Finite covers of disintegrated sets

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January 1992

1 Introduction

Let X be structure. A 2-sorted structure $\bar{E} = (E, X)$ is a *finite cover* of X if

1. there is 0-definable surjection $\pi : E \to X$ whose fibers $E_x = \pi^{-1}(x)$ have a bounded finite cardinality
2. There is no new structure induced on X, i.e. every in $\bar{E}$ 0-definable relation on X is 0-definable in X.

In this note we determine what the finite covers are in the simplest of all cases: where X is a countable set without structure, a disintegrated set.

Remark 1.1 *Let $\bar{E}$ be a finite cover of X. Since $\bar{E}$ is algebraic over X every automorphism of X extends to an automorphism of $\bar{E}$. This shows that a finite cover of an ω–categorical structure is also ω–categorical.*

Since countable ω–categorical structures are - up to interdefinability - determined by their automorphism groups we will instead of giving the actual structures of the finite covers give their automorphism groups.

The following lemma is easy to prove.

Lemma 1.2 *A permutation group G on $E \cup X$ is the automorphism group of a finite cover of X if the following conditions are satisfied:*

1. *G is a closed subgroup of* $\mathrm{Sym}(E \cup X)$
2. *Every element of G respects the fibration π.*
3. *Every permutation of X is induced by an element of G.*

The *kernel* of a finite cover $\bar{E}$ is the set of all automorhisms σ which leave X pointwise fixed i.e. $\mathrm{Aut}(\bar{E}/X)$. All reducts $\bar{E}'$ of $\bar{E}$ which are also finite covers are determined solely by their kernel K'. For one computes easily that $\mathrm{Aut}(\bar{E}') = \mathrm{Aut}(\bar{E})K'$.

Let F be a set with f_0 elements. We make $(F \times X, X)$ to a finite covering by adding the natural projection $\pi : F \times X \to X$ and for every $f \in F$ a predicate for the set $\{(f, x) \mid x \in X\}$. We denote this structure by $F \times X$. The reducts of $F \times X$ (and the covers sets isomorphic to such reducts) are called *splitting* covers. Splitting covers are determined by their kernels. One sees easily that this are exactly the closed subgroups of $\prod_{x \in X} \mathrm{Sym}(F)$ which are invariant under conjugation with elements of $id \times \mathrm{Sym}(X)$.

In the next section we will prove that all finite covers of the disintegrated set split. In the last section we determine the kernels of the splitting covers.

2 The covers split

Let $\bar{E} = (E, X)$ be a finite cover of X. For subsets A of X let E_A denote the union of the fibers above A i.e. $\pi^{-1}(A)$.

Lemma 2.1 *Let A be a subset of X. Then E_A and X are* orthogonal *over A in the sense that*

$$\mathrm{tp}(E_A/A) \vdash \mathrm{tp}(E_A/X)$$

Proof:
We can assume that A is finite. Then by restriction the finite group $\mathrm{Aut}(E_A/A)/\mathrm{Aut}(E_A/X)$ is a homomorphic image of $Aut(E/A)/\mathrm{Aut}(E/X)$, which is again by restriction isomorphic to $\mathrm{Aut}(X/A) \cong \mathrm{Sym}(X \setminus A)$. But an infinite symmetric group has no non-trivial finite quotients. Whence $\mathrm{Aut}(E_A/A) = \mathrm{Aut}(E_A/X)$, which we had to show: two tuples from E_A which are conjugate over A are conjugate over X.

This is just a group-proof of a more general fact: Let $\bar{E}$ be a finite cover of a strongly minimal set X. If X weakly eliminates imaginairies and A is an algebraically closed subset of X then E_A and X are orthogonal over A.

Corollary 2.2 *Let $\sigma \in \mathrm{Sym}(X)$ have support B. Then σ can be extended to $\bar{\sigma} \in \mathrm{Aut}(E)$ which is the identity on $E_{X \setminus B}$.*

Proof:
Since both X and E_A are $\bigwedge$–definable over A

$$\mathrm{tp}(E_A/A) \vdash \mathrm{tp}(E_A/X)$$

is equivalent to

$$\mathrm{tp}(X/A) \vdash \mathrm{tp}(X/E_A).$$

This, applied to $A = X \setminus B$ proves the assertion.

Lemma 2.3 *For all $a \neq b \in X$ there is a $\sigma_{ab} \in \mathrm{Aut}(E)$ such that*

1. *σ_{ab} induces the transposition (ab) on X,*
2. *σ_{ab} fixes $E_{X \setminus \{ab\}}$ pointwise,*
3. *$\sigma_{ab}^2 = id$*

Proof:
By the corollary choose τ_{ab} which satisfies 1 and 2. Then $\rho = \tau_{ab}^2$ leaves everything fixed except possibly E_a and E_b. If we conjugate ρ by τ_{bc} we obtain an automorphism ρ_c which leaves everything fixed except possibly E_a and E_c. Furthermore ρ_c agrees with ρ on E_a. If we fix a and b and let c tend to infinity the sequence (ρ_c) will converge to an automorphism σ which agrees with ρ on E_a and is the identity else. Now set $\sigma_{ab} = \sigma^{-1}\tau_{ab}$

Theorem 2.4 *Every finite cover of a disintegrated set splits*

Proof:
We have to prove that there is a bijection between $\bar{E}$ and $F \times X$ which respects the fibration and such that every 0-definable relation of $\bar{E}$ is mapped to a 0-definable relation of $F \times X$ i.e. every automorphism of $F \times X$ is the image of an automorphism of $\bar{E}$. This amounts to finding a

family $(\beta_x)_{x\in X}$ of bijections $\beta_x : F \to E_x$ such that every permutation σ of X can be lifted to an automorphism $\bar{\sigma}$ of $\bar{E}$ such that every diagram

$$\begin{tikzcd} & F \arrow[dl, "\beta_x"'] \arrow[dr, "\beta_{\sigma(x)}"] & \\ E_x \arrow[rr, "\bar{\sigma}"'] & & E_{\sigma(x)} \end{tikzcd}$$

commutes. Fix an element $a \in X$ and a bijection $\beta_a : F \to E_a$. Now chose for every $b \neq a$ the bijection $\beta_b : F \to E_b$ in such a way that the diagram above commutes for $\sigma = (ab)$, $\bar{\sigma} = \sigma_{ab}$ and $x = a$. But then the diagram commutes for all $x \in X$. This is clear if $x \neq b$ and follows from $\sigma_{ab}^2 = id$ if $x = b$. If $\sigma \in \mathrm{Sym}(X)$ has finite support it can be written as a product of transpositions (ab_i). If we define $\bar{\sigma}$ to be product of the σ_{ab_i} the diagram commutes. Finally if σ is the limit of permutations σ_i of finite support let $\bar{\sigma}$ be an accumulation point of the $\bar{\sigma}_i$.

3 The kernels

Finally we determine the possible kernels of finite covers of disintegrated sets. Since all covers split this is done by the following theorem:

Theorem 3.1 *The closed subgroups K of $\prod_{x\in X} \mathrm{Sym}(F)$ which are invariant under conjugation with elements of $id \times \mathrm{Sym}(X)$ are exactly the groups of the form*

$$K_H^G = \{\alpha \in \prod_{x\in X} G \mid \forall x, y\; \alpha_x H = \alpha_y H\},$$

where G is a subgroup of $\mathrm{Sym}(F)$ and H is a normal subgroup of G.

Proof:
Clearly all groups K_H^G are closed and invariant under conjugation with elements of $id \times \mathrm{Sym}(X)$. Let conversely K be a group with this property. Fix $a \in X$ and set

$$G = \{\sigma_a \mid \sigma \in K\}$$

and

$$H = \{\alpha \mid \exists \sigma \in K\; \sigma_a = \alpha \wedge \forall x \neq a\; \sigma_x = id\}.$$

We will prove that $K = K_H^G$.
Let σ be an element of K. For each $x \in X$ the a-component of $\sigma^{(ax)}$ is σ_x. This shows that all σ_x lie in G. If x and y are given the components of the commutator $\kappa = [\sigma, (xy)]$ are $\sigma_x\sigma_y^{-1}$ at x, $\sigma_y\sigma_x^{-1}$ at y and the identity everywhere else. The limit $\tau = \lim_{z\to\infty} \kappa^{(yz)}$ is an element of K whose components are the identity except that $\tau_x = \sigma_x\sigma_y^{-1}$. Then $\tau^{ax} \in K$ shows that $\sigma_x\sigma_y^{-1} \in H$.
For the converse we remark first that by conjugation G and H do not depend on the choice of a. This implies immediately that $\prod_{x\in X} H$ is contained in K. If now σ is an arbitrary element of K_H^G choose an element τ of K such that $\sigma_a = \tau_a$. Since all σ_x and all τ_x (by the first part of the proof) are congruent mod H to $\sigma_a = \tau_a$ the quotient $\sigma\tau^{-1}$ belongs to $\prod_{x\in X} H$. This shows $\sigma \in K$.