \frac{1}{2\pi i}\int_{\Gamma^1_{r,t}\cup\Gamma^2_{r,t}\cup\Gamma^3_{r,t}} &\frac{f'(\sqrt{t}\,\lambda)}{\lambda - r^{-\frac{k}{2}}\tilde{x}_{0,r}}\\ &\sum_{j=0}^{\infty}\left(t^{-\frac{1}{2}}\left(\tilde{x}_{1,r} + O\Big(r^{\frac{k}{2}} t^{-\frac{1}{2}}\Big)\right)\frac{1}{\lambda - r^{-\frac{k}{2}}\tilde{x}_{0,r}}\right)^j d\lambda\,.\end{aligned}$$

Let us note that because $\Gamma_{r,t}$ depends explicitly on t, the real part of $\sqrt{t}\,\lambda$ remains bounded by ε, so that $f\big(\sqrt{t}\,\lambda\big)$ remains uniformly bounded, too.

For $\lambda \in \Gamma^1_{r,t}$ and $k \geq 1$, Theorem 7.48 implies that

$$\left\|\frac{f'(\sqrt{t}\,\lambda)}{\lambda - r^{-\frac{k}{2}}\tilde{x}_{0,r}}\sum_{j=0}^{\infty}\left(t^{-\frac{1}{2}}\left(\tilde{x}_{1,r} + O\Big(r^{\frac{k}{2}}t^{-\frac{1}{2}}\Big)\right)\frac{1}{\lambda - r^{-\frac{k}{2}}\tilde{x}_{0,r}}\right)^j - \frac{f'(\sqrt{t}\,\lambda)}{\lambda - t^{-\frac{1}{2}}Y_{k,t}}\right\| \leq C\sqrt{r}\,t\,.$$

Integrating over the contour $\Gamma^1_{r,t}$ of length $O\big(t^{-\frac{1}{2}}\big)$ gives

$$\left\|\int_{\Gamma^1_{r,t}}\left(\frac{f'(\sqrt{t}\,\lambda)\,d\lambda}{\lambda - t^{-\frac{1}{2}}\widetilde{X}_{r,r^{-k}t}} - \frac{f'(\sqrt{t}\,\lambda)\,d\lambda}{\lambda - t^{-\frac{1}{2}}Y_{k,t}}\right)\right\| \leq C\sqrt{rt}\,.$$

For $t < r^{-\frac{1}{2}}$, we have $\sqrt{r} < t^{-1}$, so the integral over $t \in \left[1, r^{-\frac{1}{2}}\right]$ of the difference above converges. By [BL], we already know that

$$\int_1^\infty\left(\int_{\Gamma^1_{r,t}}\operatorname{str}_V\left(N^V\frac{f'(\sqrt{t}\,\lambda)\,d\lambda}{\lambda - t^{-\frac{1}{2}}Y_{k,t}}\right) - \chi'(E_{k+1})\right)\frac{dt}{2t}$$

converges, so we have uniform convergence for the contribution from $\Gamma^1_{r,t}$ to (7.50) for $t \in \left[1, r^{-\frac{1}{2}}\right]$.

The reasoning for the integral over $\Gamma^2_{r,t}$ is similar, but we use that $|\operatorname{Im}\lambda| > c > 0$, so $f\big(\sqrt{t}\,\lambda\big)$ decays rapidly as $t \to \infty$ uniformly on $\Gamma^2_{r,t}$. Finally the integral over $\Gamma^3_{r,t}$ vanishes as $r \to 0$ because $|\operatorname{Im}\lambda| > r^{-\frac{1}{2}}c$. This shows that

$$\lim_{r\to 0}\int_1^{r^{-\frac{1}{2}}}\alpha'_{k,r,t} = \int_1^\infty\Big(\operatorname{str}\big(N^V f'(Y_{k,t})\big) - \chi'(E_{k+1})\Big)\frac{dt}{2t}\,.$$

For $k=0$, we similarly recover the remaining contribution to $T(\nabla^{E_0}, D_0', g^{E_0})$ up to divergencies and correction terms, because

(7.51)

$$\begin{aligned}\lim_{r\to 0}\biggl(\int_1^{r^{-\frac12}} \alpha_{r,t} + \frac{\log r}{4}\,\bigl(\chi'(E_1)-\chi'(E_\infty)\bigr)\biggr) \\ = \int_1^\infty \Bigl(\operatorname{str}_V\Bigl(N^{E_0}\, f'\bigl(\widetilde X_{0,t}\bigr)\Bigr) - \chi'(E_1) - \bigl(\chi'(E_0)-\chi'(E_1)\bigr)\, f'\bigl(\sqrt{-t}/2\bigr)\Bigr)\,\frac{dt}{2t} \\ + \bigl(\chi'(E_0)-\chi'(E_1)\bigr)\int_1^\infty f'\bigl(\sqrt{-t}/2\bigr)\,\frac{dt}{2t}\,.\end{aligned}$$

We use the contour $\Gamma_{r,1}$ for all $t\in\bigl[r^{\frac12},1\bigr]$ and $k\geq 1$. Using Theorem 7.48 and the results of [BL], we can again prove uniform convergence, and we find that

$$\begin{aligned}\lim_{r\to 0}\int_{r^{\frac12}}^1 \Bigl(\operatorname{str}\Bigl(N^V\, f'\bigl(\widetilde X_{r,r^{-k}t}\bigr)\Bigr) - \chi'(E_k)\, f'\bigl(\sqrt{-t}/2\bigr)\Bigr)\,\frac{dt}{2t} \\ = \int_0^1 \Bigl(\operatorname{str}\Bigl(N^V\, f'\bigl(Y_{k,t}\bigr)\Bigr) - \chi'(E_k)\, f'\bigl(\sqrt{-t}/2\bigr)\Bigr)\,\frac{dt}{2t}\,.\end{aligned}$$

Thus for $k\geq 1$,

(7.52)

$$\begin{aligned}\lim_{r\to 0}\biggl(\int_{r^{\frac12}}^{r^{-\frac12}} \alpha_{r,r^{-k}t} + \frac{\log r}{4}\,\bigl(\chi'(E_k)-\chi'(E_{k+1})\bigr) + \frac{\log r}{2}\,\bigl(\chi'(E_{k+1})-\chi'(E_\infty)\bigr)\biggr) \\ = \int_0^\infty \Bigl(\operatorname{str}_V\bigl(N^{E_k}\, f'(Y_{k,t})\bigr) - \chi'(E_{k+1}) - \bigl(\chi'(E_k)-\chi'(E_{k+1})\bigr)\, f'\bigl(\sqrt{-t}/2\bigr)\Bigr)\,\frac{dt}{2t} \\ + \bigl(\chi'(E_k)-\chi'(E_{k+1})\bigr)\biggl(\int_1^\infty f'\bigl(\sqrt{-t}/2\bigr)\,\frac{dt}{2t} + \int_0^1 \Bigl(f'\bigl(\sqrt{-t}/2\bigr)-1\Bigr)\,\frac{dt}{2t}\biggr)\,.\end{aligned}$$

Modulo divergencies and correction terms in degree 0, we find $T^{\mathrm{BL}}(D_k', g^{E_k})$ in the limit.

Finally, we regard the last contribution

$$\int_{\sqrt r}^\infty \alpha_{r,r^{-k_0}t}\,.$$

Because $E_{k_0}=E_\infty$, we know that

$$\operatorname{Spec}\Bigl(r^{-\frac{k_0}{2}}\,\tilde x_{0,r}\Bigr) \subset \{0\}\cup i\Bigl(-\infty,-r^{-\frac12}\frac{c_1}{2}\Bigr)\cup i\Bigl(r^{-\frac12}\frac{c_1}{2},\infty\Bigr)\,.$$

For $\lambda\in\Gamma^1_{r,t}$, we therefore have

$$\Bigl\|\Bigl(\lambda - r^{-\frac{k_0}{2}}\,\tilde x_{0,r}\Bigr)^{-1} - \Pi_{\ker\tilde x_{0,r}}\,\lambda^{-1}\,\Pi_{\ker\tilde x_{0,r}}\Bigr\| \leq C\sqrt r$$

and

$$\Bigl\|\Bigl(\lambda - r^{-\frac{k_0}{2}}\,\widetilde X_{r,t}\Bigr)^{-1} - \Pi_{\ker\tilde x_{0,r}}\,\lambda^{-1}\,\Pi_{\ker\tilde x_{0,r}}\Bigr\| \leq C\sqrt{r/t}\,.$$

Since only integrals of $f'\big(\sqrt{t}\,\lambda\big)\,\big(\lambda - r^{-\frac{k_0}{2}}\widetilde{X}_{r,t}\big)^{-1}$ over $\Gamma^1_{r,t}$ and $\Gamma^3_{r,t}$ contribute to $f'\big(\widetilde{X}_{r,r^{-\frac{k_0}{2}}t}\big)$, we conclude that

$$\lim_{r\to 0}\int_{r^{\frac12}}^{\infty}\alpha_{r,r^{-k_0}t} = 0\;. \tag{7.53}$$

Combining (7.43), (7.44), (7.51), (7.52) and (7.53), we can compute the asymptotic behaviour of $T(\nabla^V, A', g^V_r)$ by

$$\begin{aligned}\lim_{r\to 0}\Biggl(T\big(\nabla^V, A', g^V_r\big) - \frac{\log r}{2}\sum_{k=1}^{k_0-1}\big(\chi(E_k)-\chi(E_\infty)\big)\Biggr)&\\ = T\big(\nabla^{E_0}, D'_0, g^{E_0}\big) + \sum_{k=1}^{k_0-1} T^{\mathrm{BL}}\big(D'_k, g^{E_k}\big)&\;.\end{aligned} \tag{7.54}$$

Moreover, we can extend both summations over k to infinity.

It remains to study the variation of $\widetilde{\mathrm{ch}}^{\mathrm{o}}(\nabla^H, g^H, g^H_{V,r})$ as $r\to 0$. Identifying H and E_∞ using g^H, we find

$$\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^H_{V,r}\big) = \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^{E_\infty}\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{E_\infty}, g^{E_\infty}, g^{E_\infty}_r\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^{E_\infty}_r, g^H_{V,r}\big)\;. \tag{7.55}$$

Clearly

$$\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{E_\infty}, g^{E_\infty}, g^{E_\infty}_r\big) = \frac{\log r}{2}\,\mathrm{str}_{E_\infty}(N^F)\;. \tag{7.56}$$

If we put $H_r = \ker a_{0,r}$ and $\widetilde{H} = \ker \tilde{a}_{0,r}$ and let ∇^{H_r} and $\nabla^{\tilde{H}_r}$ denote the connections induced by A' and $\widetilde{A}'_r$, then

$$\begin{aligned}\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^{E_\infty}_r, g^H_{V,r}\big) &= \widetilde{\mathrm{ch}}^{\mathrm{o}}\Big(\nabla^{H_r}, \big(r^{\frac{N^F}{2}}\,\Pi_{k_0}\,r^{-\frac{N^F}{2}}|_{H_r}\big)^* g^V_r, g^V_r|_{H_r}\Big)\\ &= \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{\tilde{H}_r}, (\Pi_{k_0}|_{\tilde{H}_r})^* g^V, g^V|_{\tilde{H}_r}\big)\end{aligned}$$

because $E_{k_0} = E_\infty$. On the other hand, let $\Pi_{\tilde{H}_r}$ denote the g^V-orthogonal projection onto $\widetilde{H}_r$. Then

$$\lim_{r\to 0}\Pi_{\tilde{H}_r} = \frac{1}{2\pi i}\lim_{r\to 0}\int_{\Gamma^1_{r,t}}\frac{d\lambda}{\lambda - r^{-\frac{k_0}{2}}\tilde{a}_{0,r}} = \Pi_{k_0}$$

in all C^k-norms, so

$$\lim_{r\to 0}\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^{E_\infty}_r, g^H_{V,r}\big) = \lim_{r\to 0}\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{\tilde{H}_r}, (\Pi_{k_0}|_{\tilde{H}_r})^* g^V, g^V|_{\tilde{H}_r}\big) = 0\;.$$

By (7.56) and (7.55),

$$\lim_{r\to 0}\Bigl(\widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^H_{V,r}\big) - \frac{\log r}{2}\,\mathrm{str}_{E_\infty}(N^F)\Bigr) = \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^{E_\infty}\big)\;. \tag{7.57}$$

Now it follows from (7.40), (7.54) and (7.57) that

$$\begin{aligned}&T\big(\nabla^V, A', g^V\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^H_V\big)\\ &\qquad= \lim_{r\to 0}\Bigl(T\big(\nabla^V, A', g^V_r\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^V, g^V_r, g^V\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^H_{V,r}\big)\Bigr)\\ &\qquad= T\big(\nabla^{E_0}, D'_0, g^{E_0}\big) + \sum_{k=1}^{\infty} T^{\mathrm{BL}}\big(D'_k, g^{E_k}\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^{E_\infty}\big)\;.\end{aligned}$$

This completes the proof of Theorem 7.37. $\square$

7.e. A proof of the Berthomieu-Bismut convergence results. This chapter should be considered an appendix containing the proofs of some results that are analogous to results proved in [BB]. In particular, we prove Theorem 7.48 for $k \geq 1$, however regarding the family of self-adjoint operators $\tilde{a}_{0,r}$ instead of $\tilde{x}_{0,r}$. The case $k = 0$ is an easy exercise that we leave to the reader.

7.58. Proposition ([BB], Theorem 6.1). *For each $r \geq 1$ and each $k \geq 1$, there are inclusions $E^{p,q}_{k,r} \subset E^{p,q}_0$ such that $E^{p,q}_{k,r} = \ker(d'_{k-1,r} + d''_{k-1,r}) \subset E^{p,q}_{k-1,r}$, and $E^{p,q}_{k,r}$ is given by*

$$(1) \quad E^{p,q}_{k,r} = \left\{ v_0 \in E^{p,q}_0 \;\middle|\; \text{there exist } v_1, \dots, v_{k-1} \text{ such that } \sum_{i=0}^{j} \tilde{a}^{(i)}_{0,r} v_{j-i} = 0 \text{ for } j = 0, \dots, k-1 \right\} .$$

Let $\Pi_{k,r}\colon E_0 \to E_{k,r}$ denote the orthogonal projection with respect to g. The operator $d_{k,r} = \frac{1}{2}(d'_{k,r} + d''_{k,r})$ acts on $E_{k,r}$ by

$$(2) \qquad d_{k,r} v_0 = \Pi_{k,r} \sum_{i=1}^{k} \tilde{a}^{(i)}_{0,r} v_{j-i} .$$

There exists a linear map $\varphi_k\colon E_{k,T} \to V^{k-1}$ such that for $v_0 \in E_{k,r}$ and $(v_1, \dots, v_{k-1}) = \varphi_k(v_0)$, one has

$$(3) \qquad \sum_{i=0}^{j} \tilde{a}'^{(i)}_{0,r}\, r^{\frac{j-i}{2}}\, v_{j-i} = 0 \qquad \text{for } j = 0, \dots, k-1 .$$

Proof. The proof is a simple adaptation of the proof given in [BB]. □

It follows from this theorem that for all k both the subspace $E_{k,r} \subset E_0$ and the operator

$$r^{-\frac{k}{2}}\, d_{k,r} \in \operatorname{End}\bigl(E_{k,r}\bigr)$$

are independent of r. Later on, we put

$$E_k = E_{k,r} , \qquad \Pi_k = \Pi_{k,r} \qquad \text{and} \qquad d_k = r^{-\frac{k}{2}}\, d_{k,r} \in \operatorname{End}\bigl(E_k\bigr) .$$

If $\varphi_k(v_0) = (v_1, \dots, v_{k-1})$, we define $\varphi_{k,r}$ by putting

$$\varphi_{k,r}(v_0) = \Bigl(r^{\frac{1}{2}} v_1, \dots, r^{\frac{k-1}{2}} v_{k-1}\Bigr) .$$

Next, we study the resolvent of $\tilde{a}_{0,r}$. We begin with an approximation. Because $(d'_{0,r})^2 = (d''_{0,r})^2 = 0$, the operators $\rho\, d'_{0,r} + \bar{\rho}\, d''_{0,r}$ have the same square for all $\rho \in S^1 \subset \mathbb{C}$ and thus the same spectrum. For this reason,

$$iU_{\varepsilon,k} = \bigl\{ z \in \mathbb{C} \bigm| d\bigl(z, \operatorname{Spec}(d_k)\bigr) > \varepsilon \bigr\} .$$

7.59. Proposition ([BB], Theorem 6.2). *For each $k \geq 0$ and each $\lambda \in iU_{\varepsilon,k}$, there exists a linear map*

$$\psi_{k,r,\lambda}\colon V \longrightarrow V^{k+1} \qquad \textit{with} \qquad v \longmapsto (v_{0,r}, \dots, v_{k,r})$$

such that

$$v_{0,r} = (\lambda - d_k)^{-1} \Pi_k v \in E_k \tag{1}$$

and

$$\sum_{i=0}^{j} \tilde{a}_{0,r}^{(j-i)} v_{i,r} = 0 \qquad \textit{for } j = 0, \dots, k-1, \textit{ and} \tag{2}$$

$$r^{-\frac{k}{2}} \sum_{i=0}^{k} \tilde{a}_{0,r}^{(k-i)} v_{i,r} = \lambda v_{0,r} - v\,. \tag{3}$$

Moreover there exists a constant $C > 0$ such that

$$|v_{i,r}| \leq C\, r^{\frac{i}{2}}\, \varepsilon^{-1}\, |v| \qquad \textit{for } i = 1, \dots, k\,. \tag{4}$$

Proof. As in the proof of Proposition 7.58, we first solve the equations (2) and

$$\Pi_{k-l,r}\left(\lambda v_{0,r} - \sum_{i=0}^{k} \tilde{a}_{0,r}^{(k-i)} v_{i,r}\right) = \Pi_{k-l,r} v \tag{2'}$$

with $v_{k,r} = 0$ by induction over $l = 0, \dots, k-1$, and finally give a solution for $l = k$, where (3') is equivalent to (3).

For $l = 0$, we have to find $v_{0,r} \in E_k$ with

$$\left(\lambda - r^{-\frac{k}{2}}\, d_{k,r}\right) v_{0,r} = \left(\lambda - d_k\right) v_{0,r} = \Pi_{k,r} v\,,$$

which gives (1). We put $(v_{1,r}, \dots, v_{k-1,r}) = \varphi_{k,r}(v_0)$. Then by Proposition 7.58, equations (2) and (3') hold for $l = 0$, and there exists a constant C such that

$$|v_{0,r}| \leq C\left(\varepsilon\,(1 + |\lambda|)\right)^{-1}\,|v| \qquad \text{and} \qquad |v_{i,r}| \leq C\, r^{\frac{i}{2}}\, \varepsilon^{-1}\,|v| \qquad \text{for } i = 1, \dots, k-1\,.$$

In the following steps, we have to choose $w_0 \in E_{k-l}$ such that

$$d_{k-l} w_0 = r^{\frac{l}{2}} \Pi_{k-l,r}\left(\lambda v_{0,r} - v - r^{-\frac{k}{2}} \sum_{i=0}^{k-1} \tilde{a}_{0,r}^{(k-i)} v_{i,r}\right) \quad \in \quad E'^{\perp}_{k-l+1} \subset E_{k-l}\,.$$

The right hand side is still of order $C\, r^{\frac{l}{2}}\, \varepsilon^{-1}\,|v|$. We have $\left|d_{k-l}^{-1}|_{E'^{\perp}_{k-l+1}}\right| \leq C'$, so that w_0 is of order $C\, C'\, r^{\frac{l}{2}}\, \varepsilon^{-1}\,|v|$. For definiteness, we may demand that $s_l + w_0 \in E'^{\perp}_{k-l+1}$ without changing the estimate above. We take $(w_1, \dots, w_{l-1}) = \varphi_{k-l,r}(w_0)$ and replace $(v_{0,r}, \dots, v_{k-1,r})$ by

$$\left(v_{0,r}, \dots, v_{l-1,r}, v_{l,r} + w_0, \dots, v_{k-1,r} + w_{l-1}\right)\,.$$

For this new tupel, (2) and (3') hold for l and (4) holds with a new constant C.

Finally for $k=l$, it remains to determine $v_{k,r} \in E_1'^{\perp}$ with

$$\tilde{a}_0^{(0)}\, v_{k,r} = r^{\frac{k}{2}} \left(\lambda v_{0,r} - v\right) - \sum_{i=0}^{k-1} \tilde{a}_{0,r}^{(k-i)} v_{i,r} \quad \in \quad E_1'^{\perp} \ .$$

This implies that (2) and (3) hold, and (4) holds with a new constant C, so we are done. □

If we put

$$\bar{\psi}_{k,r,\lambda} = v_{0,r} + \cdots + v_{k,r} \ ,$$

then Proposition 7.59 immediately implies

$$\left(\lambda - r^{-\frac{k}{2}}\, \tilde{a}_{0,r}\right) \bar{\psi}_{k,r,\lambda}(v) = v + \lambda \sum_{i=1}^{k} v_{i,k} - r^{-\frac{k}{2}} \sum_{j=k+1}^{\infty} \sum_{i=0}^{k} \tilde{a}_{0,r}^{(j-i)}\, v_{i,r} \ . \tag{7.60}$$

Note that the first sum is finite because our filtration F^pV has finite length. As in [BB], this gives the main result on the asymptotics of the operator $\tilde{a}_{0,r}$.

7.61. Theorem ([BB], Theorem 6.5). *There exist $r_0 > 0$, $c > 0$ and $C > 0$ such that for all $k \geq 1$, all $r \in (0, r_0)$ and all $\lambda \in iU_{\varepsilon,k}$ with $\max(|\lambda|, \varepsilon^{-1}) < c\, r^{-\frac{1}{2}}$, one has*

$$\left\| \left(\lambda - r^{-\frac{k}{2}}\, \tilde{a}_{0,r}\right)^{-1} - \Pi_k\, (\lambda - d_k)^{-1}\, \Pi_k \right\| \leq C\, \sqrt{r}\, \varepsilon^{-1} \left(1 + \max(|\lambda|, \varepsilon^{-1})\right) . \qquad \square$$

For $k = 0$, we may replace $\max(|\lambda|, \varepsilon^{-1})$ simply by ε^{-1} in the theorem. Theorem 7.48 follows from Theorem 7.61 by replacing $\tilde{a}_{0,r}$ by $\tilde{x}_{0,r}$ etc.

Proof. Write (7.60) as

$$\left(\lambda - r^{-\frac{k}{2}}\, \tilde{a}_{0,r}\right) \bar{\psi}_{k,r,\lambda}(v) = v + R_{k,r,\lambda}(v) \ ,$$

then Proposition 7.59 implies that the operator norms of $\psi_{k,r,\lambda}$ and $R_{k,r,\lambda}(v)$ satisfy the estimates

$$\left\| \psi_{k,r,\lambda} - \Pi_k\, (\lambda - d_k)^{-1}\, \Pi_k \right\| = O\big(\sqrt{r}/\varepsilon\big) \qquad \text{and} \qquad \|R_{k,r,\lambda}\| = O\big(\sqrt{r}\, \max(|\lambda|, \varepsilon^{-1})\big) \ .$$

The "max" comes from the fact that the ε^{-1} in Proposition 7.59 becomes relevant only if $|\lambda|$ is small (for large $|\lambda|$, we may may choose a large value of ε). Let us fix $c > 0$ and $r_0 > 0$ suitably small and let us assume that $r < r_0$ and $\max(|\lambda|, \varepsilon^{-1}) \leq c\, r^{-\frac{1}{2}}$. Then (7.60) gives us a series converging to the resolvent of $r^{\frac{k}{2}}\, \tilde{a}_{0,r}$, and we estimate

$$\begin{aligned} \left(\lambda - r^{-\frac{k}{2}}\, \tilde{a}_{0,r}\right)^{-1} &= \sum_{k=0}^{\infty} \psi_{k,r,\lambda} \left(-R_{k,r,\lambda}\right)^k \\ &= \Big(\Pi_k\, (\lambda - d_k)^{-1}\, \Pi_k + O\big(\sqrt{r}/\varepsilon\big)\Big) \Big(1 + O\big(\sqrt{r}\, \max(|\lambda|, \varepsilon^{-1})\big)\Big) \ . \qquad \square \end{aligned}$$

As in [BB], this implies that for r sufficiently small, the operator $\tilde{a}_0$ has eigenvalues of order $r^{\frac{k}{2}}$ that correspond precisely to the eigenvalues of d_k. The condition on $|\lambda|$ guarantees that λ stays away from the eigenvalues of $r^{-\frac{k}{2}}\tilde{a}_{0,r}$ corresponding to those of $r^{\frac{k'-k}{2}}\, d_{k'}$ with $k' < k$ when taking the resolvent. Similarly, the restriction to ε^{-1} guarantees that λ avoids the eigenvalues of $r^{-\frac{k}{2}}\tilde{a}_{0,r}$ corresponding to those of $r^{\frac{k'-k}{2}}\, d_{k'}$ with $k' > k$.

8. Igusa's higher Franz-Reidemeister torsion

In this chapter, we state and prove Theorem 8.1, by which Igusa's higher Franz-Reidemeister torsion coincides with the higher torsion constructed in Definition 7.29.

Using Igusa's two-index theorem, the proof of Theorem 8.1 is reduced to the case of a family Thom-Smale complex $V = V^0 \oplus V^1$ living in only two degrees. We also give a sketch of the proof in Section 8.a.

We construct a family of flat superconnections from a given flat superconnection on a bundle $V = V^0 \oplus V^1$ in Section 8.b. This family is used to decompose $T(\nabla^V, A', g^V)$ into four terms in Section 8.c.

One of these terms is a Bismut-Lott torsion class, and we show in Section 8.d that it is equal to Igusa's class F in [I2]. The remaining three terms are polynomials in the components of A', and their sum forms an L-class à la [BG], cf. Section 2.e. We identify these terms with Igusa's classes G, K and $\overline{K}^*$ in Subsections 8.e and f.

8.a. Igusa's two-index theorem and a proof of Theorem 8.1. We formulate the main result of this section and reduce its proof to the special case of a family Thom-Smale complex living in only two degrees.

Let $\tau = \tau(\mathcal{M}_r(\mathbb{C}), U(r))$ and $T = T(\mathcal{M}_r(\mathbb{C}), U(r)) \in H^{\text{even}}\big(Wh^h(\mathcal{M}_r(\mathbb{C}), U(r))\big)$ denote Igusa's universal higher Franz-Reidemeister torsion, and the class constructed in Definition 7.29 above.

8.1. Theorem. *The classes τ and T in $H^{\text{even}}\big(Wh^h(\mathcal{M}_r(\mathbb{C}), U(r))\big)$ coincide.*

A proof of this theorem is given at the end of this subsection.

8.2. Definition. Let $\mathcal{W}h^{\text{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$ be the full simplicial subcategory of the differential Whitehead category $\mathcal{W}h^{\text{diff}}(\mathcal{M}_r(\mathbb{C}), U(r))$ consisting in degree k of all objects (P, a') such that $P = P^0 \cup P^1$ and such that $a_0'\colon (\mathbb{C}^r)^{P^0} \to (\mathbb{C}^r)^{P^1}$ is invertible at one (and thus at all) points of Δ_k. Let the acyclic differential two-index Whitehead space $Wh^{\text{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$ be the geometric realisation of $\mathcal{W}h^{\text{diff}}(\mathcal{M}_r(\mathbb{C}), U(r))$.

8.3. Remark. We can also construct simplicial and mixed models $Wh^{\text{simp},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$ and $Wh^{\text{mix},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$ of the acyclic two-index Whitehead space as in Section 6. However, the proof of Theorem 6.31 shows that these spaces are homotopy equivalent.

8.4. Fact (Two-Index Theorem). *The acyclic two-index Whitehead space $Wh^{h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$ is a deformation retract of the full acyclic Whitehead space $Wh^h(\mathcal{M}_r(\mathbb{C}), U(r))$.*

This implies that it is sufficient to prove Theorem 8.1 for the restrictions of τ and T to the acyclic two-index Whitehead space $Wh^{h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$.

Proof. of Theorem 8.1. By the two-index theorem, it is enough to check the equality of τ and T on $Wh^{h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$. All constructions in this and the following subsections will be natural with respect to pull-back. Thus, me may and we will replace $Wh^{h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$ for notational convenience by an arbitrary manifold B equipped with a family Thom Smale complex of the special form

$$(V, \nabla^V, A') = \left(V^0 \oplus V^1, \nabla^{V^0} \oplus \nabla^{V^1}, \nabla^{V^0} \oplus \nabla^{V^1} + a_0 + a_1^0 + a_1^1 + a_2\right), \tag{8.5}$$

where a_0, a_1^0, a_1^1 and a_2 are the components of $A' - \nabla^V$. We assume that V carries an adapted metric g^V as in Definition 2.37 that is parallel. We will forget about the family of endomorphisms h^V;

we will however remember that the coefficients of $A' - \nabla^V$ take values in a nilpotent subalgebra of $\operatorname{End} V$ in a neighbourhood of each point of B, cf. Remark 7.28 (3) above.

There is a natural construction of Igusa's higher Franz-Reidemeister torsion $\tau = \tau(\nabla^V, A', g^V)$ in this situation, which we will introduce in Subsections 8.d–f. In fact, it follows from Proposition 8.18 and Lemmata 8.38, 8.48 and 8.72 that

$$\begin{aligned} T(\nabla^V, A', g^V)^{[\geq 2]} &= \big(F + G + K - \overline{K}^*\big)^{[\geq 2]} + d\Big(\widetilde{T} + \widetilde{G} + \widetilde{K} - \overline{\widetilde{K}}^*\Big) \\ &= \tau(\nabla^V, A', g^V)^{[\geq 2]} + d\Big(\widetilde{T} + \widetilde{G} + \widetilde{K} - \overline{\widetilde{K}}^*\Big) \end{aligned}$$

for certain naturally defined forms $\widetilde{T}$, $\widetilde{G}$ and $\widetilde{K}$ depending only on ∇^V, A', and g^V. This implies that the cochains obtained by integrating $\tau(\nabla^{\mathcal{V}}, \mathcal{A}', g^{\mathcal{V}})^{[\geq 2]}$ and $T(\nabla^{\mathcal{V}}, \mathcal{A}', g^{\mathcal{V}})^{[\geq 2]}$ over the simplices of $\big|\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))\big|$ differ by an exact cochain. On the other hand, we know by [I2] and Theorem 2.48 that both $\tau(\nabla^{\mathcal{V}}, \mathcal{A}', g^{\mathcal{V}})^{[0]}$ and $T(\nabla^{\mathcal{V}}, \mathcal{A}', g^{\mathcal{V}})^{[0]}$ coincide with the classical fibre-wise Franz-Reidemeister torsion of the complex (V, a_0). Because both τ and T are even classes, this proves Theorem 8.1. $\square$

8.b. A family of flat superconnections. We now restrict our attention to family Thom-Smale complexes of the form (8.5).

Let us start by classifying all flat superconnections of total degree one with a prescribed term in degree zero on a vector bundle $V = V^0 \oplus V^1$. If ∇^{V^0} and ∇^{V^1} are connections on V^0 and V^1 and $a_0 \colon V^0 \to V^1$ is an isomorphism, we define new connections

$$a_0^{\#}\nabla^{V^1} = -a_0^{-1}\,\nabla^{V^1}\,a_0 \qquad \text{and} \qquad a_{0\#}\nabla^{V^0} = -a_0\,\nabla^{V^0}\,a_0^{-1}\,. \tag{8.6}$$

By our sign convention, these are indeed connections, with

$$\big(a_0^{\#}\nabla^{V^1}\big)_v = a_0^{-1}\,\nabla^{V^1}_v\,a_0 \qquad \text{and} \qquad \big(a_{0\#}\nabla^{V^0}\big)_v = a_0\,\nabla^{V^0}_v\,a_0^{-1}$$

for all $v \in TB$.

8.7. Lemma. *Let $V = V^0 \oplus V^1 \to B$ be a $\mathbb{Z}$-graded vector bundle, let $a_0 \colon V^0 \to V^1$ be a vector bundle isomorphism, and let ∇^1 be a connection on V^1. Then*

$$A' = a_0 + a_0^{\#}\nabla^1 \oplus \nabla^1 - a_0^{-1}\,(\nabla^1)^2$$

defines a flat superconnection of total degree 1 on V. Moreover, all flat superconnections of total degree 1 on V with a given invertible a_0 arise this way.

Proof. Let $A' = a_0 + \nabla^0 \oplus \nabla^1 + a_2$ be a superconnection of total degree 1. Flatness of A' is equivalent to the following three conditions:

$$a_0\,\nabla^0 + \nabla^1\,a_0 = 0\,, \tag{1}$$

$$a_2\,a_0 + (\nabla^0)^2 = 0 \qquad \text{and} \qquad a_0\,a_2 + (\nabla^1)^2 = 0\,, \tag{2}$$

$$\text{and} \qquad \nabla^0\,a_2 + a_2\,\nabla^1 = 0\,. \tag{3}$$

We assume that a_0 is invertible. Then equation (1) is equivalent to

$$\nabla^0 = a_0^{\#}\nabla^1\,. \tag{1'}$$

Given (1) or (1′), we have $a_0\,(\nabla^0)^2 = (\nabla^1)^2\,a_0$, so equation (2) is equivalent to

$$a_2 = -a_0^{-1}\,(\nabla^1)^2\,. \tag{2′}$$

Finally if (1) or (1′) and (2) or (2′) hold, then (3) follows from the Bianchi identity for ∇^1, because

$$\nabla^0\,a_2 + a_2\,\nabla^1 = -a_0^{-1}\left[\nabla^1,(\nabla^1)^2\right] = 0\,. \qquad \square$$

In order to compare Igusa's torsion class in the Chern normalisation with $T(\nabla^V, A', g^V)$, we consider a family of flat superconnections linking A' to a "trivial" flat superconnection. Consider the bundle

$$\overline{V} = V \times [0,1] \quad\longrightarrow\quad \overline{B} = B \times [0,1]\,,$$

and extend the connection $\nabla^V = \nabla^{V^0} \oplus \nabla^{V^1}$ and the superconnection A' trivially to $\overline{V}$. Let r be the coordinate of $[0,1]$, and suppose that the original superconnection takes the form

$$A' = \nabla^V + a_0 + a_1^0 + a_1^1 + a_2$$

with an invertible a_0. We define a new superconnection on $\overline{V}$ by

$$\overline{A}' = a_0 + a_0^{\#}\big(\nabla^{\overline{V}^1} + r\,a_1^1\big) \oplus \big(\nabla^{\overline{V}^1} + r\,a_1^1\big) - a_0^{-1}\big(\nabla^{\overline{V}^1} + r\,a_1^1\big)^2\,. \tag{8.8}$$

8.9. Lemma. *The superconnection $\overline{A}'$ is flat, and its restriction to $r=1$ equals A'.*

Proof. The first statement is a consequence of Lemma 8.7. The second statement also follows easily from Lemma 8.7, because in both cases it suffices to compare a_0 and the connection acting on V^1. $\square$

8.10. Remark. Note that the coefficients of $\overline{A}'|_{r=1} - \nabla^{\overline{V}} = A' - \nabla^V$ take values in a nilpotent subalgebra of $\operatorname{End} V$ by assumption. Similarly, we have

$$\overline{A}'|_{r=0} - a_0^{\#}\nabla^{V^1} \oplus \nabla^{V^1} = a_0 \quad\in\quad \operatorname{Hom}(V^0,V^1) \subset \operatorname{End} V\,,$$

and $\operatorname{Hom}(V^0,V^1)$ is of course a nilpotent subalgebra. Thus it makes sense to talk about the higher torsions

$$T\Big(a_0^{\#}\nabla^{V^1} \oplus \nabla^{V^1}, \overline{A}'|_{r=0}, g^V\Big) = T^{\mathrm{BL}}\Big(\overline{A}'|_{r=0}, g^V\Big) \qquad\text{and}\qquad T\Big(\nabla^V, \overline{A}'|_{r=1}, g^V\Big)\,. \tag{8.11}$$

Note however that in general there will be no meaningful torsion of $A'|_r$ for some $r \in (0,1)$. Even worse, in general there will be no flat connection on $\overline{V}$ which restricts to $a_0^{\#}\nabla^{V^1} \oplus \nabla^{V^1}$ at $r=0$ and to ∇^V at $r=1$.

Let $\overline{A}''$ be the adjoint of $\overline{A}'$ with respect to g^V. Since we assume that g^V is unitarily flat with respect to ∇^V, we have $\nabla^{V^1} = \nabla^{V^1,*}$. We find that

$$A'' = a_0^* + a_{0\#}^*\big(\nabla^{V^1} + r a_1^{1*}\big) \oplus \big(\nabla^{V^1} + r a_1^{1*}\big) - \big(\nabla^{V^1} + r a_1^{1*}\big)^2\,(a_0^*)^{-1}\,,$$

where we agree that $a_1^{1*}(v) = -(a_1^1(v))$ for $v \in TB$.

We extend the restrictions to the hyperplane $T = 1$ of the superconnections ${}^1\widetilde{A}$, ${}^2\widetilde{A}$ and ${}^3\widetilde{A}$ of (7.4), (7.13) und (7.19) to the bundles

$$\begin{aligned}
&{}^1\overline{V} \longrightarrow {}^1\overline{B} = B \times [0,1]^2 \times [1,\infty) &&\text{with coordinates } (r,s,t) \in [0,1]^2 \times [1,\infty)\ ,\\
&{}^2\overline{V} \longrightarrow {}^2\overline{B} = B \times [0,1]^3 &&\text{with coordinates } (r,s,u) \in [0,1]^3\ ,\\
\text{and}\quad &{}^3\overline{V} \longrightarrow {}^3\overline{B} = B \times [0,1]^3 &&\text{with coordinates } (r,s,v) \in [0,1]^3\ .
\end{aligned}$$

Therefore, set

$$\begin{aligned}
{}^1\widetilde{A} = {}&\frac{\sqrt{t}}{2}\left(a_0 + a_0^*\right) + (1-s)\, a_0^{\#}\big(\nabla^{\overline{V}^1} + r a_1^1\big) \oplus \big(\nabla^{\overline{V}^1} + r a_1^1\big)\\
&+ s\, a_{0\#}^*\big(\nabla^{\overline{V}^1} + r a_1^{1*}\big) \oplus \big(\nabla^{\overline{V}^1} + r a_1^{1*}\big)\\
&- \frac{2}{\sqrt{t}}\Big((1-s)^2\, a_0^{-1}\big(\nabla^{\overline{V}^1} + r a_1^1\big)^2 + s\,\big(\nabla^{\overline{V}^1} + r a_1^{1*}\big)^2 (a_0^*)^{-1}\Big)\\
&- 2N^V\, ds + (2s-1)\,\frac{N^V}{2t}\, dt\ ,
\end{aligned} \tag{8.12}$$

$$\begin{aligned}
{}^2\widetilde{A} = {}&\frac{1}{2}\left((1-s+us)\, a_0 + (u+s-us)\, a_0^*\right)\\
&+ (1-s)\, a_0^{\#}\big(\nabla^{\overline{V}^1} + r a_1^1\big) \oplus \big(\nabla^{\overline{V}^1} + r a_1^1\big)\\
&+ s\, a_{0\#}^*\big(\nabla^{\overline{V}^1} + r a_1^{1*}\big) \oplus \big(\nabla^{\overline{V}^1} + r a_1^{1*}\big)\\
&- 2\,(1-us)\,(1-s)\, a_0^{-1}\big(\nabla^{\overline{V}^1} + r a_1^1\big)^2\\
&- 2\,(1-u+us)\, s\,\big(\nabla^{\overline{V}^1} + r a_1^{1*}\big)^2 (a_0^*)^{-1} - 2u\, N^V\, ds\ ,
\end{aligned} \tag{8.13}$$

$$\begin{aligned}
{}^3\widetilde{A} = {}&\frac{v}{2}\left((1-s)\, a_0 + s\, a_0^*\right)\\
&+ (1-s)\Big((1-r)\, a_0^{\#}\nabla^{\overline{V}^1} + r\,\big(\nabla^{\overline{V}^0} + v\, a_1^0\big)\Big) \oplus \big(\nabla^{\overline{V}^1} + rv a_1^1\big)\\
&+ s\,\Big((1-r)\, a_{0\#}^*\nabla^{\overline{V}^1}\big) + r\,\big(\nabla^{\overline{V}^0} + v\, a_1^{0*}\big)\Big) \oplus \big(\nabla^{\overline{V}^1} + rv a_1^{1*}\big)\\
&- 2\,(1-s) v\, a_0^{-1}\big(\nabla^{\overline{V}^1} + r a_1^1\big)^2 - 2\, sv\,\big(\nabla^{\overline{V}^1} + r a_1^{1*}\big)^2 (a_0^*)^{-1}\ .
\end{aligned} \tag{8.14}$$

We state some elementary properties of these superconnections.

8.15. Proposition. *The superconnections ${}^1\widetilde{A}$, ${}^2\widetilde{A}$ and ${}^3\widetilde{A}$ above satisfy the following properties.*

1. *We have ${}^1\widetilde{A}|_{t=1} = {}^2\widetilde{A}|_{u=1}$ and ${}^2\widetilde{A}|_{u=0} = {}^3\widetilde{A}|_{v=1}$.*
2. *The zero degree term of ${}^1\widetilde{A}^2$ with respect to N^B is given by $\frac{t}{4}\,[a_0, a_0^*]$.*
3. *The zero degree terms of ${}^3\widetilde{A}|_{s=0}$, ${}^3\widetilde{A}|_{s=1}$ and ${}^3\widetilde{A}|_{v=0}$ are nilpotent.*
4. *The curvatures of ${}^1\widetilde{A}|_{s=0}$, ${}^1\widetilde{A}|_{s=1}$ and ${}^2\widetilde{A}|_{s=0}$, ${}^2\widetilde{A}|_{s=1}$ do not contain the exterior variable dr.*

Proof. Claims (1), (2) and (3) are immediate from the definitions and Lemma 8.7.

To prove (4), note that at $s = 0$, we have

$$\begin{aligned}
{}^1\widetilde{A}^2|_{s=0} &= \left[\left(\frac{t}{4}\right)^{\frac{N^V}{2}} A' \left(\frac{t}{4}\right)^{-\frac{N^V}{2}},\ \frac{\sqrt{t}}{2}\, a_0^*\right],\\
\text{and}\qquad {}^2\widetilde{A}^2|_{s=0} &= \left[2^{-N^V} A'\, 2^{N^V},\ \frac{u}{2}\, a_0^*\right].
\end{aligned}$$

The only term that could involve dr is in both cases a multiple of

$$\left[a_0^{-1}\, a_1^1\, dr\,,\ a_0^*\right]\,.$$

But the only nozero components of $a_0^{-1} a_1^1\, dr$ and a_0^* map V^1 to V^0, so the supercommutator vanishes. Clearly, a similar statement holds at $s=1$. $\square$

8.c. A comparison of two finite-dimensional higher torsions. In this subsection, we describe how we can use the above superconnections to construct and compare the two higher torsions of (8.11) in Proposition 8.18. This allows us to compare the definitions of Section 2 with those of Bismut and Lott in [BL] in Remark 8.19 (2). It also gives an expression for $T(\nabla^V, A', g^V)$ as a sum of four terms. Our strategy in the proof of Theorem 8.1 will be a comparison of each of these terms with one term in Igusa's definition of τ, which will take the remainder of this section.

We start by constructing some elements in $\Omega^*(B;\operatorname{End} V^0)$ from the coefficient of the original superconnections A' and its dual. These elements are used by Igusa to construct certain correction terms. If we write $A' = a_0 + (\nabla^{V^0} + a_1^0) \oplus (\nabla^{V^1} + a_1^1) + a_2$ as before, we set

$$\begin{aligned}
&(8.16) && x = a_0^{-1}\left[\nabla^V, a_0\right]\,, && y = -a_1^0\,, && z = -a_2 a_0 = \left(\nabla^{V^0} + a_1^0\right)^2\,,\\
&\text{and} && x^* = \left[\nabla^V, a_0^*\right](a_0^*)^{-1}\,, && y^* = -a_1^{0*}\,, && z^* = -a_0^* a_2^* = \left(\nabla^{V^1} + a_1^{0^*}\right)^2\,.
\end{aligned}$$

Here we have used that ∇^V was assumed to be parallel with respect to g^V. Note that x, y, z above correspond up to sign to the $\operatorname{End} V^0$-valued forms X, V, W in [I2], Section 2.4, whereas x^* above corresponds to Y in [I2]. Igusa's U can be expressed in terms of X, V, W and a_0.

Using flatness of A' and the Bianchi-Identity, we can also compute the derivatives of the forms x, y, z. With our sign convention, we find

$$\begin{aligned}
(8.17)\qquad && \left[\nabla^{V^0}, x\right] &= a_0^{-1}\left[\nabla^V a_0\right] a_0^{-1}\left[\nabla^V, a_0\right] = x^2\,,\\
&& \left[\nabla^{V^0}, y\right] &= -\left(\nabla^{V^0} - y\right)^2 + y^2 = y^2 - z\,,\\
\text{and} && \left[\nabla^{V^0}, z\right] &= \left[\nabla^{V^0} - y, \left(\nabla^{V^0} - y\right)^2\right] + \left[y, \left(\nabla^{V^0} - y\right)^2\right] = [y,z]\,.
\end{aligned}$$

The derivatives of the adjoint forms x^*, y^*, z^* are given by similar formulas.

With these preliminaries, we can give yet another expression of the higher Chern analytic torsion of Definition 2.91 and Theorem 7.23. We restrict our attention to classes of forms on B in degree ≥ 2, because we want to ignore all the correction terms in Theorem 7.23.

8.18. Proposition. *The higher Bismut-Lott torsion of $\overline{A}'|_{r=0} = a_0 + a_0^{\#}\nabla^{V^1} \oplus \nabla^{V^1}$ is up to exact forms given by*

$$\begin{aligned}
(1)\qquad T^{\mathrm{BL}}\left(\overline{A}'|_{r=0}, g^V\right)^{[\geq 2]} &= -(2\pi i)^{-\frac{N^B}{2}} \int_{(s,t)\in[0,1]\times[1,\infty)} \frac{1}{2}\operatorname{str}_V\left(e^{-\,{}^1\widetilde{A}^2|_{r=0}}\right)^{[\geq 2]}\\
&\quad - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,u)\in[0,1]^2} \frac{1}{2}\operatorname{str}_V\left(e^{-\,{}^2\widetilde{A}^2|_{r=0}}\right)^{[\geq 2]}\\
&\quad - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,v)\in[0,1]^2} \frac{1}{2}\operatorname{str}_V\left(e^{-\,{}^3\widetilde{A}^2|_{r=0}}\right)^{[\geq 2]}\,.
\end{aligned}$$

The higher torsion of $\overline{A}'|_{r=1} = A'$ *is up to exact forms given by*

$$(2)\qquad T\big(\nabla^V, A', g^V\big)^{[\geq 2]} = -(2\pi i)^{-\frac{N^B}{2}} \int_{(s,t)\in[0,1]\times[1,\infty)} \frac{1}{2}\,\mathrm{str}_V\Big(e^{-\,{}^1\widetilde{A}^2|_{r=1}}\Big)^{[\geq 2]} - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,u)\in[0,1]^2} \frac{1}{2}\,\mathrm{str}_V\Big(e^{-\,{}^2\widetilde{A}^2|_{r=1}}\Big)^{[\geq 2]} - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,v)\in[0,1]^2} \frac{1}{2}\,\mathrm{str}_V\Big(e^{-\,{}^3\widetilde{A}^2|_{r=1}}\Big)^{[\geq 2]} .$$

There is a naturally defined form $\widetilde{T}$ *such that*

$$(3)\qquad T^{\mathrm{BL}}\big(\overline{A}'|_{r=0}, g^V\big)^{[\geq 2]} - T\big(\nabla^V, A', g^V\big)^{[\geq 2]} = \big(G' + K' - \overline{K'}^*\big)^{[\geq 2]} + d\widetilde{T} ,$$

$$\text{with}\qquad G' = (2\pi i)^{-\frac{N^B}{2}} \int_{(r,s)\in[0,1]^2} \frac{1}{2}\,\mathrm{str}_V\Big(e^{-\,{}^3\widetilde{A}^2|_{v=0}}\Big) ,$$

$$\text{and}\qquad K' = -(2\pi i)^{-\frac{N^B}{2}} \int_{(r,v)\in[0,1]^2} \frac{1}{2}\,\mathrm{str}_V\Big(e^{-\,{}^3\widetilde{A}^2|_{s=0}}\Big) ,$$

where $\overline{K'}^*$ *is constructed as in (2.5).*

The forms G', K' *and* $\overline{K'}^* \in \Omega^*(B)$ *are even. The form* G'_{2k} *is the supertrace of a polynomial in* x *and* x^* *for each* k, *and*

$$(4)\qquad dG' = \mathrm{ch}^{\mathrm{o}}\big(a_0^{\#}\nabla^{V^0}, g^{V^0}\big) - (2\pi i)^{\frac{1-N^B}{2}} \int_0^1 \frac{1}{2}\,\mathrm{tr}_{V^0}\Big(x\, e^{r(1-r)\,x^2}\Big)\,dr + (2\pi i)^{\frac{1-N^B}{2}} \int_0^1 \frac{1}{2}\,\mathrm{tr}_{V^0}\Big(x^*\, e^{r(1-r)\,(x^*)^2}\Big)\,dr .$$

The form K'_{2k} *is the supertrace of a polynomial in* x, y, z, *and* w *for each* k, *and*

$$(5)\qquad dK' = (2\pi i)^{\frac{1-N^B}{2}} \int_0^1 \frac{1}{2}\,\mathrm{tr}_{V^0}\Big(x\, e^{r(1-r)\,x^2}\Big)\,dr .$$

The form $\overline{K'}^*_{2k}$ *is the supertrace of a polynomial in* x^*, y^*, z^*, *and* w^* *for each* k, *and*

$$(6)\qquad d\overline{K'}^* = (2\pi i)^{\frac{1-N^B}{2}} \int_0^1 \frac{1}{2}\,\mathrm{tr}_{V^0}\Big(x^*\, e^{r(1-r)\,(x^*)^2}\Big)\,dr .$$

Note that x, y, z do not a priori commute pairwise. Thus, in each monomial, the order of variables is important.

8.19. Remark.

(1) This proposition is the first step in a comparison of Igusa's torsion class τ with our T. The finite-dimensional Bismut-Lott torsion class $T^{\mathrm{BL}}(\overline{A}'|_{r=0}, g^V)$ corresponds to Igusa's class F, see Lemma 8.38 below. The class G' corresponds to the class G in [I2], Chapter 1, which is called K in [I2], Chapter 2, see Lemma 8.48 below. The class K' corresponds to L in [I2], see Lemma 8.72 below. We have chosen to use the letter K instead, so that we do not confuse the class L of [BG] and Definition 2.19 above with Igusa's L.

(2) We want to show that the classes G, K, and $\overline{K}^*$ together can be interpreted as an L-class in the sense of [BG], Section 2. Recall that we defined another superconnection ${}^3\widetilde{A}$ in (7.19), this fits with our present ${}^3\widetilde{A}$ along the common domain of definition $B \times \{1\}^2 \times [0,1]^2$, where $r = T = 1$ and the coordinates of $[0,1]^2$ are s and v. Let us define a family of flat superconnections $\left(\overline{A}'_\ell\right)_{\ell\in[0,1]}$ using (7.19) and (8.14) as

$$\overline{A}'_\ell = \begin{cases} {}^3\widetilde{A}|_{(r,s,v)=(0,0,3\ell)} & 0 \le \ell \le \frac{1}{3}\,, \\ {}^3\widetilde{A}|_{(r,s,v)=(3\ell-1,0,1)} & \frac{1}{3} \le \ell \le \frac{2}{3}\,, \\ {}^3\widetilde{A}|_{(s,T,v)=(0,3-3\ell,1)} & \frac{2}{3} \le \ell \le 1\,. \end{cases} \qquad \text{and}$$

Since the coefficient a_0 degenerates at $\ell = 0$ and at $\ell = 1$, the integral defining the L-class as in Definition 2.19 and Definition 2.91 diverges in degree 0. Still, we can apply Stokes' theorem to $B \times D_3$ and proceed as in the proof of Theorem 7.23 to find that

$$\begin{aligned}
&\left(G' + K' - \overline{K'}^*\right)^{[\ge 2]} \\
&\quad\equiv \frac{1}{2}\,(2\pi i)^{-\frac{N^B}{2}} \Biggl(-\int_{(s,v)\in[0,1]^2} \mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{r=0}}\Bigr) + \int_{(r,s)\in[0,1]^2} \mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{v=1}}\Bigr) \\
&\qquad\qquad + \int_{(s,v)\in[0,1]^2} \mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{r=1}}\Bigr)\Biggr)^{[\ge 2]} \\
&\quad\equiv \frac{1}{2}\,(2\pi i)^{-\frac{N^B}{2}} \Biggl(-\int_{(s,v)\in[0,1]^2} \mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{r=0}}\Bigr) + \int_{(r,s)\in[0,1]^2} \mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{v=1}}\Bigr) \\
&\qquad\qquad + \int_{(s,T)\in[0,1]^2} \mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{v=1}}\Bigr)\Biggr)^{[\ge 2]} \\
&\quad= -L_{\mathrm{ch}}\left(\overline{A}'_\ell, g^V\right)^{[\ge 2]}
\end{aligned}$$

modulo exact forms. This allows us to compare the finite-dimensional Bismut-Lott torsion classes with our torsion classes of Definition 2.46 and Remark 7.28 by

$$T\bigl(\nabla^V, A', g^V\bigr) = T^{\mathrm{BL}}\bigl(a_0 + a_0^{\#}\nabla^{V^1} \oplus \nabla^{V^1}, g^V\bigr) + L_{\mathrm{ch}}\left(\overline{A}'_\ell, g^V\right)^{[\ge 2]}\,. \tag{8.20}$$

Proof of Proposition 8.18. We recall that by Corollary 7.24 and Remark 8.10, we have

$$T^{\mathrm{BL}}\bigl(\overline{A}'|_{r=0}, g^V\bigr) = T\bigl(a_0^{\#}\nabla^{V^1} \oplus \nabla^{V^1}, \overline{A}'|_{r=0}, g^V\bigr)\,.$$

Now, our claims (1) and (2) are simple consequences of Theorem 7.23 and the definitions of ${}^1\widetilde{A}$, ${}^2\widetilde{A}$ and ${}^3\widetilde{A}$ in (8.12), (8.13) and (8.14).

To prove assertion (3), we apply Stokes theorem. We use (1) and (2) to find that

(8.21)
$$\begin{aligned}
&T\bigl(\nabla^V, A', g^V\bigr)^{[\ge 2]} - T^{\mathrm{BL}}\bigl(\overline{A}'|_{r=0}, g^V\bigr)^{[\ge 2]} \\
&\quad= -(2\pi i)^{-\frac{N^B}{2}} \lim_{T\to\infty} \int_{(s,t)\in[0,1]\times[1,T]} \frac{1}{2}\Bigl(\mathrm{str}_V\Bigl(e^{-{}^1\widetilde{A}^2|_{r=1}}\Bigr) - \mathrm{str}_V\Bigl(e^{-{}^1\widetilde{A}^2|_{r=0}}\Bigr)\Bigr) \\
&\qquad - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,u)\in[0,1]^2} \frac{1}{2}\Bigl(\mathrm{str}_V\Bigl(e^{-{}^2\widetilde{A}^2|_{r=1}}\Bigr) - \mathrm{str}_V\Bigl(e^{-{}^2\widetilde{A}^2|_{r=0}}\Bigr)\Bigr) \\
&\qquad - (2\pi i)^{-\frac{N^B}{2}} \int_{(s,v)\in[0,1]^2} \frac{1}{2}\Bigl(\mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{r=1}}\Bigr) - \mathrm{str}_V\Bigl(e^{-{}^3\widetilde{A}^2|_{r=0}}\Bigr)\Bigr)\,.
\end{aligned}$$

We now investigate the integrals over the remaining faces of $B\times\partial([0,1]^2\times[1,T])$ and $B\times\partial([0,1]^3)$, respectively. By Proposition 8.15 (1), we have

$$\begin{aligned}0&=\int_{(r,s)\in[0,1]^2}\frac{1}{2}\left(\operatorname{str}_V\left(e^{-{}^1\widetilde{A}^2|_{t=1}}\right)-\operatorname{str}_V\left(e^{-{}^2\widetilde{A}^2|_{u=1}}\right)\right)\\&=\int_{(r,s)\in[0,1]^2}\frac{1}{2}\left(\operatorname{str}_V\left(e^{-{}^2\widetilde{A}^2|_{u=0}}\right)-\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2|_{v=1}}\right)\right).\end{aligned}\tag{8.22}$$

By Proposition 8.15 (2),

$$0=\lim_{T\to\infty}\int_{(r,s)\in[0,1]^2}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^1\widetilde{A}^2|_{t=T}}\right)\tag{8.23}$$

because $[a_0,a_0^*]\in\operatorname{End}V$ is invertible. By Proposition 8.15 (4), we find

$$\begin{aligned}0&=\int_{(r,t)\in[0,1]\times[1,T]}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^1\widetilde{A}^2|_{s=0}}\right)=\int_{(r,t)\in[0,1]\times[1,T]}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^1\widetilde{A}^2|_{s=1}}\right)\\&=\int_{(r,u)\in[0,1]^2}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^2\widetilde{A}^2|_{s=0}}\right)=\int_{(r,u)\in[0,1]^2}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^2\widetilde{A}^2|_{s=1}}\right).\end{aligned}\tag{8.24}$$

By Stokes theorem and (8.21)–(8.24),

$$\begin{aligned}&T^{\mathrm{BL}}\big(\overline{A}'|_{r=0},g^V\big)^{[\geq2]}-T\big(\nabla^V,A',g^V\big)^{[\geq2]}\\&\quad=(2\pi i)^{-\frac{N^B}{2}}\int_{(r,s)\in[0,1]^2}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2|_{v=0}}\right)^{[\geq2]}\\&\qquad-(2\pi i)^{-\frac{N^B}{2}}\int_{(r,v)\in[0,1]^2}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2|_{s=0}}\right)^{[\geq2]}\\&\qquad+(2\pi i)^{-\frac{N^B}{2}}\int_{(r,v)\in[0,1]^2}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2|_{s=1}}\right)^{[\geq2]}.\end{aligned}\tag{3}$$

Because

$$\overline{K'}^*=-(2\pi i)^{-\frac{N^B}{2}}\int_{(r,v)\in[0,1]^2}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2|_{s=1}}\right),$$

this proves (3), with

$$\begin{aligned}\widetilde{T}&=(2\pi i)^{-\frac{N^B+1}{2}}\int_{(r,s,t)\in[0,1]^2\times[1,\infty)}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^1\widetilde{A}^2}\right)\\&\quad+(2\pi i)^{-\frac{N^B+1}{2}}\int_{(r,s,u)\in[0,1]^3}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^2\widetilde{A}^2}\right)\\&\quad+(2\pi i)^{-\frac{N^B+1}{2}}\int_{(r,s,v)\in[0,1]^3}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2}\right).\end{aligned}$$

We want to express G' in terms of x and x^*. The connection ${}^3\widetilde{A}|_{v=0}$ acts on $\overline{V}^1$ as $\nabla^{\overline{V}^1}$, and on $\overline{V}^0$ as

$$\begin{aligned}\big({}^3\widetilde{A}|_{v=0}\big)\big|_{\overline{V}^0}&=(1-r)\left((1-s)\,a_0^{\#}\nabla^{\overline{V}^1}+s\,a_{0\#}^*\nabla^{\overline{V}^1}\right)+r\,\nabla^{V^0}\\&=\nabla^{\overline{V}^0}-(1-r)\left((1-s)\,x+sx^*\right).\end{aligned}\tag{8.25}$$

Together with (8.17), we find that $({}^3\widetilde{A}|_{v=0})^2|_{\overline{V}^0}$ is an element of the subalgebra of $\Omega^*(B;\operatorname{End} V^0)$ generated by x, x^* and r, s, dr, ds. Because the term ${}^3\widetilde{A}^2|_{v=0}$ is of total degree 2, the form G'_k is the trace of a polynomial in x, x^* for each k.

By Stokes' theorem and because ${}^3\widetilde{A}|_{v=0}|_{\overline{V}^1} = \nabla^{\overline{V}^1}$ is flat,

$$dG' = (2\pi i)^{\frac{1-N^B}{2}} \int_{s=0}^1 \frac{1}{2} \operatorname{tr}_{V^0}\Bigl(e^{-\,{}^3\widetilde{A}^2|_{\substack{r=0\\v=0}}} - e^{-\,{}^3\widetilde{A}^2|_{\substack{r=1\\v=0}}}\Bigr) - (2\pi i)^{\frac{1-N^B}{2}} \int_{r=0}^1 \frac{1}{2} \operatorname{tr}_{V^0}\Bigl(e^{-\,{}^3\widetilde{A}^2|_{\substack{s=0\\v=0}}} - e^{-\,{}^3\widetilde{A}^2|_{\substack{s=1\\v=0}}}\Bigr)\;. \tag{8.26}$$

We analyse the right hand side term by term.

At $r=0$, we have the connection

$${}^3\widetilde{A}|_{\substack{r=0\\v=0}}|_{\overline{V}^0} = (1-s)\, a_0^{\#}\nabla^{\overline{V}^1} + s\, a_{0\#}^*\nabla^{\overline{V}^1}\;.$$

Note that $a_{0\#}^*\nabla^{\overline{V}^1}$ is the adjoint connection of $a_0^{\#}\nabla^{\overline{V}^1}$ wit respect to g^{V^0} by (8.6). By Definition 2.89, this implies that

$$(2\pi i)^{\frac{1-N^B}{2}} \int_{s=0}^1 \frac{1}{2} \operatorname{str}_V\Bigl(e^{-\,{}^3\widetilde{A}^2|_{\substack{r=0\\v=0}}}\Bigr) = \operatorname{ch}^o\bigl(a_0^{\#}\nabla^{V^1}, g^{V^0}\bigr)\;. \tag{8.27}$$

At $r=1$, the connection ${}^3\widetilde{A}|_{\substack{r=1\\v=0}} = \nabla^{\overline{V}}$ is flat, and we have

$$\int_{s=0}^1 \frac{1}{2} \operatorname{tr}_{V^0}\Bigl(e^{-\,{}^3\widetilde{A}^2|_{\substack{r=1\\v=0}}}\Bigr) = 0\;. \tag{8.28}$$

At $s=0$, we find the connection ${}^3\widetilde{A}|_{\substack{s=0\\v=0}}|_{\overline{V}^0} = \nabla^{\overline{V}^0} - (1-r)\,x$ with curvature given by

$${}^3\widetilde{A}^2|_{\substack{s=0\\v=0}} = -r(1-r)\,x^2 - x\,dr$$

by (7.5). Thus, we have

$$-(2\pi i)^{\frac{1-N^B}{2}} \int_{r=0}^1 \frac{1}{2} \operatorname{str}_V\Bigl(e^{-\,{}^3\widetilde{A}^2|_{\substack{s=0\\v=0}}}\Bigr) = -(2\pi i)^{\frac{1-N^B}{2}} \int_0^1 \frac{1}{2} \operatorname{tr}_{V^0}\Bigl(x\, e^{r(1-r)\,x^2}\Bigr)\, dr\;. \tag{8.29}$$

Passing to adjoints with respect to g^{V^1} proves the analogous equation

$$(2\pi i)^{\frac{1-N^B}{2}} \int_{r=0}^1 \frac{1}{2} \operatorname{str}_V\Bigl(e^{-\,{}^3\widetilde{A}^2|_{\substack{s=1\\v=0}}}\Bigr) = (2\pi i)^{\frac{1-N^B}{2}} \int_0^1 \frac{1}{2} \operatorname{tr}_{V^0}\Bigl(x^*\, e^{r(1-r)\,(x^*)^2}\Bigr)\, dr\;. \tag{8.30}$$

Now, statement (4) follows from (8.26)–(8.30).

The remaining statements are proven analogously. Since we wish to describe the superconnection ${}^3\widetilde{A}|_{s=0}$ and its curvature by elements of $\Omega^*(B;\operatorname{End} V^0)$, we use a_0 to identify V^1 with V^0, and continue to work with two copies of V^0. We may write ${}^3\widetilde{A}|_{s=0} - \nabla^{V^0} \oplus a_{0\#}\nabla^{V^0}$ as a 2×2-matrix with coefficients in $\Omega^*(B;V^1)$ by

$${}^3\widetilde{A}|_{s=0} = \begin{pmatrix} \nabla^{\overline{V}^0} - (1-r)\,x - rv\,y & -2v\,\bigl(\nabla^{\overline{V}^0} - (1-r)\,x - r\,y\bigr)^2 \\ \frac{v}{2} & -\nabla^{\overline{V}^0} + (1-rv)\,x + rv\,y \end{pmatrix}\;. \tag{8.31}$$

Recall that due to our sign convention, $a_0^{-1}\,\nabla^{V^1}\,a_0$ is the negative of a connection; this fact is responsible for the strange signs in the lower right corner. Together with (8.17), this shows that the curvature of ${}^3\widetilde{A}|_{s=0}$ is a 2×2-matrix with coefficients in the subalgebra of $\Omega^*(B;\operatorname{End}V^0)$ generated by x, y, z and r, v, dr, dv. Because the term ${}^3\widetilde{A}^2|_{v=0}$ of degree 0 with respect to N^B is nilpotent,

$$\left(\int_{(r,v)\in[0,1]^2}\operatorname{str}\left(e^{-{}^3\widetilde{A}^2|_{s=0}}\right)\right)^{[k]}$$

is the supertrace of a polynomial in x, y, z for each k.

We prove (5) using Stokes' theorem. We have

$$dK' = (2\pi i)^{\frac{1-N^B}{2}}\int_{r=0}^{1}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2|_{\substack{s=0\\v=0}}}-e^{-{}^3\widetilde{A}^2|_{\substack{s=0\\v=1}}}\right) - (2\pi i)^{\frac{1-N^B}{2}}\int_{s=0}^{1}\frac{1}{2}\operatorname{str}_V\left(e^{-{}^3\widetilde{A}^2|_{\substack{r=0\\s=0}}}-e^{-{}^3\widetilde{A}^2|_{\substack{r=1\\s=0}}}\right). \tag{8.32}$$

The contribution from the line with $s=0$ and $v=0$ has been given in (8.29). We will show that the three remaining faces of $\partial([0,1]^2)$ do not contribute to dK'.

At $v=1$, we have the flat superconnection

$$\begin{aligned}{}^3\widetilde{A}\Big|_{\substack{s=0\\v=1}} &= \frac{1}{2}\,a_0 + a_0^{\#}\left(\nabla^{\overline{V}^1}+r\,a_1^1\right)\oplus\left(\nabla^{\overline{V}^1}+r\,a_1^1\right) - 2a_0^{-1}\left(\nabla^{\overline{V}^1}+r\,a_1^1\right)\\ &= 2^{-\frac{N^V}{2}}\,\overline{A}'\,2^{\frac{N^V}{2}}\;,\end{aligned}$$

by (8.14), so this face does not contribute to dK'.

At $r=0$ and $r=1$, we get

$$\begin{aligned}{}^3\widetilde{A}\Big|_{\substack{r=0\\s=0}} &= \frac{v}{2}\,a_0 + a_0^{\#}\,\nabla^{\overline{V}^1}\oplus\nabla^{\overline{V}^1}\;,\\ \text{and}\qquad {}^3\widetilde{A}\Big|_{\substack{r=1\\s=0}} &= \frac{v}{2}\,a_0 + \left(\nabla^{\overline{V}^0}+v\,a_1^0\right)\oplus\left(\nabla^{\overline{V}^1}+v\,a_1^1\right)+2v\,a_2\;.\end{aligned}$$

The curvatures ${}^3\widetilde{A}^2\Big|_{\substack{r=0\\s=0}}$ and ${}^3\widetilde{A}^2\Big|_{\substack{r=1\\s=0}}$ are constructed only from a_0, a_1, and a_2, which belong to a nilpotent subalgebra of $\operatorname{End}V$. Thus, these two faces do not contribute to dK' either, and the only contribution to dK' stems from the face $v=0$, and was calculated (up to sign) in (8.30).

Because K' and $\overline{K'}^*$ are related by (2.5), the remaining claims about $\overline{K'}^*$ follow immediately. □

8.d. Igusa's classes I and F. We prove that Igusa's class I is a variant of the odd Chern class ch^{o}, and that by transgression, the class F equals the Bismut-Lott torsion class $T^{\mathrm{BL}}(A',g^V)$. This is the first step in the programme outlined in Remark 8.19 (1).

We start with the classes c_{2k+1} and I_{2k+1} defined in [I2], Section 1.2, but using the Chern normalisation of [BG]. Let $V=V^0\oplus V^1\to B$ be a $\mathbb{Z}$-graded complex vector bundle with $\operatorname{rk}V^0=\operatorname{rk}V^1=n$, with a flat connection $\nabla^V=\nabla^{V^0}\oplus\nabla^{V^1}$ and a parallel metric $g^V=g^{V^0}\oplus g^{V^1}$. Let $a_0\in\Gamma(B,\operatorname{Iso}(V^0,V^1)\subset\Omega^0(B;\operatorname{End}^1V)$ be a family of isomorphisms.

Igusa's form $I_{2k+1}\in\Omega^{2k+1}(B)$ is the real valued differential form given by

$$I_{2k+1}=\frac{k!}{2\,(2\pi i)^k\,(2k+1)!}\operatorname{tr}\Big(\big((a_0a_0^*)^{-1}\,[\nabla^1,a_0a_0^*]\big)^{2k+1}\Big)\;. \tag{8.33}$$

We put $I=\sum_{k=0}^{\infty}I_{2k+1}\in\Omega^{\mathrm{odd}}(B)$.

The form I is closed because its exterior differential is the supertrace of a non-zero even power of an odd operator. Note that in order to have the same normalisation as in [BG], we have changed the coefficients with respect to [I2]. Integrating I and $\mathrm{ch}^{\mathrm{o}}(a_0^{\#}\nabla^{\mathcal{V}^1},g^{\mathcal{V}_0})$ over I gives simplicial cochains on $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}),U(r))$ that we still denote by I and $\mathrm{ch}^{\mathrm{o}}(a_0^{\#}\nabla^{\mathcal{V}^1},g^{\mathcal{V}_0})$.

8.34. Lemma. *We have*

$$I=\operatorname{ch}^{\mathrm{o}}\bigl(a_0^{\#}\nabla^{V^1},g^{V^0}\bigr)\;.$$

In particular, the simplicial cocycles I and $\operatorname{ch}^{\mathrm{o}}(a_0^{\#}\nabla^{\mathcal{V}^1},g^{\mathcal{V}_0})$ agree on $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}),U(r))$.

Proof. Because ∇^{V^1} is selfadjoint with respect to g^{V^1}, the adjoint connection of $a_0^{\#}\nabla^{V^1}$ with respect to g^{V^0} is given by $a_{0\#}^{*}\nabla^{V^1}$, cf. (8.6). This implies that

$$\bigl(a_0^{\#}\nabla^{V^1}\bigr)^*-a_0^{\#}\nabla^{V^1}=a_0^*\left(a_0a_0^*\right)^{-1}\left[\nabla^{V^1},a_0a_0^*\right]\left(a_0^*\right)^{-1}\;.$$

Our claims follow because by (8.27), because

$$\begin{aligned}I&=\int_0^1\operatorname{tr}_{V^1}\left(\frac{(a_0a_0^*)^{-1}\left[\nabla^{V^1},a_0a_0^*\right]}{2}\,e^{s(1-s)\frac{\left((a_0a_0^*)^{-1}\left[\nabla^{V^1},a_0a_0^*\right]\right)^2}{2\pi i}}\right)ds\\&=\int_0^1\operatorname{tr}_{V^0}\left(\frac{(a_0^{\#}\nabla^{V^1})^*-a_0^{\#}\nabla^{V^1}}{2}\,e^{s(1-s)\frac{\left((a_0^{\#}\nabla^{V^1})^*-a_0^{\#}\nabla^{V^1}\right)^2}{2\pi i}}\right)ds=\operatorname{ch}^{\mathrm{o}}\bigl(a_0^{\#}\nabla^{V^1},g^{V^0}\bigr)\;.\qquad\square\end{aligned}$$

Using I_{2k+1}, Igusa defines another class F_{2k} by transgression in [I2], (1.4). Let $\overline{B}=B\times[0,1]$, pull V and ∇^1 back to $\overline{B}$, and with the coordinate u of $[0,1]$, put

$$\overline{I}_{2k+1}=\frac{k!}{2\,(2\pi i)^k\,(2k+1)!}\operatorname{tr}\Bigl(\bigl((a_0a_0^*)^{-u}\left[\overline{\nabla}^1,(a_0a_0^*)^u\right]\bigr)^{2k+1}\Bigr)\;.$$

Igusa's form $F_{2k}\in\Omega^{\mathrm{even}}(B)$ is defined as

$$F_{2k}=-\int_{\overline{B}/B}\overline{I}_{2k+1}\;. \tag{8.35}$$

We put $F=\sum_{k=0}^{\infty}F_{2k}\in\Omega^{\mathrm{even}}(B)$.

8.36. Remark. We note that $\overline{I}_{2k+1}$ is closed and $\overline{I}_{2k+1}|_{u=0}$ vanishes identically. It thus follows from Stokes' theorem that

$$d\,F_{2k}=\overline{I}_{2k+1}|_{u=1}-\overline{I}_{2k+1}|_{u=0}=I_{2k+1}\;.$$

We define a superconnection of total degree one on $V\to B$ by

$$A_0'=\overline{A}'|_{r=0}=a_0+a_0^{\#}\nabla^{V^1}\oplus\nabla^{V^1}\;, \tag{8.37}$$

cf. Remark 8.10. Let $T^{\mathrm{BL}}(A_0',g^V)$ denote the Bismut-Lott torsion form of (8.11). As above, we denote the corresponding forms and simplicial cochains on $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}),U(r))$ by F and $T^{\mathrm{BL}}(\mathcal{A}_0',g^{\mathcal{V}})$.

8.38. Lemma. *There is a natural form $\widetilde{F}\in\Omega^{\mathrm{odd}}(B)$ such that*

$$F=T^{\mathrm{BL}}\bigl(A_0',g^V\bigr)+d\widetilde{F}\;.$$

In particular, the simplicial cochains F and $T^{\mathrm{BL}}(\mathcal{A}_0',g^{\mathcal{V}})$ on $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}),U(r))$ differ by an exact cochain.

Proof. On the pullback of V to $B \times [0,1]$, consider the metric

$$\overline{g}^V = \big(s\, g^{V^0} + (1-s)\, a_0^{\#} g^{V^1}\big) \oplus g^{V^1} = \big(s\, g^{V^0} + (1-s)\, g^{V^1}(a_0\,\cdot\,, a_0\,\cdot\,)\big) \oplus g^{V^1} \;.$$

Note that we have not used parallelity of g^{V^0} in the definitions of I and F. Thus, let $\overline{I}$, $\overline{F}$ and $T(\overline{A}_0', \overline{g}^{V^0} \oplus \overline{g}^{V^1})$ denote the corresponding forms on $B \times [0,1]$. Then clearly

$$\overline{F}|_{B\times\{0\}} = T^{\mathrm{BL}}\big(\overline{A}_0', \overline{g}^V\big)|_{B\times\{0\}} = 0 \;.$$

We put

$$\widetilde{F} = -\int_{s\in[0,1]} \Big(\overline{F} - T^{\mathrm{BL}}\big(\overline{A}_0', \bar{g}^V\big)\Big) \;.$$

Then by Lemma 8.34, (8.35), Remark 8.36 and Stokes' Theorem, we have

$$\begin{aligned}
F - T^{\mathrm{BL}}\big(A_0', g^V\big) &= \Big(\overline{F} - T^{\mathrm{BL}}\big(A_0', g^V\big)\Big)\Big|_{s=1} \\
&= \int_{s\in[0,1]} d\Big(\overline{F} - T^{\mathrm{BL}}\big(\overline{A}_0', \bar{g}^V\big)\Big) - d\int_{s\in[0,1]} \Big(\overline{F} - T^{\mathrm{BL}}\big(\overline{A}_0', \bar{g}^V\big)\Big) \\
&= \int_{s\in[0,1]} \Big(\overline{I} - \mathrm{ch}^{\mathrm{o}}\big(a_0^{\#}\nabla^1, \bar{g}^V\big)\Big) + d\widetilde{F} = d\widetilde{F} \;. \qquad \square
\end{aligned}$$

8.e. Igusa's polynomial P and the class G. We interpret the classes G_{2k} of Chapter 1 (respectively K_{2k} in Chapter 2) of [I2] in our smooth setting and show that they equal the classes G_{2k}' of Proposition 8.18. This follows from Proposition 8.18 (4) and the acyclicity of a certain cyclic cochain complex already studied in [I2].

Recall the definition of x, $x^* \in \Omega^1(B; \operatorname{End} V^0)$ in (8.16). These forms correspond to $-x$ and y in [I2], Lemma 1.3.1. It follows from the proof of Lemma 8.34 that

$$I = \mathrm{ch}^{\mathrm{o}}\big(a_0^{\#}\nabla^{V^1}, g^{V^0}\big) = -(2\pi i)^{\frac{1-N^B}{2}} \int_0^1 \frac{1}{2}\, \mathrm{tr}\Big((x^* - x)\, e^{s(1-s)\,(x^*-x)^2}\Big)\, ds \;. \tag{8.39}$$

The construction of G_{2k} relies on the existence of a certain homogeneous polynomial P_{2k} of degree $2k$ with

$$d\ \mathrm{tr}\big(P_{2k}(x,x^*)\big) = \frac{k!}{2\,(2k+1)!}\Big(\mathrm{tr}\big((x^*)^{2k+1}\big) - \mathrm{tr}\big(x^{2k+1}\big) - \mathrm{tr}\big((x^*-x)^{2k+1}\big)\Big) \;. \tag{8.40}$$

Note that x and x^* do not commute, so the order of the $x$es and x^*s in each monomial is important. There will in general be several monomials with the same number of $x$es and x^*s involved.

We recall the definition of the polynomials P_{2k} in [I2], Section 1.3.1. Let $A = \mathbb{C} \times \mathbb{C}$ be the direct sum of two copies of the algebra $\mathbb{C}$, generated by $u = (1,0)$ and $v = (0,1)$. As in [I2], we consider the $\mathbb{Z}$-graded cyclic complex

$$C^* = \prod_{j=0}^{\infty} A^{\otimes(j+1)} \;/\; \big((-1)^j z_j - \mathrm{id}\big) \;, \tag{8.41}$$

where $z_j(a_0 \otimes \cdots \otimes a_j) = (a_1 \otimes \cdots \otimes a_j \otimes a_0)$.

Let $b\colon C^* \to C^{*-1}$ denote the cyclic boundary operator,

$$b\big(a_0 \cdots a_j\big) = \sum_{i=0}^{j-1} (-1)^i a_0 \cdots (a_i a_{i+1}) \cdots a_j + (-1)^j \,(a_j a_0)\, a_1 \cdots a_{j-1} \;. \tag{8.42}$$

We define a coboundary operator δ by $\delta u = u \cdot u$, $\delta v = v \cdot v$, and

$$\delta\big(a_0 \cdots a_j\big) = \sum_{i=0}^{j} (-1)^i a_0 \cdots a_{i-1} \cdot \delta a_i \cdot a_{i+1} \cdots a_j \;.$$

Clearly $\delta u^{\cdot k} = u^{\cdot (k+1)}$ if k is odd and $\delta u^{\cdot k} = 0$ if k is even, and similarly for v. Let the "weight" $w \in \operatorname{End} C^*$ be the linear endomorphism that multiplies a monomial $a_0 \cdots a_j$ with $a_i \in \{u, v\}$ for all i by the number of indices i modulo $j+1$, such that $a_i \neq a_{i+1}$. By [I2], we have

$$b \circ \delta + \delta \circ b = w \;. \tag{8.43}$$

Note that w commutes with δ and b. In particular, the subcomplex generated by all mixed monomials (i.e., by all monomials of non-zero weight) is acyclic both with respect to b and with respect to δ.

Because $\delta(v-u) = [v+u, v-u]$, we have

$$\delta\big((v-u)^{2k+1}\big) = \sum_{i=0}^{2k} (-1)^i (v-u)^i \cdot [v+u, v-u] \cdot (v-u)^{2k-i} = 0 \;.$$

Similarly,

$$\delta\big(u^{2k+1}\big) = \delta\big(v^{2k+1}\big) = 0 \;.$$

Igusa defines

$$P_{2k}(u,v) = -\frac{k!}{2\,(2k+1)!}\,\frac{b}{w}\,\big((v-u)^{2k+1} + u^{2k+1} - v^{2k+1}\big) \tag{8.44}$$

in [I2]. Because $(v-u)^{2k+1} + u^{2k+1} - v^{2k+1}$ contains no mixed monomials, the polynomial P satisfies

$$\partial P = -\frac{k!}{2\,(2k+1)!}\,\big((v-u)^{2k+1} + u^{2k+1} - v^{2k+1}\big) \;. \tag{8.45}$$

by (8.43) and (8.46). We put $P = \sum_{k=1}^{\infty} P_{2k}$.

There is a $\mathbb{C}$-linear map $\varphi\colon A \to \Omega^1(B, \operatorname{End} F)$ with $\varphi(u) = x$ and $\varphi(v) = x^*$. Define

$$\Phi\colon\ C^* \longrightarrow \Omega^*(B) \qquad \text{with} \qquad \Phi\big(a_0 \cdots a_j\big) = (2\pi i)^{-\frac{N^B}{2}} \operatorname{tr}\big(\varphi(a_0) \cdots \varphi(a_j)\big) \;.$$

The sign factor $(-1)^j$ in (8.41) is compatible with this construction because x and y are of odd degree. By (8.41), we clearly have

$$d \circ \Phi = \sqrt{2\pi i}\,\Phi \circ \delta \;. \tag{8.46}$$

Igusa's form G is defined as

$$G = \Phi(P) \;. \tag{8.47}$$

Let G and G' also denote the cochains obtained by integrating G and G' over each simplex in $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$.

8.48. Lemma. *The forms G and G' agree up to the differential of a naturally defined form $\widetilde{G}$. In particular, the cochains G and G' on $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}),U(r))$ agree up to an exact cochain.*

Proof. We are going to prove the existence of polynomials $P'_{2k} \in C^{2k}$ in u and v with the following properties.

(1) $G'_{2k} = \Phi(P'_{2k})$.
(2) $\partial P'_{2k} = \frac{k!}{2\,(2k+1)!}\big(v^{2k+1} - v^{2k+1} - (v-u)^{2k+1}\big)$, and
(3) P'_{2k} contains only mixed monomials.

In this case, we concluce that $\partial(P-P') = 0$ by (8.45) and (2). Moreover, by (8.44) and (3), the polynomial $P_{2k} - P'_{2k}$ contains no mixed monomials. Putting $\widetilde{P} = \frac{b}{w}\,(P-P')$, we find $\partial\widetilde{P} = P - P'$ by (8.43). Using (8.46), (8.47) and (1), our claim follows with $\widetilde{G} = (2\pi i)^{-\frac{1}{2}}\,\Phi\big(\widetilde{P}\big)$, because

$$d\widetilde{G} = \Phi\big(\partial\widetilde{P}\big) = \Phi\big(P - P'\big) = G - G' \;.$$

We work in the algebra $\Omega^*([0,1]^2, C^*)$. Let us set

$$P' = \int_{(r,s)\in[0,1]^2} \frac{1}{2}\, e^{-\left(\delta + d - (1-r)\,((1-s)\,u + s\,v)\right)^2} \quad\in\quad C^* \;, \tag{8.49}$$

where d denotes the exterior differential on $[0,1]^2$. By Proposition 8.18 and (8.25), we get (1).

In the proof of Proposition 8.18 (4), we have only used (8.17), and the cyclic properties of the trace. This implies that all calculations remain valid in $\Omega^*([0,1]^2; C^*)$, in particular, (2) follows from

(8.50)
$$\begin{aligned}
\delta P' &= \int_{(r,s)\in[0,1]^2} \frac{1}{2}\left[\delta + d - (1-r)\,\big((1-s)\,u + s\,v\big)\,,\; e^{-(\delta+d-(1-r)\,((1-s)\,u+s\,v))^2}\right] \\
&\quad - \int_{(r,s)\in[0,1]^2} \frac{1}{2}\left[d\,,\; e^{-(\delta+d-(1-r)\,((1-s)\,u+s\,v))^2}\right] \\
&= -\int_0^1 \frac{v-u}{2}\, e^{s(1-s)\,(v-u)^2}\,ds + \int_0^1 \frac{v}{2}\, e^{r(1-r)\,v^2}\,dr - \int_0^1 \frac{u}{2}\, e^{r(1-r)\,u^2}\,dr \\
&= \sum \frac{k!}{2\,(2k+1)!}\left(v^{2k+1} - u^{2k+1} - (v-u)^{2k+1}\right) .
\end{aligned}$$

Because positive even powers of u and v vanish in C^*, the polynomial P' contains only mixed monomials. This finishes the proof. ☐

8.f. Igusa's polynomial Q and the class K. We translate some definitions of [I2], Chapter 2, to the language of flat superconnections. The reader is warned at this point that for compatibility with [BL], we work with bundles of cochain complexes. This means that the bundles C_0, C_1 of [I2] are now named V^1 and V^0. This is compatible with our definitions in Subsections d and e.

Let V^0, $V^1 \to B$ be flat bundles with connection ∇, let $f\colon V^0 \to V^1$ be a vector bundle isomorphism. Igusa assumes that for nearby points p, $q \in B$, there are smooth families E^0, E^1 of isomorphisms $E^i(p,q)\colon V^i(q) \to V^i(p)$ with $E^i(q,p) = (E^i(p,q))^{-1}$ such that

$$\begin{array}{ccc}
V^1(p) & \xleftarrow{E^1(p,q)} & V^1(q) \\
{\scriptstyle f(p)}\big\uparrow & & {\scriptstyle f(q)}\big\uparrow \\
V^0(p) & \xleftarrow{E^0(p,q)} & V^0(q)
\end{array} \tag{8.51}$$

commutes.

Igusa also assumes that there is a family S of linear maps $S(p,q,r)\colon V^1(r) \to V^0(p)$ for nearby points p, q, $r \in B$ such that

$$
\begin{aligned}
S(p,q,r)\,f(r) &= E^0(p,q)\,E^0(q,r) - E^0(p,r)\;, \qquad \text{and} \\
f(p)\,S(p,q,r) &= E^1(p,q)\,E^1(q,r) - E^1(p,r)\;.
\end{aligned} \tag{8.52}
$$

Moreover, he requires the following behaviour under permutations of p, q, r:

$$
S(q,p,r) = -E^0(q,p)\,S(p,q,r)\;, \qquad \text{and} \qquad S(p,r,q) = -S(p,q,r)\,E^1(r,q)\;. \tag{8.53}
$$

Igusa demands that the coefficients of f, $E^0 - \mathrm{id}$, $E^1 - \mathrm{id}$ and S are given by upper triangular matrices with respect to some local bases of V. This corresponds to property (3) in Definition 2.35.

We use the data f, E and S to construct a flat superconnection of total degree 1. We denote by $\nabla^{V^i,q}$, $\nabla^{V^i,r}$ the derivation with respect to the variables q and $r \in B$. Then we define $a_0 \in \mathrm{Hom}(V^0, V^1)$, $a_1^i \in \Omega^1(B; \mathrm{End}(V^i))$ and $a_2 \in \Omega^2(B; \mathrm{Hom}(V^1, V^0))$ by

$$
\begin{aligned}
a_0(p) &= f(p)\;, \\
a_1^i(p)(X) &= \nabla_X^{V^i,q}\big|_{q=p} E^i(p,q)\;, \\
\text{and} \qquad a_2(p)(X,Y) &= -\big(\nabla_X^{V,q}\,\nabla_Y^{V,r} - \nabla_Y^{V,q}\,\nabla_X^{V,r} - \nabla_{[X,Y]}^{V,r}\big)\big|_{r=q=p} S(p,q,r)\;,
\end{aligned}
$$

and define a superconnection of total degree 1 on $V = V^0 \oplus V^1$ by

$$
A' = \nabla + a_0 + a_1 + a_2\;. \tag{8.54}
$$

Then the coefficients of A' are all strictly upper triangular.

8.55. Proposition. *The superconnection A' is flat.*

Proof. Note that A' is of total degree 1, so $(A')^2$ could have contributions in $\Omega^k(B, \mathrm{End}\,V^*)$, where $k = 1$, 2, 3.

Because $E(q,p) = E(p,q)^{-1}$, we have $E(p,p) = \mathrm{id}$. We derive (8.51) with respect to q at $q = p$ and obtain

$$
0 = \nabla_X^{V,q}\big|_{q=p}\big(E(p,q)\,f(q) - f(p)\,E(p,q)\big) = \big[\nabla + a_1, a_0\big](p)\;, \tag{8.56}
$$

so $(A')^2$ has no component in degree 1.

By (8.53), we have $S(p,p,q) = S(p,q,q) = 0$. We derive (8.52) twice, and find

$$
\begin{aligned}
0 = \big(\nabla_X^{V,q}\,\nabla_Y^{V,r} - \nabla_Y^{V,q}\,\nabla_X^{V,r} - \nabla_{[X,Y]}^{V,r}\big)\big|_{r=q=p}& \\
\big(E(p,q)\,E(q,r) - E(p,r) - f(p)\,S(p,q,r) - S(p,q,r)\,f(r)\big)& \\
= \big((\nabla^V + a_1)^2 + [a_0, a_2]\big)(p)(X,Y)&\;,
\end{aligned} \tag{8.57}
$$

so the homogeneous component of $(A')^2$ of degree 2 vanishes, too.

As in the proof of Lemma 8.7, the Bianchi identity for $\nabla + a_1$ and equations (8.56) and (8.57) imply that $(A')^2$ has no component in degree 3 either. This proves the Proposition. ☐

8.58. Remark. In other words, there is a functor from Igusa's category $\mathcal{W}^{\mathrm{diff}}(\mathbb{C}^m,n)$ introduced in [I2], Chapter 2.2, to our category $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(R,G)$. Igusa proved that $\mathcal{W}^{\mathrm{diff}}(\mathbb{C}^m,n)$ is homotopy equivalent to the category $\mathcal{W}h^{\mathrm{simp},h,\{0,1\}}(R,G)$. It is easy to check that these functors are compatible with the homotopy equivalences constructed in Theorem 6.31.

Moreover, the final correction term K (called L in [I2]) is constructed using the variables x, y, z of (8.16) that are defined using the coefficients of the superconnection A'. In Igusa's notation in [I2], Section 2.4.1, we have $X=x$, $V=-y$, and $W=z$. The form $U\in\Omega^1(B,\operatorname{End}V^1)$ can be represented by $U=a_0\,(y-x)\,a_0^{-1}$, however, a careful analysis of Igusa's construction shows that U is not really used in the definition of K. It is now clear from the above that Igusa's torsion class on $\mathcal{W}^{\mathrm{diff}}(\mathbb{C}^m,n)$ pulls back from $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(R,G)$.

The construction of K relies on a sequence of homogeneous polynomials Q_{2k} in x, y, z of degree $2k$, where $\deg x=\deg y=1$ and $\deg w=2$. These polynomials should satisfy

$$d\ \operatorname{tr}\bigl(Q_{2k}(x,y,z)=\frac{k!}{2\,(2k+1)!}\operatorname{tr}\bigl(x^{2k+1}\bigr)\ .$$

Again, the order of the variables in each monomial is important.

Let us recall some details of the construction of Q in [I2]. We consider the subalgebra $A\subset M_3(\mathbb{C})$ generated by the elements

$$o=\begin{pmatrix}1&&\\&0&\\&&0\end{pmatrix}\ ,\qquad p=\begin{pmatrix}0&&\\&0&\\&&1\end{pmatrix}\ ,\qquad\text{and}\qquad q=\begin{pmatrix}0&&\\&0&1\\&&0\end{pmatrix}\ .$$

This is a subalgebra of the algebra used in [I2], Section 2.4.1, where o, p, and q correspond to the generators x, v, w in [I2]. The corresponding algebra in [I2] has another generator u, however, u is never used in [I2], so we do not introduce a name for it here. We assign degrees $|o|=|p|=1$ and $|q|=2$. As in [I2], Section 2.4.1, we consider a new cyclic complex

$$C^*=\prod_{j=0}^{\infty}A^{\otimes(j+1)}/(z_j-\mathrm{id})\ , \tag{8.59}$$

$$\text{where}\qquad z_j\bigl(a_0\otimes\cdots\otimes a_j\bigr)=(-1)^{|a_0|\,(|a_1|+\cdots+|a_j|)}\,(a_1\otimes\cdots\otimes a_j\otimes a_0)\ .$$

We define a coboundary operator δ on generators by

$$\delta o=o\cdot o\ ,\qquad \delta p=p\cdot p-q\ ,\qquad\text{and}\qquad \delta q=p\cdot q-q\cdot p\ , \tag{8.60}$$

and in general by

$$\delta(a_0\cdots a_j)=\sum_{i=0}^{j}(-1)^i\,a_0\cdots a_{i-1}\cdot\delta a_i\cdot a_{i+1}\cdots a_j\ .$$

Then δ increments the total degree by 1. Let b be the cyclic boundary operator given as in (8.42), and note that b decrements the total degree by 1 because $q\cdot q=0$. Let $w\in\operatorname{End}C^*$ be the *colour weight* that multiplies each monomial $a_0\cdots a_j$ by twice the number of indices i modulo $j+1$ such that $a_i=o$ and $a_{i+1}\neq o$. It is proved in [I2], Lemma 2.4.2, that

$$b\circ\delta+\delta\circ b=w\ . \tag{8.61}$$

Note that w commutes with b and δ, and that the subcomplex spanned by all mixed monomials $a_0\cdots a_j$ is acyclic. Here, a monomial is called *mixed* iff $a_i=o$ for at least one index i, and $a_k\neq o$ for at least one index k, i.e., iff $w(a_0\cdots a_j)\neq 0$.

We now regard the subcomplex $(C'^*,\delta)\subset(C^*,\delta)$ spanned by p and q. It follows that if a monomial has colour weight $w=0$, then it is either of the form x^{2k+1}, or it belongs to C'^*. By (8.60), the subcomplex C'^* admits a filtration

$$C'^*=F'^0\supset F'^1\supset\dots\,, \tag{8.62}$$

where F'^k is spanned by all monomials in p and q that contain q at least k times. As in [I2], we may write $\delta|_{C'^*}=\delta'+\delta''$, where

$$\begin{aligned} &\delta'\colon F'^k\longrightarrow F'^k\,, &&\text{with}\quad p\longmapsto p^2\,, &&q\longmapsto pq-qp\,,\\ \text{and}\quad &\delta''\colon F'^k\longrightarrow F'^{k+1}\,, &&\text{with}\quad p\longmapsto -q\,, &&q\longmapsto 0\,. \end{aligned}$$

We define a new cyclic boundary operator b' on C'^* of total degree -1 by

$$\begin{aligned} b'(a_0\cdots a_j)=\sum_{i=0}^{j-1}&(-1)^{|a_0|+\dots+|a_{i-1}|}\,a_0\cdots(a_i*a_{i+1})\cdots a_j\\ &+(-1)^{|a_j|\,(|a_0|+\dots+|a_{j-1}|)}\,(a_j*a_0)\cdot a_1\cdots a_{j-1}\,, \end{aligned}$$

which is on generators given by

$$p*p=p\,,\qquad p*q=q*p=q\,,\qquad\text{and}\qquad q*q=0\,.$$

Let w' be the weight operator that multiplies each monomial $a_0\cdots a_j$ by the number of indices i modulo $j+1$ such that $a_i=p$, $a_{i+1}=q$. Then it it proved in [I2], Lemma 2.4.4, that

$$b'\circ\delta'+\delta'\circ b'=-w'\,. \tag{8.63}$$

Because the only monomial of weight $w'=0$ of total degree $2k+1$ is v^{2k+1}, a simple spectral sequence argument gives

$$H^{\mathrm{odd}}\bigl(F'^k,\delta\bigr)=0\qquad\text{for all } k\geq 1. \tag{8.64}$$

Note that C^* admits an involution ι given by

$$o\overset{\iota}{\longleftrightarrow}-o\,,\qquad p\overset{\iota}{\longleftrightarrow}p-o\,,\qquad\text{and}\qquad q\overset{\iota}{\longleftrightarrow}q\,. \tag{8.65}$$

To show that ι is a cochain map, note that

$$\delta\overset{\iota}{\longleftrightarrow}\delta-[o,\,\cdot\,]$$

on generators, however

$$\begin{aligned} \sum_i(-1)^{|a_0|+\dots+|a_{i-1}|}\,a_0\cdots a_{i-1}\cdot\bigl(\delta a_i-o\cdot a_i-(-1)^{|a_i|}a_i\cdot o\bigr)\cdot a_{i+1}\cdots a_j&\\ =\sum_i(-1)^{|a_0|+\dots+|a_{i-1}|}\,a_0\cdots a_{i-1}\cdot\delta a_i\cdot a_{i+1}\cdots a_j\,,& \end{aligned}$$

so the action of δ on C^* commutes with ι. However, the multiplication on A and thus the operators b and w are changed. In fact, the involution ι corresponds to passing from the superconnection A' of degree 1 to a superconnection

$$a_0^{-1} + \big(\nabla^{V^0} + a_1^0\big) \oplus \big(\nabla^{V^1} + a_1^1\big) a_0\, a_2\, a_0$$

of degree -1.

Finally, we define a linear map $\varphi\colon A \to \Omega^*(B; \operatorname{End} V^0)$ on generators by $\varphi(o) = x$, $\varphi(p) = y$, and $\varphi(q)$. Define

$$\Phi\colon (C^*, \delta) \longrightarrow (\Omega^*(B), d) \qquad \text{by} \qquad \Phi\big(a_0 \cdots a_j\big) = (2\pi i)^{-\frac{N^B}{2}} \operatorname{tr}\big(\varphi(a_0) \cdots \varphi(a_1)\big)\ . \tag{8.66}$$

Then by (8.17) and (8.60), we have

$$d \circ \Phi = \sqrt{2\pi i}\, \Phi \circ \delta\ . \tag{8.67}$$

The map Φ has a large kernel. We note that $\varphi(p) = a_1^0$ and $\varphi(q) = a_2 a_0$ are strictly triangular with respect to the eigenspaces of h^{V^0}. We also note that

$$\varphi(o - p) = \big(\nabla^{V^0} + a_1^0\big) + a_0^{-1}\, \nabla^{V^1}\, a_0 = -a_0^{-1}\, a_1^1\, a_0$$

and $\varphi(q) = a_0^{-1}\,(a_0 a_2)\, a_0$ are strictly triangular with respect to the eigenspaces of $a_0^{-1} h^{V^1} a_0$. In other words,

$$C'^* \subset \ker(\Phi) \qquad \text{and} \qquad \iota(C'^*) \subset \ker(\Phi)\ . \tag{8.68}$$

As noted after Lemma 2.4.5 in [I2], Igusa's polynomials Q_{2k} in our normalisation satisfy

$$\delta Q_{2k} = \frac{k!}{2\,(2k+1)!}\, o^{2k+1} + R_{2k+1} - \iota R_{2k+1} \tag{8.69}$$

for some $R_{2k+1} \in C'^*$, where

$$R_{2k+1} \equiv \frac{k!}{2(2k+1)!}\, p^{2k+1} \qquad \text{modulo } F'^1\ . \tag{8.70}$$

Igusa's form K_{2k} is defined as

$$K_{2k} = \Phi(Q_{2k}) \quad \in \Omega^{2k}\big(Wh^{h,\{0,1\}}\big(\mathcal{M}_r(\mathbb{C}), U(r)\big)\big)\ . \tag{8.71}$$

We put $K = \sum_k K_{2k}$. Let K and K' also denote the cochains obtained by integrating K and K' over each simplex in $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$.

8.72. Lemma. *The forms K and K' agree up to the differential of a naturally defined form $\widetilde{K}$. In particular, the cochains K and K' on $\mathcal{W}h^{\mathrm{diff},h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$ agree up to an exact cochain.*

Proof. We want to construct polynomials $Q'_{2k} \in C^*$ and $R'_{2k+1} \in C'^*$ with the following properties.

(1) $K'_{2k} = \Phi(Q'_{2k})$.
(2) $\delta Q'_{2k} = \frac{k!}{2\,(2k+1)!}\, o^{2k+1} + R'_{2k+1} - \iota R'_{2k+1}$.
(3) $R'_{2k+1} \equiv \frac{k!}{2\,(2k+1)!}\, p^{2k+1}$ modulo F'^1.

Given Q'_{2k} and R'_{2k+1} as above, we proceed as follows. First, using (8.64), (8.70), and (3), we find an even formal power series $\widetilde{R} \in F'^1 \subset C'^*$ such that

$$R - R' = \delta\widetilde{R}\,.$$

From (8.69) and (2) we conclude that $\delta(Q - Q' - \widetilde{R} + \iota\widetilde{R}) = 0$. We let $(\,\cdot\,)^{(w>0)}$ denote the components of colour weight $\neq 0$ and define

$$\widetilde{K} = (2\pi i)^{-\frac{1}{2}}\,\Phi\biggl(\frac{b}{w}\,(Q - Q' - \widetilde{R} + \iota\widetilde{R})^{(w>0)}\biggr)\,.$$

By (8.61), (8.67), (8.71) and (1), and because $o^{2k} = 0 \in C^*$ for all $k > 0$ and C'^*, $\iota C'^* \subset \ker\Phi$ by (8.68), we find that

$$d\widetilde{K} = \Phi\bigl(Q - Q' - \widetilde{R} + \iota\widetilde{R}\bigr) = \Phi\bigl(Q - Q'\bigr) = K - K'\,,$$

which proves the lemma.

It remains to construct Q', R' satisfying (1–3). Our final formula for Q' will be the sum of two formal power serieses ${}^1Q' + {}^2Q'$, such that

$$\Phi({}^1Q') = K' \qquad\text{and}\qquad \Phi({}^2Q') = 0\,, \tag{8.73}$$

which implies (1). To construct ${}^1Q'$, we work in $\Omega^*([0,1]^2; C^*)$. As in (8.31), we construct a superconnection on the trivial bundle with fibre $(C^*)^2$ by

$${}^3\widetilde{A}' = \begin{pmatrix} d + \delta - (1-r)\,o - rv\,p & -2v\,\bigl(d + \delta - (1-r)\,o - r\,p\bigr)^2 \\ v/2 & -d - \delta + (1 - rv)\,o + rv\,p \end{pmatrix}\,, \tag{8.74}$$

where d denotes the exterior differential on $[0,1]^2$. Again, the signs in the lower right corner make sense if we agree that the total degree in the second copy of C^* is shifted by 1. We can now translate parts of the proof of Proposition 8.18 (5).

We set

$${}^1Q' = \int_{(r,v)\in[0,1]^2} \frac{1}{2}\,\mathrm{str}\Bigl(e^{-{}^3\widetilde{A}'^2}\Bigr)\,.$$

The first equation in (8.73) follows from Proposition 8.18 (3), (8.14), (8.66), (8.74). By Stokes' theorem,

$$\begin{aligned}
\delta\,{}^1Q'^* &= \int_{(r,v)\in[0,1]^2} \frac{1}{2}\,\mathrm{str}\Bigl(\Bigl[{}^3\widetilde{A}', e^{-{}^3\widetilde{A}'^2}\Bigr]\Bigr) - \int_{(r,v)\in[0,1]^2} \frac{1}{2}\,d\,\mathrm{str}\Bigl(e^{-{}^3\widetilde{A}'^2}\Bigr) \\
&= \int_{r=0}^1 \frac{1}{2}\,\mathrm{str}\Bigl(e^{-{}^3\widetilde{A}'^2|_{v=0}} - e^{-{}^3\widetilde{A}'^2|_{v=1}}\Bigr) - \int_{v=0}^1 \frac{1}{2}\,\mathrm{str}\Bigl(e^{-{}^3\widetilde{A}'^2|_{r=0}} - e^{-{}^3\widetilde{A}'^2|_{r=1}}\Bigr)\,.
\end{aligned} \tag{8.75}$$

As in (7.5), we have

$${}^3\widetilde{A}'^2|_{v=0} = \begin{pmatrix} d + \delta - (1-r)\,o & 0 \\ 0 & -d - \delta + o \end{pmatrix}^2 = \begin{pmatrix} -r(1-r)\,o^2 - o\,dr & \\ & 0 \end{pmatrix}\,.$$

Thus,

$$\int_{r=0}^{1} \frac{1}{2}\,\mathrm{str}\Bigl(e^{-\,{}^3\widetilde{A}'^2|_{v=0}}\Bigr) = \int_{r=0}^{1} \frac{o}{2}\, e^{r(1-r)\,o^2}\,dr\;. \tag{8.76}$$

At $v=1$, we find

$${}^3\widetilde{A}'^2|_{v=1} = \begin{pmatrix} d+\delta-(1-r)\,o-r\,p & -2\bigl(d+\delta-(1-r)\,o-r\,p\bigr)^2 \\ 1/2 & -d-\delta+(1-r)\,o+r\,p \end{pmatrix}^2 = 0$$

by Lemma 8.7, so there is no contribution from this side.

The formal connection $d+\delta-o$ is flat either by (8.60) or because it corresponds to $a_0^{\#}\nabla^{\overline{V}^1}$. This implies

$${}^3\widetilde{A}'^2|_{r=0} = \begin{pmatrix} d+\delta-o & 0 \\ v/2 & -d-\delta+o \end{pmatrix}^2 = \begin{pmatrix} 0 & \\ dv/2 & 0 \end{pmatrix}\,,$$

so

$$\int_{r=0}^{1} \frac{1}{2}\,\mathrm{str}\Bigl(e^{-\,{}^3\widetilde{A}'^2|_{v=1}}\Bigr) = \int_{v=0}^{1} \frac{1}{2}\,\mathrm{str}\Bigl(e^{-\,{}^3\widetilde{A}'^2|_{r=0}}\Bigr) = 0\;. \tag{8.77}$$

In the proof of Proposition 8.18 (5), the face $r=1$ did not contribute because the corresponding curvature had coefficients in a nilpotent subalgebra of $\operatorname{End}V$. We do not have this argument here. Instead, the contribution to $\delta\,{}^1Q'$ at $r=1$ will be cancelled by the second polynomial ${}^2Q'$. To that end, we define yet another superconnection on $V\times[0,1]^2$ by

$${}^4\widetilde{A} = \frac{v}{2}\,a_0 + \bigl(\nabla^{V^0} + v\,a_1^0\bigr) \oplus \bigl(\nabla^{V^1} + v\,a_1^1\bigr) + 2v(1-w)\,a_2\;,$$

and as its preimage under Φ, we take

$${}^4\widetilde{A}' = \begin{pmatrix} d+\delta-v\,p & -2v(1-w)\,\bigl(d+\delta-p\bigr)^2 \\ v/2 & -d-\delta+(1-v)\,o+v\,p \end{pmatrix}\,. \tag{8.74}$$

We set

$${}^2Q' = \int_{(w,v)\in[0,1]^2} \frac{1}{2}\,\mathrm{str}\Bigl(e^{-\,{}^4\widetilde{A}'^2}\Bigr)\,.$$

Note that the definition of ${}^4\widetilde{A}$ only involves the coefficients a_0, a_1^0, a_1^1 and a_2, which are all strictly triangular with respect to h^V. This gives the second equation in (8.73).

The analogue of (8.75) holds for $\delta\,{}^2Q'$. One easily checks that

$${}^3\widetilde{A}'|_{r=1} = {}^4\widetilde{A}'|_{w=0}\;,$$

so the contributions to $\delta\,{}^1Q'$ at $r=1$ and to $\delta\,{}^2Q'$ at $w=0$ cancel as announced above.

We recall that by (8.60), the formal connection $d+\delta-p$ has curvature q. At $v=1$, we thus have

$${}^4\widetilde{A}'^2 = \begin{pmatrix} d+\delta-p & -2(1-w)\,q \\ 1/2 & -d-\delta+v\,p \end{pmatrix}^2 = \begin{pmatrix} w\,q & * \\ 0 & w\,q \end{pmatrix}\,.$$

Because this matrix is triangular and the diagonal terms do not involve dw, there will be no contribution to $\delta\,{}^2Q'$ from the side $v = 1$. At $v = 0$, things are even simpler because ${}^4\widetilde{A}'$ is evidently flat. So, we find

$$\int_{w=0}^{1} \frac{1}{2}\,\mathrm{str}\Big(e^{-\,{}^4\widetilde{A}'^2|_{v=0}}\Big) = \int_{w=0}^{1} \frac{1}{2}\,\mathrm{str}\Big(e^{-\,{}^4\widetilde{A}'^2|_{v=1}}\Big) = 0\,. \tag{8.78}$$

Finally, at $w = 1$, we compute

$$\begin{aligned} {}^4\widetilde{A}'^2 &= \begin{pmatrix} d+\delta-v\,p & 0 \\ v/2 & -d-\delta+o+v\,(p-o) \end{pmatrix}^2 \\ &= \begin{pmatrix} -v(1-v)\,p^2+v\,q+p\,dv & 0 \\ * & -v(1-v)\,(p-o)^2+v\,q+(p-o)\,dv \end{pmatrix}. \end{aligned}$$

Let us define

$$R' = \int_0^1 \frac{p}{2}\,e^{r\,(1-r)\,p^2-r\,q}\,dr\;,$$

then by (8.65), we find

$$\int_{v=0}^{1} \frac{1}{2}\,\mathrm{str}\Big(e^{-\,{}^4\widetilde{A}'^2|_{w=1}}\Big) = -R' + \iota R'\,. \tag{8.79}$$

Combining equations (8.75)–(8.77), (8.78), (8.79), we obtain (2). Moreover, R' clearly satisfies (3). This finishes the proof of Lemma 8.72. $\square$

9. The unitary Whitehead space and its torsion class

We generalise the constructions of Section 6 to allow family Thom-Smale complexes such that the fibrewise cohomology carries a unitarily flat metric. In Section 9.a, we contruct a unitary Whitehead space $Wh^u(R,G)$ that naturally covers the full Whitehead space $Wh(R,G)$. The cohomology bundle $\mathcal{H} \to Wh^u(R,G)$ carries a natural metric $g^{\mathcal{H}}$. In Section 9.b, we extend the universal torsion class defined in Section 2 and studied in Section 8 to a class $\tau(R,G) \in H^{\mathrm{even}}\,Wh^u(R,G)$.

Suppose that $p\colon M \to B$ is a family equipped with a fibre-wise Morse function h and a flat vector bundle $F \to M$. Then the lifts ξ^u of the associated map $\xi = \xi_{p,h,F}\colon B \to Wh(R,G)$ to $Wh^u(R,G)$ are in one-to-one correspondence with the parallel metrics on the fibre-wise cohomology $H = H^*(M/B;F) \to B$. The class $(\xi^u)^*\tau(R,G)$ does not depend on the choice of the lift in degree ≥ 2, and we recover the unitary Morse torsion class $T(M/B;F,h)$ of Definition 2.64 as $(\xi^u)^*\tau(R,G)^{[\geq 2]}$. Because all our constructions are compatible with stabilisation, we can generalise Igusa's constructions in [I2] to arbitrary families of manifolds equipped with unitarily flat vector bundles such that the fibrewise cohomology carries a parallel metric.

Finally, we give a natural map $\pi\colon Wh^u(R,G) \to Wh^h(R,G)$, such that $\tau(R,G)$ is preserved by π^*. This shows in particular that we recover Igusa's torsion $\tau(M/B;F)$, which is defined similarly under the assumption that $H \to B$ is trivial or has nilpotent structure group.

9.a. The unitary Whitehead space. We construct the unitary Whitehead space $Wh^u(R;G)$ as a covering of the Whitehead space $Wh(R,G)$. Note that the unitary Whitehead space should be viewed as a generalisation of Igusa's acyclic Whitehead space $Wh^h(R;G)$.

We have seen that the higher torsion forms of Section 2 can be used to construct cohomology classes if both flat bundles $F|_C \to C$ and $H \to B$ carry parallel metrics. Therefore, we need a variant of the Whitehead space that contains information about metrics on the tautological bundle $\mathcal{V} \to Wh(R;G)$ and the universal cohomology bundle $\mathcal{H} \to Wh(R;G)$.

The bundle $F|_C$ admits a parallel metric iff we can choose its discrete structure group to be subgroup of the unitary group, so it is sufficient to choose a group $G \subset R^\times$ and a complex representation $\rho: G \to GL_n(\mathbb{C})$ such that $\rho(G) \subset U(r)$. On the other hand, if we regard $\rho(G) \subset U(r)$, then the complexified bundle $\mathcal{V} \to Wh(R,G)$ carries a natural Hermitian metric induced by the standard metric on $\mathbb{C}^r$.

All fibre-wise metrics on $\mathcal{H}$ form a fibre bundle over $Wh(R,G)$ with a natural flat bundle connection for any R and G. We can take the étale space of the sheaf of horizontal sections of this bundle, which become parallel metrics on $\mathcal{H}$, as a classifying space for Morse vector bundles with a given parallel metric on the fibre-wise cohomology. In fact, we will refine this concept just a little by fixing a parallel Hermitian basis of $\mathcal{H}$ over each simplex of our unitary Whitehead space $Wh^u(R,G)$.

To keep things simple, we will take the point of view that unitary higher torsion is defined on $Wh^u(\mathcal{M}_r(\mathbb{C}),U(r))$. For arbitrary R and G, we obtain the unitary Whitehead space $Wh^u(R,G)$ and its universal torsion class $\tau(R,G) \in H^{\mathrm{even}}(R,G)$ from a pullback construction

$$\begin{tikzcd} Wh^u(R,G) \arrow[r] \arrow[d] & Wh^u(\mathcal{M}_r(\mathbb{C}),U(r)) \arrow[d] \\ Wh(R,G) \arrow[r] & Wh(\mathcal{M}_r(\mathbb{C}),U(r)) \end{tikzcd} ,$$

where the lower right arrow is induced by a representation $\rho: R \to \mathcal{M}_r(\mathbb{C})$ with $\rho(G) \subset U(r)$, the space $Wh^u(R,G)$ is constructed as a pullback bundle, and $\tau(R,G)$ is the pullback of $\tau = \tau(\mathcal{M}_r(\mathbb{C}),U(r))$ by the upper right arrow.

For convenience, we give a simplicial model for $Wh^u(\mathcal{M}_r(\mathbb{C}),U(r))$. Recall that the $\mathbb{Z}$-graded flat vector bundles $\mathcal{V} \to Wh(\mathcal{M}_r(\mathbb{C}),U(r))$ and $\mathcal{H} \to Wh(\mathcal{M}_r(\mathbb{C}),U(r))$ were described in Remark 6.11.

9.1. Definition. The *unitary simplicial pre-Whitehead category* $\mathcal{W}h'^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ is defined as follows.

(1) Objects of $\mathcal{W}h_k^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ are given by triples (P,a,e), where (P,a) is an object of $\mathcal{W}h_k^{\mathrm{simp}}(\mathcal{M}_r(\mathbb{C}),U(r))$ and $e = (e^p(i))_{p,i}$ is a family of complex bases $e_1^p(i), \dots, e_{\mathrm{rk}\,H^p}^p(i)$ on the spaces $H^p\bigl((\mathbb{C}^r)^P, a(i)\bigr)$ that are compatible with the natural isomorphisms
$$\bigl(\mathrm{id} + a(i,j)\bigr)_*\colon\ H^*\bigl((\mathbb{C}^r)^P, a(j)\bigr) \longrightarrow H^*\bigl((\mathbb{C}^r)^P, a(i)\bigr)$$
for $0 \le i \le j \le k$.

(2) Morphisms from (P,a,e) to $(Q,b,e') \in \mathcal{W}h^{\mathrm{simp},u}$ are given by pairs (f,g) such that (f,g) is a $\mathcal{W}h^{\mathrm{simp}}(\mathcal{M}_r(\mathbb{C}),U(r))$-morphism from (P,a) to (Q,b), and the induced isomorphisms
$$(f,g)_*\colon\ H^*\bigl((\mathbb{C}^r)^P, a(i)\bigr) \longrightarrow H^*\bigl((\mathbb{C}^r)^Q, b(i)\bigr)$$
are given in the bases e and e' by unitary matrices for one (and thus for all) $0 \le i \le k$.

(3) Generalised face operators are given by re-indexing as in Definition 6.10.

The *simplicial unitary pre-Whitehead space* $Wh'^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ is the geometric realisation of the simplicial unitary pre-Whitehead category.

We stabilise the unitary Whitehead space as in Definition 6.14. Note that if $(P^{[l]},a^{[l]})$ is the elementary object of vertical degree l, then we naturally identify
$$H^*\bigl((\mathbb{C}^r)^P, a(j)\bigr) \cong H^*\Bigl((\mathbb{C}^r)^{P \cup P^{[l]}}, \bigl(a \oplus a^{[l]}\bigr)(j)\Bigr) .$$

In particular, we can compare bases on both sides.

9.2. Definition. The *(stabilised) simplicial unitary Whitehead category* $\mathcal{W}h^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ has the same objects and face operators as $\mathcal{W}h'^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$. Its morphisms are iterated formal compositions of $\mathcal{W}h'^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$-morphisms and elementary expansions

$$(P,a,e) \longrightarrow (P \cup P^{[j]}, a \oplus a^{[j]}, e) .$$

The *simplicial unitary Whitehead space* $Wh^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ is the geometric realisation of the simplicial unitary Whitehead category.

9.3. Remark. We collect some properties of the definition above.

(1) By its construction in Remark 6.11 (1), the bundle $\mathcal{V} \to Wh'(\mathcal{M}_r(\mathbb{C}),U(r))$ inherits a parallel metric $g^{\mathcal{V}}$ from the standard metric on $\mathbb{C}^r$. A similar construction yields a unitarily flat bundle $(\mathcal{V},g^{\mathcal{V}}) \to Wh'^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$. We also let $\mathcal{H}$ denote fibre-wise cohomology as in Remark 6.11 (2), which is a flat vector bundle that is compatible with stabilisation, so we get $\mathcal{H} \to Wh^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$.
(2) The bundle $\mathcal{H} \to Wh^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ carries a unique parallel metric $g^{\mathcal{H}}$ for which the bases e are unitary.
(3) Recall that $\mathcal{W}h^{\mathrm{simp},h}(\mathcal{M}_r(\mathbb{C}),U(r))$ is the full subcategory of $\mathcal{W}h^{\mathrm{simp}}(\mathcal{M}_r(\mathbb{C}),U(r))$ consisting of those objects $(P,a) \in \mathcal{W}h^{\mathrm{simp}}(\mathcal{M}_r(\mathbb{C}),U(r))$ for which $a(i)$ is acyclic for one (and thus for all) i. Then $\mathcal{W}h^{\mathrm{simp},h}(\mathcal{M}_r(\mathbb{C}),U(r))$ is a full subcategory of $\mathcal{W}h^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$, and the forgetful functor to $\mathcal{W}h^{\mathrm{simp},h}(\mathcal{M}_r(\mathbb{C}),U(r))$ is just the identity. This way, we may view $\mathcal{W}h^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ as a generalisation of the category $\mathcal{W}h^{\mathrm{simp},h}(\mathcal{M}_r(\mathbb{C}),U(r))$ considered in [I2].
(4) There are similar definitions for $\mathcal{W}h^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r))$, $\mathcal{W}h^{\mathrm{mix},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ and their unstabilised variants if R is an $\mathbb{R}$-algebra. Again, $Wh^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ is the classifying space for Morse vector bundles $(V,A',h^V) \to B$ whose fibre-wise cohomology $H \to B$ is equipped with a parallel metric g^H. It is now easy to check that all results of Section 6.d carry over to unitary Whitehead categories, so we have homotopy equivalences

$$Wh'^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r)) \xleftarrow[\sim]{\Phi_1^u} Wh'^{\mathrm{mix},u}(\mathcal{M}_r(\mathbb{C}),U(r)) \xrightarrow[\sim]{\Phi_2^u} Wh'^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r))$$

and $$Wh^{\mathrm{simp},u}(\mathcal{M}_r(\mathbb{C}),U(r)) \xleftarrow[\sim]{\Phi_1^u} Wh^{\mathrm{mix},u}(\mathcal{M}_r(\mathbb{C}),U(r)) \xrightarrow[\sim]{\Phi_2^u} Wh^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r)) .$$

In particular, we may talk about "the" unitary pre-Whitehead space $Wh'^u(\mathcal{M}_r(\mathbb{C}),U(r))$, which is well-defined up to homotopy equivalence and carries unitarily flat vector bundles $(\mathcal{V},\nabla^{\mathcal{V}},g^{\mathcal{V}})$ and $(\mathcal{H},\nabla^{\mathcal{H}},g^{\mathcal{H}})$.

Let us examine the homotopy type of the unitary Whitehead space. We denote by $\mathcal{G}(\mathcal{H})$ the space of Hermitian metrics on $\mathcal{H} \to Wh(\mathcal{M}_r(\mathbb{C}),U(r))$. We topologize $\mathcal{G}(\mathcal{H})$ as the étale space of the sheaf of locally parallel metrics on $\mathcal{H}$. In other words, the map $\mathcal{G}(\mathcal{H}) \to Wh(\mathcal{M}_r(\mathbb{C}),U(r))$ is a covering, and local sections are given precisely by parallel metrics. A simplicial model for $\mathcal{G}(\mathcal{H})$ can be constructed as in Definition 9.1, but using parallel metrics g^H on $H^*\big((\mathbb{C}^r)^P,a_0'\big)$ over each simplex of $Wh(\mathcal{M}_r(\mathbb{C}),U(r))$ instead of parallel bases e of this bundle.

9.4. Proposition. *There is a natural map $Wh^u(\mathcal{M}_r(\mathbb{C}),U(r)) \to \mathcal{G}(\mathcal{H})$ given by assigning to a local parallel basis e of $\mathcal{H}$ the Hermitian metric for which e is unitary. This map is a fibre bundle with contractible fibres.*

In particular, if $\xi\colon B \to Wh(\mathcal{M}_r(\mathbb{C}),U(r))$ is a map and $(V,\nabla^V,A',g^V) = \xi^*(\mathcal{V},\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}})$, then homotopy classes of lifts $\xi^u\colon B \to Wh^u(\mathcal{M}_r(\mathbb{C}),U(r))$ are in bijection with globally parallel metrics g^H on $H = H^*(V,a_0') = \xi^*\mathcal{H} \to B$.

In other words, we could have defined $Wh^u(\mathcal{M}_r(\mathbb{C}),U(r))$ simply as the covering space $\mathcal{G}(\mathcal{H}) \to Wh(\mathcal{M}_r(\mathbb{C}),U(r))$. Nevertheless, the extra information about a local framing on $\mathcal{H}$ will be helpful later on.

Proof. Over each simplex $\sigma \in Wh^{\mathrm{simp},u}$, the fibre of the map $Wh^u(\mathcal{M}_r(\mathbb{C}),U(r)) \to \mathcal{G}(\mathcal{H})$ is the geometric realisation of the category of all unitary basis and unitary base changes with respect to the tautological metric $g^{\mathcal{H}}$ on the pullback of $\mathcal{H}$ to $\mathcal{G}(\mathcal{H})$. But this fibre is just the classifying bundle $E(U(\mathcal{H}|_\sigma))$ of the unitary group acting on the fibre of $\mathcal{H}|_\sigma$, which is contractible.

The second claim follows because globally parallel metrics g^H on $H \to B$ correspond precisely to lifts of ξ to the space $\mathcal{G}(\mathcal{H})$. $\square$

9.b. The universal torsion class on the unitary Whitehead space. We generalise our definition of $T(\mathcal{M}_r(\mathbb{C}),U(r)) \in H^{\mathrm{even}}(Wh^h(\mathcal{M}_r(\mathbb{C}),U(r)))$ in Definition 7.29 to the unitary setting. Because we have seen in Theorem 8.1 that $T(\mathcal{M}_r(\mathbb{C}),U(r)) = \tau(\mathcal{M}_r(\mathbb{C}),U(r))$ on $Wh^h(\mathcal{M}_r(\mathbb{C}),U(r))$, we will name our new class $\tau(\mathcal{M}_r(\mathbb{C}),U(r)) \in H^{\mathrm{even}}(Wh^u(\mathcal{M}_r(\mathbb{C}),U(r)))$, or just τ. The letter T will be reserved for torsion forms associated directly to superconnections as in Section 2.d. Note that in general, the forms $T(\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}})$ are not closed, so they do not define cohomology classes. However, since $g^{\mathcal{V}}$ and $g^{\mathcal{H}}$ are unitarily flat, the form $T(\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}}) + \widetilde{\mathrm{ch}}^{\mathrm{o}}(\nabla^{\mathcal{H}},g^{\mathcal{H}},g^{\mathcal{H}}_{\mathcal{V}})$ is closed, and defines a class τ on $Wh^u(\mathcal{M}_r(\mathbb{C}),U(r))$ by Stokes theorem. Finally, we show that τ is compatible with respect to stabilisation in r and thus extends to a form on $Wh^u(\mathcal{M}(\mathbb{C}),U)$ with $\mathcal{M}(\mathbb{C}) = \lim_\to \mathcal{M}_r(\mathbb{C})$ and $U = \lim_\to U(r)$.

9.5. Definition. Define $\tau = \tau(\mathcal{M}_r(\mathbb{C}),U(r)) \in H^{\mathrm{even}}\big(Wh'^u(\mathcal{M}_r(\mathbb{C}),U(r)),\mathbb{R}\big)$ to be the class represented by the cocycle which assigns to every simplex Δ in $\mathcal{W}h'^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ the integral

$$\int_\Delta \Big(T\big(\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}}\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{\mathcal{H}},g^{\mathcal{H}},g^{\mathcal{H}}_{\mathcal{V}}\big)\Big) .$$

Then $\tau^{[0]}$ measures the difference of the Franz-Reidemeister metric on $\det H$ and the metric $g^{\det H}$ induced by $g^{\mathcal{H}}$. Let (V,A',h^V) be a family Thom-Smale complex over B such that ∇^V is unitarily flat and such that $H \to B$ admits a parallel metric. If S is a simplicial structure on B, there exists a "classifying functor" $\Xi\colon S \to \mathcal{W}h'^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r))$, inducing a classifying map $\xi\colon B \to Wh'^u(\mathcal{M}_r(\mathbb{C}),U(r))$ up to homotopy, such that

$$\big(V,\nabla^V,A',g^V\big) \cong \xi^*\big(\mathcal{V},\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}}\big) \qquad \text{and} \qquad \big(H,\nabla^H,g^H\big) \cong \xi^*\big(\mathcal{H},\nabla^{\mathcal{H}},g^H\big) .$$

We have an immediate generalisation of Corollary 7.30.

9.6. Corollary. *Let S be a simplicial complex on B and let $\Xi\colon S \to \mathcal{W}h'^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r))$ be a simplicial map. Let $(V,\nabla^V,A',g^V) = |\Xi|^*(\mathcal{V},\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}})$ and $(H,\nabla^H,g^H) = |\Xi|^*(\mathcal{H},\nabla^{\mathcal{H}},g^H)$, then*

$$T\big(\nabla^V,A',g^V\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H,g^H,g^H_V\big) = |\Xi|^*\tau \quad \in H^{\mathrm{even}}(|S|,\mathbb{R}) . \tag{1}$$

In particular, if $p\colon M \to B$ is a proper submersion with a fibrewise Morse-function $h\colon M \to \mathbb{R}$, if $(F,\nabla^F,g^F) \to M$ is a unitarily flat vector bundle, such that the bundle $H = H^*(V,a_0) \to B$ admits a parallel metric, and if $\xi^u\colon B \to Wh'^u(\mathcal{M}_r(\mathbb{C}),U(r))$ is constructed as in Remark 9.3 (3), then

$$T(M/B,F,h) = \xi^{u*}\tau^{[\geq 2]} . \tag{2}$$

This follows from the same kind of arguments as Corollary 7.30. Note here that since the parallel metric on H can be varied, we cannot define the ordinary Franz-Reidemeister torsion in degree 0. On the other hand, $\xi^{u*}\tau^{[\geq 2]}$ is independent of the particular lift ξ^u by Corollary 7.27. For this reason, the constant function $\xi^{u*}\tau^{[0]}$ is explicitly excluded from (2).

9.7. Proposition. *The unitary torsion class $\tau = \tau(\mathcal{M}_r(\mathbb{C}), U(r)) \in H^{\mathrm{even}}(Wh'^u(\mathcal{M}_r(\mathbb{C}), U(r)), \mathbb{R})$ is compatible with stabilisation and descends to $Wh^u(\mathcal{M}_r(\mathbb{C}), U(r))$.*

Proof. We have to show that τ is compatible with the elementary expansions in $\mathcal{W}h^{\mathrm{diff},u}$. Therefore, let $P^{[l]}$ and $a'^{[l]} = \left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right)$ as in Definition 6.28. We take the trivial metric on $(\mathbb{C}^r)^{P^{[l]}}$. Then by Corollary 7.24,

$$T\big(d, d + a'^{[l]}, g^{\mathrm{triv}}\big) = T^{\mathrm{BL}}\big(d + a'^{[l]}, g^{\mathrm{triv}}\big) = 0 \; .$$

It follows that the explicit cocycle for τ in Definition 9.5 takes the same value on (P, a, g^H) as on $(P \cup P^{[l]}, a' \oplus a'^{[l]}, g^H)$. The proposition follows from the definition of the stabilised differential unitary Whitehead space. $\square$

We will use the universal torsion class $\tau \in H^{\mathrm{even}}(Wh^u(\mathcal{M}_r(\mathbb{C}), U(r)))$ in Section 10 to extend Igusa's higher Franz-Reidemeister torsion to smooth fibre bundles with unitarily flat fibre-wise cohomology.

For all r, $r' \geq 0$, there are well-known natural homomorphisms

$$\mathcal{M}_r(\mathbb{C}) \longrightarrow \mathcal{M}_{r+r'}(\mathbb{C}) \qquad \text{and} \qquad U(r) \longrightarrow U(r+r')$$

given by extending a matrix in $\mathcal{M}_r(\mathbb{C})$ by zero, and an element $g \in U(r)$ to $g \oplus \mathrm{id}_{\mathbb{C}^{r'}}$. Let

$$\mathcal{M}(\mathbb{C}) = \lim_{r\to\infty} \mathcal{M}_r(\mathbb{C}) \qquad \text{and} \qquad U = \lim_{r\to\infty} U(r)$$

denote the direct limits with respect to these maps.

The maps $\mathcal{M}_r(\mathbb{C}) \to \mathcal{M}_{r+r'}(\mathbb{C})$ and $U(r) \to U(r+r')$ induce a functor

$$\mathcal{W}h\big(\mathcal{M}_r(\mathbb{C}), U(r)\big) \longrightarrow \mathcal{W}h\big(\mathcal{M}_{r+r'}(\mathbb{C}), U(r+r')\big) \; . \tag{9.8}$$

This gives rise to a pullback diagramme

$$\begin{array}{ccc} \mathcal{V} \oplus \mathbb{C}^{r' \,\mathrm{rk}\, \mathcal{V}/r} & \longrightarrow & \mathcal{V} \\ \big\downarrow & & \big\downarrow \\ Wh(\mathcal{M}_r(\mathbb{C}), U(r)) & \longrightarrow & Wh(\mathcal{M}_{r+r'}(\mathbb{C}), U(r+r')) \; . \end{array}$$

The additional trivial bundle $\mathbb{C}^{r' \,\mathrm{rk}\, \mathcal{V}/r}$ carries the trivial connection d, and since all coefficients a'_* of $\mathcal{A}'$ are extended trivially, the pull-back of $\mathcal{A}'$ acts as d on $\mathbb{C}^{r' \,\mathrm{rk}\, \mathcal{V}/r}$. Thus, we have another pullback diagramme

$$\begin{array}{ccc} \mathcal{H} \oplus \mathbb{C}^{r' \,\mathrm{rk}\, \mathcal{V}/r} & \longrightarrow & \mathcal{H} \\ \big\downarrow & & \big\downarrow \\ Wh(\mathcal{M}_r(\mathbb{C}), U(r)) & \longrightarrow & Wh(\mathcal{M}_{r+r'}(\mathbb{C}), U(r+r')) \; . \end{array} \tag{9.9}$$

We extend the functor (9.8) to a functor

$$\mathcal{W}h^u\big(\mathcal{M}_r(\mathbb{C}), U(r)\big) \longrightarrow \mathcal{W}h^u\big(\mathcal{M}_{r+r'}(\mathbb{C}), U(r+r')\big) \tag{9.10}$$

by specifying the standard basis on the additional summand $\mathbb{C}^{r' \,\mathrm{rk}\, \mathcal{V}/r}$ in (9.9).

9.11. Proposition. *The universal unitary higher torsion class τ is compatible with stabilisation*

$$Wh^u\bigl(\mathcal{M}_r(\mathbb{C}),U(r)\bigr) \longrightarrow Wh^u\bigl(\mathcal{M}_{r+r'}(\mathbb{C}),U(r+r')\bigr)\;.$$

Proof. We use the differential model of the Whitehead space. On a single simplex $\sigma=(P,a',e)\in\mathcal{W}h^{\mathrm{diff},u}(\mathcal{M}_r(\mathbb{C}),U(r))$, the pullback of $\tau(\mathcal{M}_{r+r'}(\mathbb{C}),U(r+r'))$ is given by

$$\begin{aligned}\int_\Delta\Bigl(T\bigl(\nabla^{\mathcal{V}}\oplus d,\mathcal{A}'\oplus d,g^{\mathcal{V}}\oplus g^{\mathrm{triv}}\bigr)+\widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^{\mathcal{H}}\oplus d,g^{\mathcal{H}}\oplus g^{\mathrm{triv}},g^{\mathcal{H}}_{\mathcal{V}}\oplus g^{\mathrm{triv}}\bigr)\Bigr)&\\ =\int_\Delta\Bigl(T\bigl(\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}}\bigr)+T^{\mathrm{BL}}\bigl(d,g^{\mathrm{triv}}\bigr)+\widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^{\mathcal{H}},g^{\mathcal{H}},g^{\mathcal{H}}_{\mathcal{V}}\bigr)\Bigr)&\\ =\int_\Delta\Bigl(T\bigl(\nabla^{\mathcal{V}},\mathcal{A}',g^{\mathcal{V}}\bigr)+\widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^{\mathcal{H}},g^{\mathcal{H}},g^{\mathcal{H}}_{\mathcal{V}}\bigr)\Bigr)&=\tau\bigl(\mathcal{M}_r(\mathbb{C}),U(r)\bigr)(\sigma)\;,\end{aligned}$$

because the Bismut-Lott torsion vanishes for a trivial superconnection. ☐

9.12. Definition. We define the *universal unitary Whitehead category* by

$$\mathcal{W}h^u\bigl(\mathcal{M}(\mathbb{C}),U\bigr)=\lim_{r\to\infty}\mathcal{W}h^u\bigl(\mathcal{M}_r(\mathbb{C}),U(r)\bigr)$$

using the functors (9.10), and let the *universal unitary Whitehead space* $Wh^u(\mathcal{M}(\mathbb{C}),U)$ be its geometric realisation. We still denote the *universal unitary higher torsion class* by

$$\tau\in H^{\mathrm{even}}\bigl(Wh^u\bigl(\mathcal{M}(\mathbb{C}),U\bigr)\bigr)\;.$$

9.c. The cone construction. In this section, let $(V,A',h^V)\to B$ be an arbitrary family Thom-Smale complex and let g^V be an arbitrary adapted metric on V which need not be parallel. We construct a fibre-wise acyclic family Thom-Smale complex $\bigl(\widehat{V},\widehat{A}',h^{\hat{V}}\bigr)$ on the vector bundle $\widehat{V}=V\oplus H$. In the next subsection, we will compare the torsion classes $T(\nabla^V,A',g^V)$ and $T\bigl(\nabla^{\hat{V}},\widehat{A}',g^{\hat{V}}\bigr)$.

We fix an adapted metric g^V on V in the sense of Definition 2.37 and let a_0'' denote the adjoint of a_0' with respect to g^V. By finite-dimensional Hodge theory, we have a g^V-orthogonal splitting

$$V=\ker\bigl(a_0'+a_0''\bigr)\oplus\operatorname{im}\bigl(a_0'\bigr)\oplus\operatorname{im}\bigl(a_0''\bigr)\;,$$

which is preserved by the combinatorial Laplacian $[a_0',a_0'']$. Let $\iota\colon H\to\ker(a_0'+a_0'')\subset V$ denote the identification of H with the harmonic elements, which we extend $\Omega^*(B)$-linearly to a map $\iota\colon\Omega^*(B;H)\to\Omega^*(B;V)$.

We can find a smooth function $\varepsilon\colon B\to\mathbb{R}_>$ such that $|\lambda|>\varepsilon$ for all non-zero eigenvalues λ of $[a_0',a_0'']$. Then we define a generalised spectral projection Π on $\Omega^*(B;V)$ by

$$\Pi=\frac{1}{2\pi i}\int_{\varepsilon S^1}\frac{d\lambda}{\lambda-(A'+a_0'')^2}\;.$$

Similar operators were already considered in [BG], Definition 10.13. The analogue of Theorem 10.9 in [BG] implies in particular that $\Pi^2=\Pi$, that Π is of total degree 0, and that Π commutes with A', a_0'' and $\Omega^*(B)$. Moreover, the projection $\Pi^{[0]}$ in degree 0 with respect to N^B coincides with the orthogonal projection onto the harmonic elements of V.

Let $W\subset\Omega^*(B;V)$ be the image of Π, then W is an A'-invariant $\Omega^*(B)$-submodule of $\Omega^*(B;V)$ and $\Pi\circ\iota\colon\Omega^*(B;H)\to W$ is an isomorphism. Let

$$A'^H=(\Pi\circ\iota)^{-1}\circ A'\circ(\Pi\circ\iota) \tag{9.13}$$

be the pullback superconnection on H.

9.14. Lemma. *The superconnection A'^H takes the form*

$$A'^H = \nabla^H + a'^H \qquad \text{with } a'^{H,[k]} \in \Omega^{\geq 2}(B; \operatorname{End} H) \; . \tag{1}$$

Moreover, $\Pi \circ \iota$ induces an $\Omega^*(B)$-linear isomorphism of cochain complexes

$$\Pi \circ \iota\colon \bigl(\Omega^*(B;H), A'^H\bigr) \longrightarrow (W, A') \subset \bigl(\Omega^*(B;V), A'\bigr) \; . \tag{2}$$

We will see later that (1) means that the difference between A'^H and ∇^H is almost negligible.

Proof. Both Π and ι are $\Omega^*(B)$-linear. Because $\Pi^{[0]}$ coincides with the projection onto the $(a_0'+a_0'')$-harmonic elements of V, it is clear that we obtain an isomorphism. Because Π commutes with A', it is clear that A' acts on W. It remains to show that the restriction of A' to W pulls back to a flat superconnection on H, which starts with ∇^H in degree 1.

We write the superconnection A'^H on H of (9.13) as

$$A'^H = \nabla^H + \sum_{i\geq 0} a_i'^H$$

with $a_i'^H \in \Omega^i(B; \operatorname{End} H)$. Let $\eta \in \Omega^0(B; \ker(a_0' + a_0''))$ represent a section of H, then clearly

$$a_0'^H \eta = \bigl(\Pi \circ \iota \circ A'^H\bigr)^{[0]} \eta = \bigl(A' \circ \iota \circ \Pi\bigr)^{[0]} \eta = a_0' \eta = 0 \; .$$

To calculate $a_1'^H$, note that since $a_0'^H = 0$, we have

$$\begin{aligned} \iota\left(\nabla^H + a_1'^H\right)\eta &= \bigl(\Pi \circ \iota \circ A'^H\bigr)^{[1]} \eta = \bigl(A' \circ \Pi \circ \iota\bigr)^{[1]} \eta \\ &= \left(\nabla^V + a_1'\right)\eta + a_0' \, \Pi^{[1]} \eta \quad \in \quad \Omega^1\bigl(B; \ker(a_0' + a_0'')\bigr) \; . \end{aligned}$$

Since the result is in $\Omega^1(B; \iota H)$ whereas $\operatorname{im}(a_0') \perp \iota H$, we also get

$$\iota\left(\nabla^H + a_1'^H\right)\eta = \Pi^{[0]}\left(\nabla^V + a_1'\right)\eta \; ,$$

but $\Pi^{[0]}\left(\nabla^V + a_1'\right)$ just represents the Gauß-Manin connection ∇^H on H, so $a_1'^H = 0$ as well. $\square$

As a consequence, we can now define a flat superconnection on $\widehat{V} = V \oplus H^{[-1]}$ of the form

$$\widehat{A}' = \begin{pmatrix} A' & \Pi \circ \iota \\ 0 & A'^H \end{pmatrix} \; , \tag{9.15}$$

where $H^{[-1]}$ denotes the $\mathbb{Z}$-graded bundle H with degree shifted down by one. The superconnection $\widehat{A}'$ is apparently flat and of total degree 1. By assumption, $A' - \nabla^V$ takes values in a nipotent subalgebra of $\operatorname{End} V$. Similarly, $A'^H - \nabla^H$ takes values in the nilpotent subalgebra of $\operatorname{End} H$ that increases the vertical degree. We conclude that $\widehat{A}' - \nabla^{\widehat{V}}$ also takes values in a nilpotent subalgebra. Moreover, the cohomology H of (V, a_0) is killed by $H^{[-1]}$, so the fibre-wise operator $\hat{a}_0' = a_0' + \Pi^{[0]} \circ \iota$ is acyclic.

9.16. Definition. Let (V, A', ∇^V, g^V) be a family Thom-Smale complex with an adapted metric, and let g^H be a metric on the vector bundle $H = H^*(V, a_0') \to B$. Then we call $\bigl(\widehat{V}, \widehat{A}', \nabla^{\hat{V}}, g^{\hat{V}}\bigr)$ the *acyclic cone* of (V, A', ∇^V).

9.17. Remark.

(1) The acyclic cone construction as a fibre-wise acyclic family Thom-Smale-complex is up to homotopy independent of the choice of metrics g^V and g^H. This follows from a standard argument involving an extension of metrics from $B \times \{0,1\}$ to $B \times [0,1]$ on the pullbacks of V and H via the projection $B \times [0,1] \to B$.

(2) If g^V and g^H are unitarily flat, then so is $g^{\hat{V}} = g^V \oplus g^H$.

(3) We would like to turn the acyclic cone construction into a functor from the universal unitary Whitehead category $\mathcal{W}h^u(\mathcal{M}(\mathbb{C}), U)$ to a suitable acyclic Whitehead category. Although the whole construction above is natural, we encounter a technical problem. We start in $\mathcal{W}h^u(\mathcal{M}_r(\mathbb{C}), U(r))$ for some fixed r. The dimension of the fibre-wise cohomology $\mathcal{H} \to Wh^u(\mathcal{M}_r(\mathbb{C}), U(r))$ depends on the connected component of $Wh^u(\mathcal{M}_r(\mathbb{C}), U(r))$ and is in general not equal to r. This means that $\mathcal{W}h^h(\mathcal{M}_r(\mathbb{C}), U(r))$ is not a good receptacle for a universal cone construction. Taking the limit $\lim_{r\to\infty} \mathcal{W}h^h(\mathcal{M}_r(\mathbb{C}), U(r))$ is difficult because our stabilisation procedure in Definition 9.12 destroys acyclicity. Similarly, if we pass to $\lim_{r\to\infty} Wh^u(\mathcal{M}(\mathbb{C}), U)$, then the dimension of the bundle $\widehat{\mathcal{V}}$ still depends on r. To solve this problem, we would need a generalisation of the acyclic Whitehead space that allows to take the limit $r \to \infty$ in a natural way.

9.d. Unitary and acyclic higher torsion. We show that the analytic torsion of a family Thom-Smale complex (V, ∇^V, A') is related to the torsion of its acyclic cone $\bigl(\widehat{V}, \nabla^{\hat{V}}, \widehat{A}', g^{\hat{V}}\bigr)$. If both V and H carry parallel metrics, this allows us to reduce the calculation of the unitary torsion to the acyclic case.

Let (V, ∇^V, A') be a family Thom-Smale complex with an adapted metric g^V, and let $\bigl(\widehat{V}, \nabla^{\hat{V}}, \widehat{A}'\bigr)$ be the acyclic cone of Definition 9.16. We have a natural filtration

$$\bigl(\widehat{V}, \widehat{A}'\bigr) \supset (V, A') \supset 0 \; , \tag{9.18}$$

to which $g^{\hat{V}} = g^V \oplus g^H$ is adapted in the sense of Definition 7.35. The metric $g^{\hat{V}}$ is parallel iff g^V and g^H are.

9.19. Theorem. *Let (V, ∇^V, A') be a family Thom-Smale complex, let $\bigl(\widehat{V}, \nabla^{\hat{V}}, \widehat{A}'\bigr)$ be its acyclic cone with respect to the metric g^V, and let $g^{\hat{V}} = g^V \oplus g^H$. Then modulo exact forms, we have*

$$T\bigl(\nabla^V, A', g^V\bigr) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^H, g^H, g^H_V\bigr) = T\bigl(\nabla^{\hat{V}}, \widehat{A}', g^{\hat{V}}\bigr) \; .$$

Proof. The bundle $\widehat{V}$ has a two-step filtration (9.18) and carries an adapted metric $g^{\hat{V}} = g^V \oplus g^H$. Moreover, $\bigl(\widehat{V}, \hat{a}_0'\bigr)$ is acyclic by construction. In other words, the spectral sequence associated to (9.18) degenerates at $E_2 = 0$.

The E_0-term of the spectral sequence is the direct sum

$$\bigl(E_0, \nabla^{E_0}, D_0', g^{E_0}\bigr) = \bigl(V, \nabla^V, A', g^V\bigr) \oplus \bigl(H^{[-1]}, \nabla^H, A'^H, g^H\bigr) \; ,$$

and its torsion is given by

$$T\bigl(\nabla^{E_0}, D_0', g^{E_0}\bigr) = T\bigl(\nabla^V, A', g^V\bigr) - T\bigl(\nabla^H, A'^H, g^H\bigr) \; .$$

We extend A'^H to a flat superconnection on $\overline{H} = H \times (0,1]$ by

$$\overline{A}'^H = s^{N^H} \left(A'^H + \frac{\partial}{\partial s} \, ds \right) s^{-N^H}$$

and set

$$\nabla^{\overline{H}} = \left(\overline{A}'^H\right)^{[1]} = \nabla^H + \left(\frac{\partial}{\partial s} - \frac{N^H}{s} \right) ds \, .$$

Then we still have

$$\overline{A}'^H = \nabla^{\overline{H}} + \sum_{i \geq 2} \bar{a}_i'^H$$

with $\bar{a}_i'^H \in \Omega^i\left(B \times (0,1], \operatorname{End} \overline{H}\right)$.

We note that $\overline{H}$ is its own fibre-wise cohomology, and if $g^{\overline{H}}$ is a metric on $\overline{H}$, then the induced metric on the fibre-wise cohomology is again $g^{\overline{H}}$, so

$$dT\left(\nabla^{\overline{H}}, \overline{A}'^H, g^{\overline{H}}\right) = \operatorname{ch}^{\mathrm{o}}\left(\nabla^{\overline{H}}, g^{\overline{H}}\right) - \operatorname{ch}^{\mathrm{o}}\left(\nabla^{\overline{H}}, g^{\overline{H}}\right) = 0 \, .$$

We may thus take $g^{\overline{H}}$ to be the pullback of g^H. It is now easy to see that

$$\lim_{s \to 0} T\left(\nabla^{\overline{H}}, \overline{A}'^H, g^{\overline{H}}\right)\big|_{B \times \{s\}} = T^{\mathrm{BL}}\left(\nabla^H, g^H\right) = 0 \, ,$$

and this is cohomologous to $T(\nabla^H, A'^H, g^H)$. We conclude that

$$T\left(\nabla^{E_0}, D_0', g^{E_0}\right) = T\left(\nabla^V, A', g^V\right) \quad \in \quad \Omega^{\mathrm{even}}(B)/d\Omega^{\mathrm{odd}}(B) \, . \tag{9.20}$$

The E_1 term of our spectral sequence is given by

$$\left(E_1, \nabla^{E_1}, D_1', g^{E_1}\right) = \left(H \oplus H^{[-1]}, \nabla^{H \oplus H}, D_1', g_V^H \oplus g^H\right) \, ,$$

where the superconnection D_1' on $H \oplus H^{[-1]}$ in matrix notation equals

$$D_1' = \begin{pmatrix} \nabla^H & \mathrm{id} \\ & \nabla^H \end{pmatrix} \, .$$

By the appendix of [BL], we conclude that

$$\begin{aligned} T^{\mathrm{BL}}\left(D_1', g^{E_1}\right) &= T^{\mathrm{BL}}\left(D_1', g_V^H \oplus g^H\right) \\ &= \widetilde{\operatorname{ch}}^{\mathrm{o}}\left(\nabla^H, g^H, g_V^H\right) \, . \end{aligned} \tag{9.21}$$

By Theorem 7.37, we now have

$$\begin{aligned} T\left(\nabla^{\hat{V}}, \widehat{A}', g^{\hat{V}}\right) &= T\left(\nabla^{E_0}, D_0', g^{E_0}\right) + T^{\mathrm{BL}}\left(\nabla^{E_1}, D_1', g^{E_1}\right) \\ &= T\left(\nabla^V, A', g^V\right) + \widetilde{\operatorname{ch}}^{\mathrm{o}}\left(\nabla^H, g^H, g_V^H\right) \, . \qquad \square \end{aligned}$$

9.22. Remark. If we assume that both g^V and g^H are parallel, then Theorem 9.19 gives us an identity of cohomology classes on B. It shows that the unitary torsion can be completely reduced to the torsion in the acyclic case. This is another justification for our definition of τ on $Wh^u(\mathcal{M}(\mathbb{C}), U)$.

We recall that the torsion on the acyclic Whitehead space $Wh^h(\mathcal{M}_r(\mathbb{C}), U(r))$ is completely determined by its restriction to the two-index Whitehead space $Wh^{h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$, which is a deformation retract of $Wh^h(\mathcal{M}_r(\mathbb{C}), U(r))$ by Igusa's two-index theorem in [I2], Theorem 5.7.1. It would be nice to know that $\tau \in H^{\mathrm{even}}\left(Wh^u(\mathcal{M}(\mathbb{C}), U)\right)$ is also determined by its restriction to $Wh^{h,\{0,1\}}(\mathcal{M}_r(\mathbb{C}), U(r))$. Unfortunately, for the reasons explained in Remark 9.17 (3), we do not obtain this conclusion immediately.

10. The Framing Principle and a higher Cheeger-Müller Theorem

We use Igusa's framing principle to establish Theorem 0.3 for fibre bundles with fibrewise Morse functions and fibrewise acyclic flat vector bundles. We establish a cohomological analogue of Igusa's splitting lemma. Using this result and Theorem 0.3, we prove the generalised second statement of Theorem 0.2. Throughout this chapter, we will use the complex matrix ring $R = \mathcal{M}(\mathbb{C}) = \lim_{\to} \mathcal{M}_r(\mathbb{C})$ and the unitary group $G = U = \lim_{\to} U(r)$. We let $\tau = \tau(\mathcal{M}(\mathbb{C}), U)$ denote the universal unitary torsion class on $Wh^u(\mathcal{M}(\mathbb{C}), U)$.

10.a. Igusa's framed function theorem. We recall the definition of generalised fibre-wise Morse functions and framed functions in the sense of [I1]. Framed functions always exist by Igusa's framed function theorem. By [I2], Chapter 4, each generalised Morse function gives rise to a classifying map $\xi_{p,h,F}: B \to Wh(\mathcal{M}(\mathbb{C}), U)$. Suppose that this map lifts to $Wh^u(\mathcal{M}(\mathbb{C}), U)$, then we can define a torsion class $\tau(M/B; F, h)$ by pulling back the universal torsion class $\tau(\mathcal{M}(\mathbb{C}), U) \in Wh^u(\mathcal{M}(\mathbb{C}), U)$.

Let $p: M \to B$ be a fibre-bundle with n-dimensional fibres, and let $h: M \to \mathbb{R}$ be a smooth function. We denote by $C \subset M$ the set of fibre-wise singularities, i.e., the zero set of $dh|_{TX}$. The subset of fibre-wise Morse singularities will be denoted C_{Morse}.

10.1. Definition. A point $p \in C_b \subset X_b$ is called a *birth-death singularity of index* k if there is a neighbourhood $V \subset B$ of b, a function $f_0: V \to \mathbb{R}$, a local trivialisation of $p^{-1}V \cong B \times X$ with coordinates $b_1, \dots, b_m$ of V and $x_0, \dots, x_n$ on X_b of X around p, such that

$$f(q) = f_0(b_1, \dots, b_m) - \frac{x_1^2}{2} - \dots - \frac{x_k^2}{2} + \frac{x_{k+1}^2}{2} + \dots + \frac{x_{n-1}^2}{2} + \frac{x_n^3}{3} - b_1 x_n$$

for all $q \in p^{-1}V$ near p. Let $C_{\text{b-d}} \subset C$ denote the set of all birth-death singularities of h.

10.2. Definition. Let $M \to B$ be a fibre bundle with fibre-wise boundary $\partial M = M_0 \cup M_1$. A *generalised fibre-wise Morse function* is a function $h: M \to [0,1]$ such that 0 and 1 are regular values with $h^{-1}(i) = M_i$ for $i \in \{0,1\}$, and such that h has only fibre-wise Morse singularities and birth-death singularities, i.e., a function with the property that $C = C_{\text{Morse}} \cup C_{\text{b-d}}$.

Then as in [I2], Chapter 4, each proper generalised fibre-wise Morse function $h: M \to \mathbb{R}$ with finitely many singularities on each fibre gives rise to a classifying map $\xi_{p,h,F}: B \to Wh(\mathcal{M}(\mathbb{C}), U)$ up to homotopy. Assume that g^F and g^H are parallel metrics on $F \to M$ and $H \to B$, then we also have a lift $\xi^u_{p,h,F}: B \to Wh^u(\mathcal{M}(\mathbb{C}), U)$. We can now combine Igusa's definition of the higher Franz-Reidemeister torsion with our definition of Morse torsion forms.

10.3. Definition. Assume that $F \to M$ and $H \to B$ admit parallel metrics. Let $h: M \to \mathbb{R}$ be a proper generalised fibre-wise Morse function with finitely many fibre-wise singularities on each fibre, and let $\xi^u_{p,h,F}: B \to Wh^u(\mathcal{M}(\mathbb{C}), U)$ be the corresponding classifying map. Then we define the *higher Franz-Reidemeister torsion*

$$\tau(M/B; F, h) = \big(\xi^u_{p,h,F}\big)^* \tau\big(\mathcal{M}(\mathbb{C}), U\big)^{[\geq 2]} \quad \in H^{\text{even},\geq 2}(B; \mathbb{R}) \ .$$

We could also define $\tau(M/B; F, h)^{[0]} = \tau(M/B; F, h, g^V, g^H)^{[0]}$ in degree 0, however, it seems better to work with Franz-Reidemeister metrics instead.

Note that $\tau(M/B; F, h)$ still depends on the choice of h. To define an independent invariant of $p: M \to B$ and $F \to M$ only, Igusa further restricts the space of admissible functions h. Up to homotopy, there is a unique splitting of the vertical tangent bundle

$$TX|_{C_{\text{Morse}}} = T^s_{\text{Morse}}X \oplus T^u_{\text{Morse}}X \ \longrightarrow\ C_{\text{Morse}}$$

such that d^2h is positive (negative) definit on $T^s_{\mathrm{Morse}}X$ ($T^u_{\mathrm{Morse}}X$). There is a similar splitting

$$TX|_{C_{\text{b-d}}} = T^s_{\text{b-d}}X \oplus T^u_{\text{b-d}}X \oplus T^c_{\text{b-d}}X \longrightarrow C_{\text{b-d}} ,$$

where $T^c_{\text{b-d}} = \ker(d^2h) \cong \mathbb{R} \times C_{\text{b-d}}$ is the unique cubic direction corresponding to the coordinate x_n in Definition 10.1, which is oriented by the third derivative of h.

In order to obtain a somewhat unique generalised fibre-wise Morse function, Igusa considers functions h such that the bundles $T^u_{\mathrm{Morse}}X$ and $T^u_{\text{b-d}}X$ admit compatible trivialisations. In the coordinates of Definition 10.1, let

$$C^{\pm} = \{\, q \mid x_1(q) = \cdots = x_{n-1}(q) = 0,\ b_1 > 0, \text{ and } x_n = \pm\sqrt{b_1} \,\} \subset C_{\mathrm{Morse}} ,$$

then C^+ has fibre-wise Morse index $k+1$ and C^- has index k.

10.4. Definition. Trivialisations $T^u_{\mathrm{Morse}}X \cong C_{\mathrm{Morse}} \times \mathbb{R}^{\operatorname{ind}_h}$ and $T^u_{\text{b-d}}X \cong C_{\text{b-d}} \times \mathbb{R}^{\operatorname{ind}_h}$ are called *compatible* if near each point $p \in C_{\text{b-d}}$, the trivialisation of $T^u_{\text{b-d}}X$ near p extends continously to the trivialisation of $T^u_{\mathrm{Morse}}X|_{C^-}$, and the trivialisation of $T^u_{\text{b-d}}X \oplus T^c_{\text{b-d}}X$ extends continously to the trivialisation of $T^u_{\mathrm{Morse}}X|_{C^+}$. A *framed function* is a generalised fibre-wise Morse function on M with fixed compatible trivialisations of $T^u_{\mathrm{Morse}}X$ and $T^u_{\text{b-d}}X$.

With this definition, we can now state a version of Igusa's framed function theorem ([I1]).

10.5. Fact (Framed Function Theorem). *Let $p\colon M \to B$ be a fibre bundle with fibre-wise boundary $\partial M = M_0 \cup M_1$. If $\dim M \geq 2\dim B$, then M carries a framed function $h\colon M \to [0,1]$ with $h^{-1}(i) = M_i$ for $i = 0, 1$. Let $p\colon M \to B$ be a fibre bundle with $\dim M > 2\dim B$, and let h_0, $h_1\colon M \to \mathbb{R}$ be framed functions as above, then there exists a framed function h on $M \times [0,1] \to B \times [0,1]$ with the same properties that restricts to the given functions and framings on $M \times \{0,1\}$.*

This implies that for $\dim M > 2\dim B$, the homotopy class of $\xi_{p,h,F}\colon B \to Wh(\mathcal{M}(\mathbb{C}),U)$ is independent of the choice of the framed function h. Of course, the same holds for any lift $\xi^u_{p,h,F}\colon B \to Wh^u(\mathcal{M}(\mathbb{C}),U)$ with respect to a fixed parallel metric on H. Igusa defines a unique higher Franz-Reidemeister torsion for unitarily flat bundles $F \to M$ such that $H^*(M/B;F) \to B$ is trivial, cf. [I2], Theorem 5.7.5. We extend Igusa's definition as follows.

10.6. Definition. Assume that $M \to B$ is a smooth fibre bundle with $\dim M > 2\dim B$, with compact fibres, and with fibre-wise boundary $\partial M = M_0 \cup M_1 \to B$. Assume that 0 and 1 are regular values of the framed function $h\colon M \to [0,1]$ with $h^{-1}(0) = M_0$ and $h^{-1}(1) = M_1$. Let $F \to M$ be a unitarily flat vector bundle such that the fibre-wise cohomology $H^*(M/B;F) \to B$ admits a parallel metric. Let $\xi^u_{p,h,F}\colon B \to Wh^u(\mathcal{M}(\mathbb{C}),U)$ be the classifying map defined by these data. Then the *unitary higher Franz-Reidemeister torsion* of $(M,M_0) \to B$ and F is defined as

$$\tau(M/B,M_0/B;F) = \tau(M/B,M_0/B;F,h) = \left(\xi^u_{p,h,F}\right)^*\tau; .$$

If $M \to B$ is as above but with $\dim M \leq 2\dim B$, then for N sufficiently large, define

$$\tau(M/B,M_0/B;F) = \tau\big(M \times \mathbb{R}P^{2N}/B\,,\ M_0 \times \mathbb{R}P^{2N}/B\,;\ F\big) .$$

It is an easy consequence of Fact 10.5 that τ is well-defined. For suppose that $h_0\colon M \times \mathbb{R}P^{2N_0} \to [0,1]$ and $h_1\colon M \times \mathbb{R}P^{2N_1} \to [0,1]$ are framed functions with fibre-wise compact sublevels and finitely many critical points on each fibre. Then both functions can be extended to framed functions on $\bar h_{0,1}\colon M \times \mathbb{R}P^{2N_0} \times \mathbb{R}P^{2N_1} \to [0,1]$ by adding a fixed Morse function $\mathbb{R}P^{2N_i} \to (-\varepsilon,\varepsilon)$ on the additional factor $\mathbb{R}^{N_1}$ or $\mathbb{R}^{N_0}$ and correcting near the boundary components. The functions $\bar{h}_0$ and $\bar{h}_1$ can then be joined by a framed function on the bundle

$$\big(M \times \mathbb{R}P^{2N_0} \times \mathbb{R}P^{2N_1}\big) \times [0,1] \longrightarrow B \times [0,1] .$$

Thus, we clearly have

$$\begin{aligned}\tau\big((M \times \mathbb{R}P^{N_0}, M_0 \times \mathbb{R}P^{N_0})/B; F, h_0\big)& \\ &= \tau\big((M \times \mathbb{R}P^{2N_0} \times \mathbb{R}P^{2N_1}, M_0 \times \mathbb{R}P^{2N_0} \times \mathbb{R}P^{2N_1})/B; F, \bar{h}_0\big) \\ &= \tau\big((M \times \mathbb{R}P^{2N_0} \times \mathbb{R}P^{2N_1}, M_0 \times \mathbb{R}P^{2N_0} \times \mathbb{R}P^{2N_1})/B; F, \bar{h}_1\big) \\ &= \tau\big((M \times \mathbb{R}P^2 N_1, M_0 \times \mathbb{R}P^{N_1})/B; F, h_1\big) .\end{aligned}$$

10.7. Remark. Assume that $M \to B$ is a fibre-bundle with closed fibres and that $h: M \to (0,1)$ is a fibre-wise Morse function. Then by Theorem 6.20 and Theorem 8.1, we clearly have

$$\tau_{\mathrm{ch}}(M/B; F, h) = \big(\xi^u_{p,h,F}\big)^* \tau\big(\mathcal{M}(\mathbb{C}), U\big) .$$

If h is framed, then

$$\tau(M/B; F) = T(M/B; F, h) .$$

Nevertheless, Igusa's Definition is much more general than Definition 2.64. One need not assume that h is fibre-wise Morse, so $\tau(M/B; F)$ is defined for all bundles with compact fibres by Fact 10.5, and one can work with fibers that have a boundary.

10.b. Igusa's framing principle. We recall Igusa's framing principle, and we use it to compare the higher analytic torsion of Bismut-Lott with Igusa's higher Franz-Reidemeister torsion if there exists a fibre-wise Morse function and a fibre-wise acyclic flat vector bundle. At this point, the identity of Igusa's and our higher torsion on $Wh^h(\mathcal{M}_r(\mathbb{C}), U(r))$ proved in Section 8 is crucial, since we need both torsions within a single equation.

Recall that the additive characteristic class ${}^0J(E)$ was defined in [BG] by

$$ {}^0J(T^uX) = \frac{1}{2} \sum_{k=1}^{\infty} \zeta'(-2k)\ \mathrm{ch}(E)^{[4k]} \in H^{4*}(M;\mathbb{R}) \tag{10.8}$$

for an arbitrary vector bundle E over a manifold M. Let now h be a fibre-wise Morse function. The pushdown or "transfer" map $\hat{p}_*: H^*(C;\mathbb{R}) \to H^*(B;\mathbb{R})$ produces a class

$$\hat{p}_*\big((-1)^{\mathrm{ind}\, h}\ {}^0J(T^uX)\big) \in H^{4*}(B;\mathbb{R}) .$$

This class can in fact be extended to the situation where h is a generalised fibre-wise Morse function, cf. [I4].

The following theorem has been proved in [I2], Chapter 6 and [I4] in the case that $H \to B$ is a trivial bundle, but for generalised fibre-wise Morse functions. We will recall Igusa's proof and show that it extends to the unitary case. For simplicity, we restrict our attention to true fibre-wise Morse functions and closed fibres.

10.9. Theorem. *Assume that $F \to M$ and $H \to B$ carry parallel metrics. Let h be a fibre-wise Morse function. Then*

$$\tau(M/B;F,h) = \tau(M/B;F) + 2\hat{p}_*\big((-1)^{\operatorname{ind} h}\,{}^0J(T^u_{\mathrm{Morse}}X)\big)\ \operatorname{rk} F\ .$$

We note that generalised fibre-wise Morse functions enter through Igusa's framed function theorem. We remark that a general version of the framing principle could also be concluded from a generalisation of Theorem 0.3 to generalised Morse functions.

Proof. We copy the proof of [I2]. Replacing M by $M \times \mathbb{R}$ and h by $\bar{h} = (p, z) \mapsto h(p) + \frac{z^2}{2}$, we may assume that $T^s X \to C$ contains a trivial subbundle corresponding to the coordinate z. We note that both h and $\bar{h}$ give rise to the same homotopy class of maps $\xi\colon B \to Wh^u(\mathcal{M}(\mathbb{C}), U)$, and thus by Definition 10.6 to the same higher torsion. We continue to write M and h for the new bundle and fibre-wise Morse function.

Igusa then constructs a generalised fibre-wise Morse function $h_1\colon M \to \mathbb{R}$ with the following properties.

(1) $h_1 = h$ outside a neighbourhood U of C.
(2) h_1 is related by cancelation of Morse-type singularities to a framed function h_2 that coincides with h_2 outside $M \setminus U$.
(3) The critical set C_1 of h_1 takes the form $C_1 = C \cup C'$, and no negative gradient flow line of h_1 originating in C_1 and leaving U or originating in C converges to C'.

By (2), the map $\xi_1\colon B \to Wh^u(\mathcal{M}(\mathbb{C}), U)$ constructed from h_1 is homotopic to the map $\xi_2\colon B \to Wh^u(\mathcal{M}(\mathbb{C}), U)$ from h_2 because the corresponding functors are homotopic up to an elementary extension. This shows that

$$\tau(M/B;F) = \tau(M/B;F,h_2) = \tau(M/B;F,h_1) \tag{10.10}$$

because h_2 is framed.

Note that h_1 is no longer a Morse function. By (3), we can construct the complex $(V_1, \nabla^{V_1}, A_1')$ outside $p(C'_{\text{b-d}})$ such that

$$\big(V_1, \nabla^{V_1}\big) = \big(V, \nabla^V\big) \oplus \big(W, \nabla^W\big)\ ,$$

where the two summands correspond to C and C', and

$$A_1' = \begin{pmatrix} B' & * \\ 0 & A' \end{pmatrix}\ .$$

We also note that V_1 carries a natural parallel metric because obviously $F|_U$ carries a parallel metric by the assumption that g^V is adapted and parallel. Again by [I2], we may assume that each birth-death singularity of h_1 locally gives rise to an elementary expansion of W in the sense of Definition 6.28. This allows us to define a filtration of $(V_1, \nabla^{V_1}, A_1')$ outside $p(C'_{\text{b-d}})$ by

$$\big(V_1, \nabla^{V_1}, A_1'\big) \supset \big(W, \nabla^W, B'\big) \supset 0\ , \tag{10.11}$$

and this filtration is compatible with stabilisation at the birth-death singularities of h_1.

The subcomplex (W, ∇^W, B') is fibre-wise acyclic. By Igusa's calculations in [I2], Chapters 6 and 7, we obtain

$$T\big(\nabla^W, B', g^W\big) = -2\,\hat{p}_*\big((-1)^{\operatorname{ind} h}\,{}^0J(T^u X)\big)\ \operatorname{rk} F\ .$$

Note that Igusa's calculations apply here because by Theorem 8.1, Igusa's higher Franz-Reidemeister torsion is equal to the torsion of Definition 10.6 in the acyclic case.

It follows from the fibre-wise acyclicity of (W, ∇^W, B') that the spectral sequence associated to (10.11) degenerates at $E_1 = H$. Theorem 7.37 and (10.10) now imply that

$$\begin{aligned} \tau(M/B;F) &= \tau(M/B;F,h_1) \\ &= \tau\bigl(\nabla^V, A', g^V\bigr) + T\bigl(\nabla^W, B', g^W\bigr) \\ &= \tau(M/B;F,h) - 2\,\hat{p}_*\bigl((-1)^{\operatorname{ind} h}\ {}^0J(T^uX)\bigr) \operatorname{rk} F\ . \qquad \square \end{aligned}$$

10.c. A Proof of Theorem 0.3. We give a short proof of Theorem 0.3 and show that it is compatible with some of Igusa's results.

Proof of Theorem 0.3. Let $M \to B$ be a family of compact manifolds, and let $F \to M$ be a flat vector bundle. We assume that there exists a fibre-wise Morse function $h: M \to \mathbb{R}$ and that the vector bundles $F \to M$ and $H^*(M/B;F) \to B$ carry parallel metrics. We construct a family Thom–smale complex (V, ∇^V, A') as usual. Theorem 0.1 then reduces to

$$\begin{aligned} \mathcal{T}\bigl(T^HM, g^{TX}, \nabla^F, g^F\bigr) - T\bigl(\nabla^V, A', g^V\bigr) - \widetilde{\operatorname{ch}}^{\circ}\bigl(\nabla^H, g^H_{L_2}, g^H_V\bigr) \\ = \hat{p}_*\bigl((-1)^{\operatorname{ind} h}\ {}^0J(T^sX - T^uX)\bigr) \operatorname{rk} F\ . \end{aligned} \tag{10.12}$$

By definition of $\mathcal{T}(M/B;F)$ and $\tau(M/B;F,h) = T(M/B;F,h)$, we have

$$\begin{aligned} \Bigl(\mathcal{T}\bigl(T^HM, g^{TX}, \nabla^F, g^F\bigr) - \tau\bigl(A', g^V, g^V\bigr) - \widetilde{\operatorname{ch}}^{\circ}\bigl(\nabla^H, g^H_{L_2}, g^H_V\bigr)\Bigr)^{[\geq 2]} \\ = \mathcal{T}(M/B;F) - \tau(M/B;F,h)\ . \end{aligned} \tag{10.13}$$

The framing principle above allows us to compute

$$\begin{aligned} &\tau(M/B;F,h) \\ &\quad = \tau(M/B;F) + \hat{p}_*\bigl((-1)^{\operatorname{ind} h}\ {}^0J(TX)\bigr) \operatorname{rk} F - \hat{p}_*\bigl((-1)^{\operatorname{ind} h}\ {}^0J(T^sX - T^uX)\bigr) \operatorname{rk} F \\ &\quad = \tau(M/B;F) + \int_{M/B} e(TX)\ {}^0J(TX) \operatorname{rk} F - \hat{p}_*\bigl((-1)^{\operatorname{ind} h}\ {}^0J(T^sX - T^uX)\bigr) \operatorname{rk} F\ , \end{aligned} \tag{10.14}$$

where the last identity holds because the singular current $(-1)^{\operatorname{ind} h}\,\delta_C$ is homologous to $e(TX)$ by the family Poincaré-Hopf theorem.

Combining (10.12)–(10.14), we obtain

$$\mathcal{T}(M/B;F) = \tau(M/B;F) + \int_{M/B} e(TX)\ {}^0J(TX) \operatorname{rk} F\ ,$$

which was the assertion of Theorem 0.3. $\square$

10.d. A glueing formula for the unitary torsion. We prove a glueing formula for the unitary torsion of a family of manifolds that are glued together from two submanifolds with a common hypersurface. The main ingredient is Theorem 7.37.

Suppose that $M^+ \to B$ and $M^- \to B$ are two proper submersions with fibre-wise boundary $\partial M^\pm = M_0^\pm \cup M_1^\pm$. Let $F^\pm \to M^\pm$ be two flat vector bundles with parallel metric $g^{F^\pm}$.

We assume that there exists a diffeomorphism $\varphi\colon M_1^- \to M_0^+$ that lifts to a parallel linear isometry $\Phi\colon F^-|_{M_1^-} \to F^+|_{M_0^+}$, so

$$\begin{array}{ccc} F^-|_{M_1^-} & \xrightarrow{\Phi} & F^+|_{M_0^+} \\ \downarrow & & \downarrow \\ M_1^- & \xrightarrow{\varphi} & M_0^+ \end{array} .$$

By glueing, we obtain

$$F := F^- \cup_\Phi F^+ \quad \longrightarrow \quad M := M^- \cup_\varphi M^+ . \tag{10.15}$$

Moreover, we put $\partial M = M_0 \cup M_1$ with

$$M_0 = M_0^- \qquad \text{and} \qquad M_1 = M_1^+ .$$

The following result is a cohomological version of Igusa's Splitting Lemma in [I2], 5.7.8, where a similar result was obtained for maps to the Whitehead space, but with more restrictive conditions on $H(M^\pm/B, M_0^\pm; F)$.

10.16. Theorem. *Assume that the fibre-wise cohomology bundles $H(M^\pm/B, M_0^\pm; F) \to B$ carry parallel metrics. Then the fibre-wise cohomology $H(M/B, M_0/B; F) \to B$ also carries a parallel metric, and*

$$\tau(M/B, M_0/B; F) = \tau(M^-/B, M_0^-/B; F^-) + \tau(M^+/B, M_0^+/B; F^+) \quad \in \quad H^{\mathrm{even},\geq 2}(B;\mathbb{R}) .$$

Proof. We may assume that there exist framed function $h^\pm\colon M^\pm \to [0,1]$ such that $i = 0$, 1 are regular values with $(h^\pm)^{-1}(i) = M_i^\pm$. Then the function $h\colon M \to [0,1]$ with

$$h(p) = \begin{cases} \frac{h^-(p)}{2} & \text{for } p \in M^-, \text{ and} \\ \frac{1+h^+(p)}{2} & \text{for } p \in M^+. \end{cases}$$

is framed and satisfies $h^{-1}(i) = M_i$ for the regular values $i = 0$, 1.

We construct a family Thom-Smale complex $(V, \nabla^V, A') \to B$ as before and let $\xi = \xi_{p,h,F}\colon M \to Wh^u((\mathbb{C}), U)$ denote the corresponding map to the universal unitary Whitehead space. Then V admits a natural filtration

$$\bigl(V, \nabla^V, A'\bigr) \supset \bigl(V^+, \nabla^{V^+}, A^{+\prime}\bigr) \supset 0 .$$

The two steps of the associated spectral sequence take the form

$$\begin{aligned} E_0 &= \bigl(V^-, \nabla^{V^-}, A^{-\prime}\bigr) \oplus \bigl(V^+, \nabla^{V^+}, A^{+\prime}\bigr) \\ \text{and} \qquad E_1 &= \bigl(H^*(V^-, a_0^{-\prime}) \oplus H^*(V^+, a_0^{+\prime}),\ \nabla^{E_1} + d_1'\bigr) . \end{aligned}$$

By our assumptions, the bundles $H^*(V^\pm, a_0^{\pm\prime}) \cong H^*(M^\pm/B, M_0^\pm; F) \to B$ carry parallel metrics $g^{H^\pm}$. Let $g^{H^+ \oplus H^-}$ denote the direct sum of these metrics on $E_1 = H^*(V^-, a_0^{-\prime}) \oplus H^*(V^+, a_0^{+\prime})$. Moreover, the coboundary

$$d_1'\colon\ H^*(V^-, a_0^{-\prime}) \ \longrightarrow\ H^*(V^+, a_0^{+\prime})$$

is parallel with respect to ∇^{E_1} by Proposition 7.33. This implies that the total fibre-wise cohomology

$$H := H^*(V, a_0') \cong H^*(M/B, M_0/B; F) \ \longrightarrow\ B$$

carries a parallel metric induced by $g^{H^+\oplus H^-}$ via finite-dimensional Hodge theory, which we denote by g^H. The appendix of [BL] implies that

$$T^{\mathrm{BL}}\big(D_1', g^{H^+\oplus H^-}\big) = T^{\mathrm{BL}}\big(\nabla^{E_1} + d_1', g^{H^+\oplus H^-}\big) = 0\,. \tag{10.17}$$

We can now apply Theorem 7.37 to find that

$$\begin{aligned}
&T\big(\nabla^V, A', g^V\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^H_V\big)\\
&\qquad = T\big(\nabla^{E_0}, D_0', g^{E_0}\big) + T^{\mathrm{BL}}\big(D_1', g^{E_1}\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^H, g^H, g^{E_\infty}\big)\\
&\qquad = T\big(\nabla^{E_0}, D_0', g^{E_0}\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{E_1}, g^{H^+\oplus H^-}, g^{E_1}\big) + T^{\mathrm{BL}}\big(D_1', g^{H^+\oplus H^-}\big)\\
&\qquad = T\big(\nabla^{V^+}, A^{+\prime}, g^{V^+}\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{H^+}, g^{H^+}, g^{H^+}_{V^+}\big)\\
&\qquad\qquad + T\big(\nabla^{V^-}, A^{-\prime}, g^{V^-}\big) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\big(\nabla^{H^-}, g^{H^-}, g^{H^-}_{V^-}\big)
\end{aligned}$$

modulo exact forms on B. The theorem follows. $\square$

For proper submersions with closed fibres, Theorem 10.16 implies a cut-and-paste formula. Here, we assume that $M_0 \to B$ and $M_1 \to B$ are two proper submersion such that

$$M_i = M_i^+ \cup_{M_i^0} M_i^-$$

for $i = 0, 1$, where $M_i^\pm$, $M_1^\pm$ are families of closed submanifolds intersecting in the common boundary $M_i^0 = M_i^+ \cap M_i^-$. Assume that $F_i\colon M_i$ is a flat vector bundle with a parallel metric g^{F_i} for $i = 0, 1$ and that there exists a diffeomorphism $\varphi\colon M_0^0 \to M_1^0$ which lifts to a linear isometry $\Phi\colon F_0|_{M_0^0} \to F_1|_{M_1^0}$,

$$\begin{array}{ccc}
F_0|_{M_0^0} & \xrightarrow{\Phi} & F_1|_{M_1^0}\\
\big\downarrow & & \big\downarrow\\
M_0^0 & \xrightarrow{\varphi} & M_1^0
\end{array}\quad .$$

Then we can construct two new families of closed submanifolds

$$M_0' = M_0^- \cup_\varphi M_1^+ \qquad \text{and} \qquad M_1' = M_1^- \cup_\varphi M_0^+$$

that carry flat vector bundles $F_i' \to M_i'$ for $i = 0, 1$.

10.18. Corollary. *Assume that the bundles $H^*(M_i^-/B; F_i) \to B$ and $H^*(M_i^+/B, M_i^0/B; F) \to B$ carry parallel metrics for $i = 0, 1$. Then $H^*(M_i/B; F_i) \to B$ and $H^*(M_i'/B; F_i) \to B$ also carry parallel metrics, and we have*

$$\tau(M_0/B; F) + \tau(M_1/B; F) = \tau(M_0'/B; F) + \tau(M_1'/B; F) \quad \in \quad H^{\mathrm{even},\geq 2}(B; \mathbb{R})\,.$$

Proof. This is an immediate consequence of Theorem 10.16. $\square$

10.e. A proof of Theorem 0.2. We want to use Corollary 10.18 to construct some more examples of "exotic" smooth structures on fibre bundles from Hatcher's example as described in Section 5.a. Assume that we are in the situation of Section 5.c. That is, we fix $k > 0$ and $l > 0$ sufficiently large with respect to k, and we let X be an arbitrary compact connected $(2l-1)$-dimensional manifold. We decompose X along a hypersurface X^0 into

$$X = X^+ \cup_{X^0} X^- \;,$$

such that X^- is diffeomorphic to the $(2l-1)$-dimensional disc D^{2l-1}.

We also let $M_\gamma \to S^{4k}$ denote Hatcher's example constructed starting with a generator $\gamma \in \pi_{4k-1}(O)$ of $\ker J$. In particular, M_γ admits a fibre-wise Morse function with six critical points (in fact, four would already be enough), such that the unique maximum has a trivial disc bundle neighbourhood, so we can decompose

$$M_\gamma = M_\gamma^+ \cup_{M_\gamma^0} M_\gamma^- \;,$$

where now $M_\gamma^+ \to S^{4k}$ is diffeomorphic to $D^{2l-1} \times S^{4k}$.

Assume that X carries a flat vector bundle $F \to X$ of rank r with parallel metric g^F, then $F|_{X^-}$ is trivial. We fix a diffeomorphism $\varphi\colon X^0 \times S^{4k} \to M_\gamma^0$ and a lift $\Phi\colon F|_{X^0} \times S^{4k} \to M_\gamma^0 \times \mathbb{C}^r$ as a parallel linear isometry. We construct

$$F'_\gamma := \bigl(F|_{X^+} \times S^{4k}\bigr) \cup_\Phi \bigl(M_\gamma^- \times \mathbb{C}^r\bigr) \qquad \longrightarrow \qquad M'_\gamma = \bigl(X^+ \times S^{4k}\bigr) \cup_\varphi M_\gamma^-$$

Glueing the other two parts gives the trivial bundle

$$\mathbb{C}^r \times S^{2l-1} \times S^{4k} \qquad \longrightarrow \qquad S^{2l-1} \times S^{4k} = X^- \cup_\varphi M_\gamma^+ \;.$$

By construction, the fibre-wise cohomology bundles of $X^- \times S^{4k} \to S^{4k}$, $(X^+, X^0) \times S^{4k} \to S^{4k}$, $M_\gamma^+ \to S^{4k}$ and $M_\gamma^- \to S^{4k}$ with coefficients in F respectively in $\mathbb{C}^r$ are all trivial. We may apply Theorem 0.3 and Corollary 10.18 and get

$$\begin{aligned} \mathcal{T}\bigl(M'_\gamma/S^{4k}; F'_\gamma\bigr) &= \tau\bigl(M'_\gamma/S^{4k}; F'_\gamma\bigr) = \tau\bigl(M_\gamma/S^{4k}, \mathbb{C}^r\bigr) \\ &= 2^{2-k}\, b_k\, \zeta'(-2k)\, u \quad \in \quad H^{4k}\bigl(S^{4k}; \mathbb{R}\bigr) \;. \end{aligned} \tag{10.19}$$

The above construction of M'_γ generalises the example of Section 5.c to the case of non-trivial F. As in Section 5.d, we may glue j copies of the bundle $M'_\gamma \to S^{4k}$ to an arbitrary fibre bundle $M \to B$ with fibre X and $4k$-dimensional compact oriented base B, to obtain a new bundle

$$p_j\colon\, M_j \;\longrightarrow\; B$$

which is homeomorphic to the original $M \to B$ as a bundle. For negative j, we glue $|j|$ copies of $M'_{-\gamma} \to S^{4k}$ to $M \to B$. We assume that $F \to M$ is a unitarily flat vector bundle, then after glueing, we have homeomorphic unitarily flat vector bundles $F_j \to M_j$ for all $j \in \mathbb{Z}$.

Let g^H be a fixed metric on the bundle

$$H^*(M_j/B; F_j) \cong H = H^*(M/B; F) \;\longrightarrow\; B \;.$$

Then as in (5.22), we compute

$$\int_{M_j/B} \mathcal{T}\bigl(M_j/B; F_j, g^H\bigr) = \int_{M/B} \mathcal{T}\bigl(M/B; F, g^H\bigr) + 2^{2-k}\, j\, b_k\, \zeta'(-2k)\, u \;,$$

where $\mathcal{T}(M_j/B; F_j, g^H) \in \Omega^{\mathrm{even}}(B)/d\Omega^{\mathrm{odd}}(B)$ is defined as in (5.20) by

$$\mathcal{T}(M/B; F, g^H) = \Bigl(\mathcal{T}\bigl(T^H M, g^{TX}, \nabla^F, g^F\bigr) + \widetilde{\mathrm{ch}}^{\mathrm{o}}\bigl(\nabla^H, g^H, g^H_{L_2}\bigr)\Bigr)^{[\geq 2]} \;.$$

This implies the following generalisations of Theorem 5.30, where *Diff*$(M; F)$ denotes the group of parallel lifts of arbitrary diffeomorphisms of M to a given flat vector bundle F, and *Diff*$^+(M; F)$ denotes the subgroup coming from orientation preserving diffeomorphisms whenever M is orientable.

10.20. Theorem. *Let $p\colon M \to B$ be a proper submersion and let $(F, \nabla^F) \to M$ be a vector bundle with a parallel metric g^F. Assume that B is compact, orientable, and of dimension $4k > 0$. Let the bundles $(q_j\colon M \to B)_{j\in\mathbb{Z}}$ with $q_0 = p$ be constructed as in Section 5.e. Assume that for i, $j \in \mathbb{Z}$ with $i \neq j$, there exists diffeomorphisms $\Phi \in \mathit{Diff}(M;F)$ and $\Psi \in \mathit{Diff}^+(B)$ such that the diagramme*

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d, "q_i"'] & M \arrow[d, "q_j"] \\ B \arrow[r, "\Psi"] & B \end{tikzcd}$$

commutes. Then $\Psi \in \mathit{Diff}^+(B;H)$ for $H = H^*(M/B;\mathbb{C}) \to B$, and

$$\mathrm{ch}^{\mathrm{o}}\bigl(\nabla^{T_\Psi H}\bigr)[T_\Psi B] = 2^{2-k}\,(j-i)\,j_0\,b_k\,\zeta'(-2k)\ \mathrm{rk}\,F\ .$$

In particular, this is only possible if $H \to B$ does not carry a parallel metric, and if Ψ does not represent a torsion element in the mapping class group.

Clearly, Theorem 10.20 implies the generalised second statement of Theorem 0.2 by the same arguments as in Section 5.e.

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