EQUATIONAL THEORIES OF FIELDS

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Abstract. A complete first-order theory is equational if every definable set is a Boolean combination of instances of equations, that is, of formulae such that the family of finite intersections of instances has the descending chain condition. Equationality is a strengthening of stability. We show the equationality of the theory of proper extensions of algebraically closed fields of some fixed characteristic and of the theory of separably closed fields of arbitrary imperfection degree. Srour showed that the theory of differentially closed fields in positive characteristic is equational. We give also a different proof of his result.

Contents

1. Introduction

Consider a sufficiently saturated model of a complete theory T. A formula $\varphi(x; y)$ is an *equation* (for a given partition of the free variables into x and y) if the family of finite intersections of instances $\varphi(x, a)$ has the descending chain condition (DCC). The theory T is *equational* if every formula $\psi(x; y)$ is equivalent modulo T to a Boolean combination of equations $\varphi(x; y)$.

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Quantifier elimination implies that the theory of algebraically closed fields of some fixed characteristic is equational. Separably closed fields of positive characteristic have quantifier elimination after adding λ-*functions* to the ring language [2]. The *imperfection degree* of a separably closed field K of positive characteristic p encodes the linear dimension of K over K^p. If the imperfection degree is finite, restricting the λ-functions to a fixed p-*basis* yields again equationality. A similar manipulation yields elimination of imaginaries for separably closed field K of positive characteristic and finite imperfection degree, in terms of the field of definition of the corresponding defining ideal. However, there is not an explicit description of imaginaries for separably closed fields K of infinite imperfection degree, that is, when K has infinite linear dimension over the definable subfield K^p.

Another important (expansion of a) theory of fields having infinite linear dimension over a definable subfield is the theory of an algebraically closed field with a predicate for a distinguished algebraically closed proper subfield. Any two such pairs are elementarily equivalent if and only if they have the same characteristic. They are exactly the models of the theory of Poizat's *belles paires* [14] of algebraically closed fields.

Determining whether a particular theory is equational is not obvious. So far, the only known *natural* example of a stable non-equational theory is the free non-abelian finitely generated group [15, 11]. In this paper, we will prove the equationality of several theories of fields: the theory of belles paires of algebraically closed fields of some fixed characteristic, as well as the theory of separably closed fields of arbitrary imperfection degree We also give a new proof of the equationality of the theory of differentially closed fields in positive characteristic, which was established by Srour [18]. In Section 9 we include an alternative proof for belles paires of characteristic 0, by showing that definable sets are Boolean combination of certain definable sets, which are Kolchin-closed in the corresponding expansion DCF_0. A similar approach appeared already in [5] using different methods. We generalise this approach to arbitrary characteristic in Section 10.

2. Equations and indiscernible sequences

Most of the results in this section come from [13, 6, 7]. We refer the avid reader to [10] for a gentle introduction to equationality.

We work inside a sufficiently saturated model $\mathbb{U}$ of a complete theory T. A formula $\varphi(x; y)$, with respect to a given partition of the free variables into x and y, is an *equation* if the family of finite intersections of instances $\varphi(x, b)$ has the descending chain condition (DCC). If $\varphi(x; y)$ is an equation, then so are $\varphi^{-1}(y; x) = \varphi(x, y)$ and $\varphi(f(x); y)$, whenever f is a $\emptyset$-definable map. Finite conjunctions and disjunctions of equations are again equations. By an abuse of notation, given an incomplete theory, we will say that a formula is an equation if it is an equation in every completion of the theory.

The theory T is *equational* if every formula $\psi(x; y)$ is equivalent modulo T to a Boolean combination of equations $\varphi(x; y)$.

Typical examples of equational theories are the theory of an equivalence relation with infinite many infinite classes, the theory of R-modules.

Example 2.1. In any field K, for every polynomial $p(X, Y)$ with integer coefficients, the equation $p(x; y) \doteq 0$ is an equation in the model-theoretic sense.

Proof. This follows immediately from Hilbert's Basis Theorem, which implies that the Zariski topology on K^n is noetherian, i.e. the system of all algebraic sets

$$\Big\{a \in K^n \;\Big|\; \bigwedge_{i=1}^{m} q_i(a) = 0\Big\},$$

where $q_i \in K[X_1, \ldots, X_n]$, has the DCC.
There is a simpler proof, without using Hilbert's Basis Theorem: Observe first that $p(x; y) \doteq 0$ is an equation, if p is linear in x, since then $p(x; a) \doteq 0$ defines a subspace of K^n. Now, every polynomial has the form $q(M_1, \ldots, M_m; y)$, where $q(u_1, \ldots, u_m; y)$ is linear in the u_i, for some monomials $M_1, \ldots, M_m$ in x. $\square$

Quantifier elimination for the incomplete) theory ACF of algebraically closed fields and the above example yield that ACF is equational.

Equationality is preserved under unnaming parameters and bi-interpretability [6]. It is unknown whether equationality holds if every formula $\varphi(x; y)$, with x a single variable, is a boolean combination of equations.

By compactness, a formula $\varphi(x; y)$ is an equation if there is no indiscernible sequence $(a_i, b_i)_{i\in\mathbb{N}}$ such that $\varphi(a_i, b_j)$ holds for $i < j$, but $\not\models \varphi(a_i, b_i)$. Thus, equationality implies stability [13]. In stable theories, *non-forking* provides a natural notion of independence. Working inside a sufficiently saturated model, we say that two sets A and B are independent over a common subset C, denoted by $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, if, for every finite tuple a in A, the type $\mathrm{tp}(a/B)$ does not fork over C. Non-forking extensions of a type over an elementary substructure M to any set $B \supset M$ are both heir and definable over M.

Definition 2.2. A type q over B is an *heir* of its restriction $q\restriction M$ to the elementary substructure M if, whenever the formula $\varphi(x, m, b)$ belongs to q, with m in M and b in B, then there is some m' in M such that $\varphi(x, m, m')$ belongs to $q\restriction M$.
A type q over B is definable over M if, for each formula $\varphi(x, y)$, there is a formula $\theta(y)$ with parameters in M such that for every b in B,

$$\varphi(x, b) \in q \text{ if and only if } \models \theta(b).$$

Observe that if q is definable over M, for any formula $\varphi(x, y)$, any two such formulae $\theta(y)$ are equivalent modulo M, so call it the *φ-definition* of q.

If φ is an equation, the φ-definition of a type q over B is particularly simple. The intersection

$$\bigcap_{\varphi(x,b)\in q} \varphi(\mathbb{U}, b)$$

is a definable set given by a formula $\psi(x)$ over B contained in q. If suffices to set

$$\theta(y) = \forall x\,(\psi(x) \to \varphi(x, y))\,.$$

By the above characterisation, a formula $\varphi(x; y)$ is an equation if and only if every instance $\varphi(a, y)$ is indiscernibly closed definable sets [7, Theorem 3.16]. A definable set is *indiscernibly closed* if, whenever $(b_i)_{i\leq\omega}$ is an indiscernible sequence such that b_i lies in X for $i < \omega$, then so does b_ω.
Extending the indiscernible sequence so that it becomes a Morley sequence over an initial segment, we conclude the following:

Lemma 2.3. *In a complete stable theory T, a definable set $\varphi(a, y)$ is indiscernibly closed if, for every elementary substructure M and every Morley sequence $(b_i)_{i\leq\omega}$ over M such that*

$$a \mathop{\smile\!\!\!\!\!\!\mid}_M b_i \text{ with } \models \varphi(a, b_i) \text{ for } i < \omega,$$

then b_ω realises $\varphi(a, y)$ as well.
We may take the sequence of length $\kappa+1$, for every infinite cardinal κ, and assume that $a \mathop{\smile\!\!\!\!\!\!\mid}_M \{b_i\}_{i<\kappa}$.

In [18, Theorem 2.5], Srour stated a different criterion for the equationality of a formula. Let us provide a version of his result. Given a formula $\varphi(x, y)$ and a type p over B, denote

$$p^+_\varphi = \{\varphi(x, b) \mid \varphi(x, b) \in p\}.$$

Lemma 2.4. *Given a formula $\varphi(x; y)$ in a stable theory T, the following are equivalent:*

(1) The formula $\varphi(x; y)$ is an equation.
(2) Given a tuple a of length $|x|$ and a subset B, there is a finite subset B_0 of B such that
$$\mathrm{tp}^+_\varphi(a/B_0) \vdash \mathrm{tp}^+_\varphi(a/B).$$
(3) There is a regular cardinal $\kappa > |T|$ such that, for any tuple a of length $|x|$ and any elementary substructures $M \subset N$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_M N$ and $|N| = \kappa$, there is a subset B_0 of N with $|B_0| < \kappa$ such that
$$\mathrm{tp}(a/MB_0) \vdash \mathrm{tp}^+_\varphi(a/N).$$

Proof. For (1) $\Longrightarrow$ (2), we observe that the intersection

$$\bigcap \{\varphi(\mathbb{U}, b) \mid \varphi(x, b) \in \mathrm{tp}^+_\varphi(a/B)\}$$

is a finite intersection with parameters in a finite subset B_0 of B. The implication (2) $\Longrightarrow$ (3) is immediate. For (3) $\Longrightarrow$ (1), it suffices to show that the set $\varphi(a, y)$ is indiscernibly closed, for every tuple a of length $|x|$. By Lemma 2.3, let M be an elementary substructure and $(b_i)_{i\leq\kappa}$ a Morley sequence over M such that

$$a \mathop{\smile\!\!\!\!\!\!\mid}_M (b_i)_{i<\kappa} \text{ and } \models \varphi(a, b_i) \text{ for } i < \kappa.$$

We construct a continuous chain of elementary substructures $(N_i)_{i<\kappa}$, each of cardinality at most κ containing M, such that:

- the sequence $(b_j)_{i\leq j\leq\kappa}$ remains indiscernible over N_i;
- $b_{<i}$ is contained in N_i;
- $a \mathop{\smile\!\!\!\!\!\!\mid}_M N_i \cup (b_j)_{i\leq j<\kappa}$.

Set $N_0 = M$. For $i < \kappa$ limit ordinal, set

$$N_i = \bigcup_{j<i} N_j.$$

Thus, we need only consider the successor case. Suppose N_i has already been constructed and let N_{i+1} be an elementary substructure of cardinality at most κ containing $N_i \cup \{b_i\}$ such that

$$N_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{N_i\cup\{b_i\}} a \cup (b_j)_{i<j\leq\kappa}.$$

Observe that the sequence $(b_j)_{i<j\leq\kappa}$ remains indiscernible over N_{i+1}. By monotonicity applied to the above independence, we have that

$$N_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{N_i\cup(b_j)_{i\leq j\leq\kappa}} a,$$

so, by transitivity,

$$a \mathop{\smile\!\!\!\!\!\!\,|}_{M} N_{i+1} \cup (b_j)_{i<j\leq\kappa},$$

as desired.

The elementary substructure $N = \bigcup_{i<\kappa} N_i$ has cardinality κ. Finite character implies that $a \mathop{\smile\!\!\!\!\!\!\,|}_M N$. By hypothesis, there is a subset B_0 of N of cardinality strictly less than κ such that

$$\mathrm{tp}(a/MB_0) \vdash \mathrm{tp}^+_\varphi(a/N).$$

Regularity of κ yields that $B_0 \subset N_i$ for some $i < \kappa$. In particular, the elements b_i and b_κ have the same type over N_i, and therefore over MB_0. Let $\tilde{a}$ such that $\tilde{a}b_i \equiv_{MB_0} ab_\kappa$. Since

$$\mathrm{tp}^+_\varphi(a/N) \supset \{\varphi(x, b_j)\}_{j<\kappa},$$

we conclude that $\models \varphi(\tilde{a}, b_i)$, and thus $\models \varphi(a, b_\kappa)$, as desired. □

Remark 2.5. Whenever $a \mathop{\smile\!\!\!\!\!\!\,|}_M N$, the type $\mathrm{tp}(a/N)$ is definable with the same definition schema as the one of $\mathrm{tp}(a/M)$. In particular, we can add a fourth equivalence to Lemma 2.4: the formula $\varphi(x; y)$ is an equation if and only if, whenever $a \mathop{\smile\!\!\!\!\!\!\,|}_M N$, then

$$\mathrm{tp}(a/M) \vdash \mathrm{tp}^+_\varphi(a/N).$$

We will finish this section with an observation on imaginaries in equational theories.

Lemma 2.6. *If every instance of an equation in an equational theory has a real canonical parameter, then the theory has elimination of imaginaries.*
In particular, an equational theory has elimination of imaginaries after adding, for each equation $\varphi(x; y)$, a sort for the equivalence relation E_φ defined as follows:

$$E_\varphi(a, b) \Longleftrightarrow \models \forall x\, (\varphi(x, a) \leftrightarrow \varphi(x, b))\, .$$

Proof. The proof is a direct application of [6, Lemma 2.8], but we include it for the sake of the presentation. Consider a class C of an $\emptyset$-definable equivalence relation E. In particular, the definable set C is a Boolean combination of instances of equations, which are again definable over the same set of parameters as C.

Let $\mathcal{X}$ be the collection of finite unions of conjugates of the instances of equations appearing in the Boolean combination for C. Clearly the family $\mathcal{X}$ is a basis of closed sets for a noetherian topology, i.e. with the DCC for closed sets. Let $\bar{C}$ be the topological closure of C. Since the intersection of finitely many equations is again an equation, the set $\bar{C}$ is definable and has a real canonical parameter $\ulcorner \bar{C} \urcorner$. As the family $\mathcal{X}$ is closed under automorphisms, the tuple $\ulcorner \bar{C} \urcorner$ is definable over the canonical parameter $\ulcorner C \urcorner$ in T^{eq}. Hence, it suffices to show that $\ulcorner C \urcorner$ is definable over $\ulcorner \bar{C} \urcorner$: Otherwise, we would obtain a different conjugate C' of C over $\ulcorner \bar{C} \urcorner$. Since equivalence classes form a partition, we have that C and C' are disjoint and both dense in $\bar{C}$, contradicting [6, Fact 2.6]. □

3. Basics on fields

In this section, we will include some basic notions of field theory and commutative algebra needed in order to prove the equationality of the theories of fields we will consider later on. We will work inside somesufficiently large algebraically closed field $\mathbb{U}$.

Two subfields L_1 and L_2 are *linearly disjoint* over a common subfield F, denoted by

$$L_1 \mathop{\smile\!\!\!\!\!|}^{\mathrm{ld}}_{F} L_2,$$

if, whenever the elements $a_1, \ldots, a_n$ of L_1 are linearly independent over F, then they remain so over L_2, or, equivalently, if L_1 has a linear basis over F which is linearly independent over L_2.
Linear disjointness implies algebraic independence and agrees with the latter whenever the base field F is algebraically closed. Let us note that linear disjointness is symmetric, and a transitive relation: If $F \subset D_2 \subset L_2$ is a subfield, denote by $D_2 \cdot L_1$ the field generated by D_2 and L_1. Then

$$L_1 \mathop{\smile\!\!\!\!\!|}^{\mathrm{ld}}_{F} L_2$$

if and only if

$$L_1 \mathop{\smile\!\!\!\!\!|}^{\mathrm{ld}}_{F} D_2 \qquad \text{and} \qquad D_2 \cdot L_1 \mathop{\smile\!\!\!\!\!|}^{\mathrm{ld}}_{D_2} L_2.$$

By multiplying with a suitable denominator, we may also use the terminology for a subring A being linearly disjoint from a field$B$ over a common subring C.

Definition 3.1. Consider a theory T of fields in the language $\mathcal{L}$ extending the language of rings $\mathcal{L}_{rings} = \{+, -, \cdot, 0, 1\}$ such that there is a predicate $\mathcal{P}$, which is interpreted in every model of T as a definable subfield. A subfield A of a sufficiently saturated model K of T is $\mathcal{P}$-*special* if

$$A \mathop{\smile\!\!\!\!\!|}^{\mathrm{ld}}_{\mathcal{P}(A)} \mathcal{P}(K),$$

where $\mathcal{P}(A)$ equals $\mathcal{P}(K) \cap A$.

It is easy to see that elementary substructures of K are $\mathcal{P}$-special.

Lemma 3.2. *Inside a sufficiently saturated model K of a stable theory T of fields in the language $\mathcal{L} \supset \mathcal{L}_{rings}$ equipped with a definable subfield $\mathcal{P}(K)$, consider a $\mathcal{P}$-special field A and a field B, both containing an elementary substructure M of K such that $A \mathop{\smile\!\!\!\!\!|}_{M} B$. The fields $\mathcal{P}(K) \cdot A$ and $\mathcal{P}(K) \cdot B$ are linearly disjoint over $\mathcal{P}(K) \cdot M$.*

Note that we write $F \cdot F'$ for the field generated by F and F'.

Proof. It suffices to show that elements $a_1, \ldots, a_n$ of A which are linearly dependent over $\mathcal{P}(K) \cdot B$ are also linearly dependent over $\mathcal{P}(K) \cdot M$. Thus, let $z_1, \ldots, z_n$ in

$\mathcal{P}(K) \cdot B$, not all zero, such that

$$\sum_{i=1}^{n} a_i \cdot z_i = 0.$$

Multiplying by a suitable denominator, we may assume that all the z_i's lie in the subring generated by $\mathcal{P}(K)$ and B, so

$$z_i = \sum_{j=1}^{m} \zeta_j^i b_j,$$

for some ζ_j^i's in$\mathcal{P}(K)$ and $b_1, \ldots, b_m$ in B, which we may assume to be linearly independent over $\mathcal{P}(K)$.
The type $\mathrm{tp}(a_1, \ldots, a_n/Mb_1, \ldots b_m)$ is a nonforking extension of $\mathrm{tp}(a_1, \ldots, a_n/M)$, so in particular is a heir over M. Thus, there are some η_j^i's in $\mathcal{P}(K)$, not all zero, and $c_1, \ldots, c_m$ in M linearly independent over $\mathcal{P}(K)$, such that

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{m} \eta_j^i c_j = 0.$$

Since A is $\mathcal{P}$-special, we may assume all the η_j^i's lie in $\mathcal{P}(A)$. As $\{c_j\}_{1 \leq j \leq m}$ are $\mathcal{P}$-linearly independent, at least one of the elements in

$$\{ \sum_{1 \leq j \leq m} \eta_j^1 c_j, \ldots, \sum_{1 \leq j \leq m} \eta_j^n c_j \}$$

is different from 0, as desired.

□

A natural example of a definable subfield is the field of p^{th} powers K^p, whenever K has positive characteristic $p > 0$. The corresponding notion of K^p-special is *separability*: A non-zero polynomial $f(T)$ over a subfield K is separable if every root (in the algebraic closure of K) has multiplicity 1, or equivalently, if f and its formal derivative $\frac{\partial f}{\partial T}$ are coprime. Whenever f is irreducible, the latter is equivalent to $\frac{\partial f}{\partial T} \neq 0$. In particular, every non-constant polynomial in characteristic 0 is separable. In positive characteristic p, an irreducible polynomial f is separable if and only if f is not a polynomial in T^p.
An algebraic extension $K \subset L$ is *separable* if the minimal polynomial over K of every element in L is separable. Algebraic field extensions in characteristic 0 are always separable. In positive characteristic p, the finite extension is separable if and only if the fields K and L^p are linearly disjoint over K^p. This explains the following definition:

Definition 3.3. An arbitrary (possibly not algebraic) field extension $F \subset K$ is *separable* if, either the characteristic is 0 or, in case the characteristic is $p > 0$, the fields F and K^p are linearly disjoint over F^p.

A field K is *perfect* if either it has characteristic 0 or if $K = K^p$, for $p = \mathrm{char}(K)$. Any field extension of a perfect field is separable. Given a field K, we define its *imperfection degree* (in $\mathbb{N} \cup \{\infty\}$), as 0 if the characteristic of K is 0, or ∞, in case of positive characteristic p if $[K : K^p]$ is infinite. Otherwise $[K : K^p] = p^e$ for e the degree of imperfection. Thus, a field is perfect if and only if its imperfection degree is 0

Another example of fields equipped with a definable subfields are differential fields. A differential field consists of a field K together with a distinguished additive morphism δ satisfying Leibniz' rule

$$\delta(xy) = x\delta(y) + y\delta(x).$$

Analogously to Zariski-closed sets for pure field, one defines *Kolchin-closed* sets in differential fields as zero sets of systems of differential-polynomials equations, that is, polynomial equations on the different iterates of the variables under the derivation. For a tuple $x = (x_1, \ldots, x_n)$ in K, denote by $\delta(x)$ the tuple $(\delta(x_1), \ldots, \delta(x_n))$.

Lemma 3.4. *In any differential field (K, δ), an algebraic differential equation*

$$p(x, \delta x, \delta^2 x \ldots; y, \delta y, \delta^2 y, \ldots) \doteq 0$$

is an equation in the model-theoretic sense.

Proof. In characteristic zero this follows from Ritt-Raudenbush's Theorem, which states that the Kolchin topology is noetherian. In arbitrary characteristic, it suffices to observe, as in Example 2.1, that $p(x, \delta x, \ldots; y, \delta y, \ldots)$ can be written as $q(M_1, \ldots; y, \delta y, \ldots)$ where $q(u_1, \ldots; y_0, y_1, \ldots)$ is linear in the u_i's, for some differential monomials M_j's in x. □

In particular, the theory DCF_0 of differentially closed fields of characteristic 0 is equational, since it has quantifier elimination [21].
In a differential field (K, δ), the set of *constants*

$$\mathrm{C}_K = \{x \in K \mid \delta(x) = 0\}$$

is a definable subfield, which contains K^p if $p = \mathrm{char}(K) > 0$. If K is algebraically closed, then so is C_K.

Fact 3.5. The elements $a_1, \ldots, a_k$ of the differential field (K, δ) are linearly dependent over $\mathrm{C}_K = \{x \in K \mid \delta(x) = 0\}$ if and only if their *Wronskian* $\mathrm{W}(a_1, \ldots, a_k)$ is 0, where

$$\mathrm{W}(a_1, \ldots, a_k) = \det \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ \delta(a_1) & \delta(a_2) & \ldots & \delta(a_k) \\ \vdots & & \vdots & \\ \delta^{k-1}(a_1) & \delta^{k-1}(a_2) & \ldots & \delta^{k-1}(a_k) \end{pmatrix}.$$

Whether the above matrix has determinant 0 does not dependon the differential field where we compute it. In particular, every differential subfield L of K is C_K-special.

Perfect fields of positive characteristic cannot have non-trivial derivations. In characteristic zero though, any field K which is notalgebraic over the prime field has a non-trivial derivation δ. Analogously to perfectness, we say that the differential field (K, δ) is *differentially perfect* if either K has characteristic 0 or, in case $p = \mathrm{char}(K) > 0$, if every constant has a p^{th}-root, that is, if $\mathrm{C}_K = K^p$.
Notice that the following well-known result generalises the equivalent situation for perfect fields and separable extensions.

Remark 3.6. Let (K, δ) be a differential field and F a differentially perfect differential subfield of K. The extension $F \subset K$ is separable.

Proof. We need only prove it when the characteristic of K is $p > 0$. By Fact 3.5, the fields F and C_K are linearly disjoint over $\mathrm{C}_F = F^p$. Since $K^p \subset \mathrm{C}_K$, this implies that F and K^p are linearly disjoint over F^p. □

In section 8, we will consider a third theory of fields equipped with a definable subfield: belles paires of algebraically closed fields. In order to show that the corresponding theory is equational, we require some basic notions from linear algebra (cf. [4, Résultats d'Algèbre]). Fix some subfield E of $\mathbb{U}$.

Let V be a vector subspace of E^n with basis $\{v_1, \ldots, v_k\}$. Observe that

$$V = \left\{v \in E^n \;\middle|\; v \wedge (v_1 \wedge \cdots \wedge v_k) = 0 \text{ in } \bigwedge^{k+1} E^n\right\}.$$

The vector $v_1 \wedge \cdots \wedge v_k$ depends only on V, up to scalar multiplication, and determines V completely. *The Plücker coordinates* $\mathrm{Pk}(V)$ of V are the homogeneous coordinates of $v_1 \wedge \cdots \wedge v_k$ with respect to the canonical basis of $\bigwedge^k E^n$. The k^{th}*-Grassmannian* $\mathrm{Gr}_k(E^n)$ of E^n is the collection of Plücker coordinates of all k-dimensional subspaces of E^n. Clearly $\mathrm{Gr}_k(E^n)$ is contained in $\mathbb{P}^{r-1}(E)$, for $r = \binom{n}{k}$. The k^{th}-Grassmannian is Zariski-closed. Indeed, given an element ζ of $\bigwedge^k E^n$, there is a smallest vector subspace V_ζ of E^n such that ζ belongs to $\bigwedge^k V_\zeta$. The vector space V_ζ is the collection of inner products $e \lrcorner \zeta$, for e in $\bigwedge^{k-1}(E^n)^*$. Recall that the inner product $\lrcorner$ is a bilinear map

$$\lrcorner : \bigwedge^{k-1}(E^n)^* \times \bigwedge^{k}(E^n) \to E.$$

A non-trivial element ζ of $\bigwedge^k E^n$ determines a k-dimensional subspace of E^n if and only if

$$\zeta \wedge (e \lrcorner \zeta) = 0,$$

for every e in $\bigwedge^{k-1}(E^n)^*$. Letting e run over a fixed basis of $\bigwedge^{k-1}(E^n)^*$, we see that the k^{th}-Grassmannian is the zero-set of a finite collection of homogeneous polynomials.

Let us conclude this section with an observation regarding projections of certain varieties.

Remark 3.7. Though the theory of algebraically closed fields has elimination of quantifiers, the projection of a Zariski-closed set need not be again closed. For example, the closed set

$$V = \{(x, z) \in E \times E \mid x \cdot z = 1\}$$

projects onto the open set $\{x \in E \,|\, x \neq 0\}$. An algebraic variety Z is *complete* if, for all varieties X, the projection $X \times Z \to X$ is a Zariski-closed map. Projective varieties are complete.

4. Model Theory of separably closed fields

Recall that a field K is *separably closed* if it has no proper algebraic separable extension, or equivalently, if every non-constant separable polynomial over K has a root in K. For each fixed degree, this can be expressed in the language of rings. Thus, the class of separably closed fields is axiomatisable. Separably closed fields of characteristic zero are algebraically closed. For a prime p, let SCF_p denote the theory of separably closed fields of characteristic p and $\mathsf{SCF}_{p,e}$ the theory of separably closed fields of characteristic p and imperfection degree e. Note that $\mathsf{SCF}_{p,0}$ is the theory ACF_p of algebraically closed fields of characteristic p.

Fact 4.1. (cf. [2, Proposition 27]) The theory $\mathsf{SCF}_{p,e}$ is complete and stable, but not superstable for $e > 0$. Given a model K and a separable field extension $k \subset K$, the type of k in K is completely determined by its quantifier-free type. In particular, the theory has quantifier elimination in the language

$$\mathcal{L}_\lambda = \mathcal{L}_{rings} \cup \{\lambda_n^i \mid 1 \leq i \leq n < \omega\},$$

where the value $\lambda_n^i(a_0, \ldots, a_n)$ is defined as follows in K. If there is a *unique* sequence $\zeta_1, \ldots, \zeta_n \in K$ with $a_0 = \zeta_1^p a_1 + \cdots + \zeta_n^p a_n$, we set $\lambda_n^i(a_0, \ldots, a_n) = \zeta_i$. Otherwise, we set $\lambda_n^i(a_0, \ldots, a_n) = 0$ and call it *undefined*,

Note that $\lambda_n^i(a_0, \ldots, a_n)$ is defined if and only if

$$K \models \neg\, p\text{-}\mathrm{Dep}_n(a_1, \ldots, a_n) \wedge p\text{-}\mathrm{Dep}_{n+1}(a_0, a_1, \ldots, a_n),$$

where $p\text{-}\mathrm{Dep}_n(a_1, \ldots, a_n)$ means that $a_1, \ldots, a_n$ are K^p-linearly dependent. In particular, the value $\lambda_n^i(a_0, \ldots, a_n)$ is undefined for $n > p^e$.

For a subfield k of a model K of SCF_p, the field extension $k \subset K$ is separable if and only if k is closed under λ-functions.

Notation. For elements $a_0, \ldots, a_n$ of K, the notation $\overline{\lambda}(a_0, a_1, \ldots, a_n) \downarrow$ is an abbreviation for $\neg\, p\text{-}\mathrm{Dep}_n(a_1, \ldots, a_n) \wedge p\text{-}\mathrm{Dep}_{n+1}(a_0, a_1, \ldots, a_n)$.

Remark 4.2. If the imperfection degree e is finite, we can fix a p-basis $\mathbf{b} = (b_1, \ldots, b_e)$ of K, that is, a tuple such that the collection of monomials

$$\bar{\mathbf{b}} = (b_1^{\nu_1} \cdots b_e^{\nu_e} \mid 0 \leq \nu_1, \ldots, \nu_e < p)$$

is a linear basis of K over K^p. All p-bases have the same type. If we replace the λ-functions by the functions $\Lambda^\nu(a) = \lambda_{p^e}^\nu(a, \bar{\mathbf{b}})$, then the theory $\mathsf{SCF}_{p,e}(\mathbf{b})$, in the language of rings with constants for $\mathbf{b}$ and equipped with the functions $\Lambda^\nu(x)$, has again quantifier elimination. Furthermore, the Λ-values of a sum or a product can be easily computed in terms of the values of each factor. In particular, the canonical base of the type (a/K) in $\mathsf{SCF}_{p,e}(\mathbf{b})$ is the field of definition of the vanishing ideal of the infinite tuple

$$(a, \overline{\Lambda}(a), \overline{\Lambda}(\overline{\Lambda}(a)), \ldots).$$

Thus, the theory $\mathsf{SCF}_{p,e}(\mathbf{b})$ has elimination of imaginaries.
As in Lemma 3.4, it follows that the formula $t(x; y) \doteq 0$ is a model-theoretic equation, for every $\mathcal{L}_\Lambda$-term $t(x, y)$. This implies that $\mathsf{SCF}_{p,e}(\mathbf{b})$, and therefore $\mathsf{SCF}_{p,e}$, is equational.

Whether there is an explicit expansion of the language of rings in which $\mathsf{SCF}_{p,\infty}$ has elimination of imaginaries is not yet known.

From now on, work inside a sufficiently saturated model K of the incomplete theory SCF_p. The imperfection degree of K may be either finite or infinite.
Since an $\mathcal{L}_\lambda$-substructure determines a separable field extension, Lemma 3.2 implies the following result:

Corollary 4.3. *Consider two subfields A and B of K containing an elementary substructure M of K. Whenever*

$$A \mathop{\smile\!\!\!\!\!|}\limits_{M}^{\mathsf{SCF}_p} B,$$

the fields $K^p \cdot A$ and $K^p \cdot B$ are linearly disjoint over $K^p \cdot M$.

Note that the field $K^p \cdot A$ is actually the ring generated by K^p and A, since A is algebraic over K^p.

Proof. The $\mathcal{L}_\lambda$-structure A' generated by A is a subfield, since $a^{-1} = \lambda_1^1(1, a^p)$ for $a \neq 0$. Since $A' \mathop{\downarrow}_M^{\mathsf{SCF}_{p,e}} B$, and A' is K^p-special, we have that $K^P \cdot A'$ and $K^p \cdot B$ are linearly disjoint over M. Whence $K^P \cdot A$ and $K^P \cdot B$ are also linearly disjoint over M □

We will now exhibit our candidate formulae for the equationality of SCF_p, uniformly on the imperfection degree.

Definition 4.4. The collection of λ-tame formulae is the smallest collection of formulae in the language $\mathcal{L}_\lambda$, containing all polynomial equations and closed under conjunctions, such that, for any natural number n and polynomials $q_0, \ldots, q_n$ in $\mathbb{Z}[x]$, given a λ-tame formula $\psi(x, z_1, \ldots, z_n)$, the formula

$$\varphi(x) = p\text{-Dep}_n(q_1(x), \ldots, q_n(x)) \ \vee \ \Big(\overline{\lambda}(q_0(x), \ldots, q_n(x))\!\downarrow \ \wedge \ \psi(x, \overline{\lambda}_n(q_0(x), \ldots, q_n(x))) \Big)$$

is λ-tame.

Note that the formula φ above is equivalent to

$$p\text{-Dep}_n(q_1, \ldots, q_n) \ \vee \ \big(p\text{-Dep}_{n+1}(q_0, \ldots, q_n) \ \wedge \ \psi(x, \overline{\lambda}_n(\overline{q}(x))) \big).$$

In particular, the formula $p\text{-Dep}_n(q_1(x), \ldots, q_n(x))$ is a tame λ-formula, since it is equivalent to

$$p\text{-Dep}_n(q_1(x), \ldots, q_n(x)) \ \ \vee \ \big(\overline{\lambda}(0, q_1(x), \ldots, q_n(x))\!\downarrow \ \wedge \ 0 \doteq 1 \big).$$

There is a natural degree associated to a λ-tame formula, in terms of the amount of nested λ-tame formulae it contains, whereas polynomial equations have degree 0. The degree of a conjunction is the maximum of the degrees of the corresponding formulae.

The next remark is easy to prove by induction on the degree of the formula:

Remark 4.5. Given a λ-tame formula φ in m many free variables and polynomials $r_1(X), \ldots, r_m(X)$ in several variables with integer coefficients, the formula $\varphi(r_1(x), \ldots, r_m(x))$ is equivalent in SCF_p to a λ-tame formula of the same degree.

Proposition 4.6. *Modulo SCF_p, every formula is equivalent to a Boolean combination of λ-tame formulae.*

Proof. By Fact 4.1, it suffices to show that the equation $t(x) \doteq 0$ is equivalent to a Boolean combination of λ-tame formulae, for every $\mathcal{L}_\lambda$-term $t(x)$. Proceed by induction on the number of occurrences of λ-functions in t. If no λ-functions occur in t, the result follows, since polynomial equations are λ-tame. Otherwise

$$t(x) = r(x, \overline{\lambda}_n(q_0(x), \ldots, q_n(x)))$$

for some $\mathcal{L}_\lambda$-term $r(x, z_1, \ldots, z_n)$ and polynomials q_i. By induction, the term $r(x, \bar{z}) \doteq 0$ is equivalent to a Boolean combination $BK(\psi_1(x, \bar{z}), \ldots, \psi_m(x, \bar{z}))$ of λ-tame formulae $\psi_1(x, \bar{z}), \ldots, \psi_m(x, \bar{z})$. Consider now the λ-tame formulae

$$\varphi_i(x) = p\text{-Dep}_n(q_1(x), \ldots, q_n(x)) \ \vee \ \big(\overline{\lambda}(\overline{q}(x))\!\downarrow \ \wedge \ \psi_i(x, \overline{\lambda}_n(\overline{q}(x))) \big).$$

Note that

$$\mathsf{SCF}_{p,e} \models \Big((\overline{\lambda}(\overline{q}(x))\!\downarrow) \longrightarrow (\psi_i(x, \overline{\lambda}_n(\overline{q}(x))) \leftrightarrow \varphi_i(x)) \Big).$$

Therefore $t(x) \doteq 0$ is equivalent to

$$\big(\neg\overline{\lambda}(\overline{q}(x))\!\downarrow \wedge\, r(x,0) \doteq 0\big) \quad \vee \quad \big(\overline{\lambda}(\overline{q}(x))\!\downarrow \wedge\, BK(\varphi_1(x), \ldots, \varphi_m(x))\big),$$

which is, by induction, a Boolean combination of λ-tame formulae. □

We conclude this section with a homogenisation result for λ-tame formulae, which will be used in the proof of the equationality of SCF_p.

Proposition 4.7. *For every λ-tame $\varphi(x, y_1, \ldots, y_n)$ there is a λ-tame formula $\varphi'(x, y_0, y_1, \ldots, y_n)$ of same degree such that*

$$\mathsf{SCF}_p \models \forall x, y_0 \ldots y_n \Big(\varphi'(x, y_0, \ldots, y_n) \longleftrightarrow \Big(\varphi\Big(x, \frac{y_1}{y_0}, \ldots, \frac{y_n}{y_0}\Big) \vee y_0 \doteq 0 \Big)\Big).$$

We call φ' a *homogenisation* of φ with respect to $y_0, \ldots, y_n$.

Proof. Let y denote the tuple $(y_1, \ldots, y_n)$. By induction on the degree, we need only consider basic λ-tame formulae, since the result is preserved by taking conjunctions. For degree 0, suppose that $\varphi(x, y)$ is the formula $q(x, y) \doteq 0$, for some polynomial q. Write

$$q(x, \frac{y}{y_0}) = \frac{q'(x, y_0, y)}{y_0^N}.$$

Then $\varphi'(x, y_0, y) = y_0 \cdot q'(x, y) \doteq 0$ is a homogenisation.
If $\varphi(x, y)$ has the form

$$p\text{-Dep}_n(q_1(x, y), \ldots, q_m(x, y)) \vee \big(\overline{\lambda}(q_0, \ldots, q_m)\!\downarrow \wedge\, \psi(x, y, \overline{\lambda}_n(q_0, \ldots, q_m))\big),$$

let $\psi'(x, y_0, y, z)$ be a homogenisation of $\psi(x, y, z)$ with respect to y_0, y. There is a natural number N such that for each $0 \leq j \leq m$,

$$q_j(x, \frac{y}{y_0}) = \frac{q_j'(x, y_0, y)}{y_0^N}$$

for polynomials q_j'. Set now $q_j'' = y_0 \cdot q_j'$ and

$$\varphi'(x, y_0, y) = p\text{-Dep}_n(q_1'', \ldots, q_m'') \vee \big(\overline{\lambda}(q_0'', \ldots, q_m'')\!\downarrow \wedge\, \psi'(x, y_0, y, \overline{\lambda}_n(q_0'', \ldots, q_m''))\big).$$

□

5. Equationality of SCF_p

By Proposition 4.6, in order to show that the theory SCF_p is equational, we need only show that each λ-tame formula is an equation in every completion $\mathsf{SCF}_{p,e}$. As before, work inside a sufficiently saturated model K of some fixed imperfection degree.

For the proof, we require *generalised λ-functions*: If the vectors $\bar{a}_0, \ldots, \bar{a}_n$ in K^N are linearly independent over K^p and the system

$$\bar{a}_0 = \sum_{i=1}^{n} \zeta_i^p \, \bar{a}_i$$

has a solution, then it is unique and denoted by $\lambda^i_{N,n}(\bar{a}_0, \ldots, \bar{a}_n)$. The notation $\overline{\lambda}_{N,n}(\bar{a}_0, \ldots, \bar{a}_n)\!\downarrow$ means that all $\lambda^i_{N,n}$'s are defined. Observe that $\lambda^i_{1,n} = \lambda^i_n$. We denote by $p\text{-Dep}_{N,n}(\bar{a}_0,\ldots,\bar{a}_n)$ the formula stating that the vectors $\bar{a}_1,\ldots,\bar{a}_n$ are linearly dependent over K^p.

Theorem 5.1. *Given any partition of the variables, every λ-tame formula $\varphi(x;y)$ is an equation in $\mathsf{SCF}_{p,e}$*

Proof. We proceed by induction on the degree D of the λ-tame formula. For $D=0$, it is clear. So assume that the theorem is true for all λ-tame formulae of degree smaller than some fixed degree $D\geq 1$. Let $\varphi(x;y)$ be a λ-tame formula of degree D.

Claim. *If*

$$\begin{aligned}\varphi(x;y) = p\text{-Dep}_{N,n}(\bar{q}_1(x^p,y),\ldots,\bar{q}_n(x^p,y)) \ \vee \\ \big(\,\overline{\lambda}_N(\bar{q}_0(x^p,y),\ldots,\bar{q}_n(x^p,y))\!\downarrow \ \wedge\ \psi(x,y,\overline{\lambda}_{N,n}(\bar{q}_0(x^p,y),\ldots,\bar{q}_n(x^p,y)))\,\big),\end{aligned}$$

where $\psi(x,y,z_1,\ldots,z_n)$ is a λ-tame formula of degree $D-1$, then $\varphi(x;y)$ is an equation.

Proof of Claim. It suffices to show that every instance $\varphi(x,b)$ is equivalent to a formula $\psi'(x,b',b)$, where $\psi'(x,y',y)$ is a λ-tame formula of degree $D-1$, for some tuple b'.
Choose a K^p-basis $b_1,\ldots,b_{N'}$ of all monomials in b occurring in the $\bar{q}_k(x^p,b)$'s and write $\bar{q}_k(x^p,b)=\sum_{j=1}^{N'}\bar{q}_{j,k}(x,b')^p\cdot b_j$. We use the notation $\mathbf{q}_k(x,b')$ for the vector of length NN' which consists of the concatenation of the vectors $\bar{q}_{j,k}(x,b')$. Let $\mathbf{Q}(x,b')$ be the $(NN'\times n)$-matrix with columns $\mathbf{q}_1(x,b'),\ldots,\mathbf{q}_n(x,b')$. The vectors $\bar{q}_1(x^p,b),\ldots,\bar{q}_n(x^p,b)$ are linearly dependent over K^p if and only if the columns of $\mathbf{Q}(x,b')$ are linearly dependent over K. Let J range over all n-element subsets of $\{1,\ldots,NN'\}$ and let $\mathbf{Q}^J(x,b')$ be the corresponding $n\times n$-submatrices. Thus

$$\mathsf{SCF}_{p,e}\models\Big(p\text{-Dep}_{N,n}(\bar{q}_1(x^p,y),\ldots,\bar{q}_n(x^p,y))\longleftrightarrow\bigwedge_J\det(\mathbf{Q}^J(x,b'))\doteq 0\Big).$$

If $\det(\mathbf{Q}^J(x,b'))$ is not zero, the vector $\overline{\zeta}=\overline{\lambda}_{N,n}(\bar{q}_0(x^p,b),\ldots,\bar{q}_n(x^p,b))$ is defined if and only if $\mathbf{q}_0(x,b')=\mathbf{Q}(x,b')\cdot\overline{\zeta}$. In that case,

$$\overline{\zeta}=\det(\mathbf{Q}^J(x,b'))^{-1}\cdot B^J(x,b')\cdot\mathbf{q}_0^J(x,b'),$$

where $B^J(x,b')$ is the adjoint of $\mathbf{Q}^J(x,b')$. Set $d^J(x,b')=\det(\mathbf{Q}^J(x,b'))$ and $r^J(x,b')=B^J(x,b')\cdot\mathbf{q}_0^J(x,b')$, so

$$\overline{\zeta}=d^J(x,b')^{-1}\cdot r^J(x,b').$$

Consider the λ-tame formula

$$\psi^J(x,b',b,\overline{z}) \ = \ \big(\mathbf{q}_0(x,b')\doteq\mathbf{Q}(x,b')\cdot\overline{z}\ \wedge\ \psi(x,b,\overline{z})\big),$$

of degree $D-1$. It follows that $\varphi(x,b)$ is equivalent to

$$\bigwedge_J\ \big(d^J(x,b')\doteq 0\ \ \vee\ \ \psi^J(x,b',b,d^J(x,b')^{-1}\cdot r^J(x,b'))\big),$$

which is equivalent to a λ-tame formula of degree $D-1$, by Remark 4.5 and Proposition 4.7. □ Claim

For the proof of the theorem, since a conjunction of equations is again an equation, we may assume that

$$\varphi(x;y) = p\text{-Dep}_n(q_1(x,y),\dots,q_n(x,y)) \;\vee \\ \big(\,\overline{\lambda}(q_0(x,y),\dots,q_n(x,y))\!\downarrow \,\wedge\, \psi(x,y,\overline{\lambda}_n(q_0(x,y),\dots,q_n(x,y)))\,\big)$$

for some λ-tame formula $\psi(x,y,z_1,\dots,z_n)$ of degree $D-1$. It suffices to show that $\varphi(a,y)$ is indiscernibly closed. By Lemma 2.3, consider an elementary substructure M of K and a Morley sequence $(b_i)_{i\leq\omega}$ over M such that

$$a \mathop{\smile\!\!\!\!\!\!\;|}_{M} b_i \text{ with } \models \varphi(a,b_i) \text{ for } i<\omega.$$

We must show that $K \models \varphi(a,b_\omega)$.
Choose a $(K^p\cdot M)$-basis $a_1,\dots,a_N$ of the monomials in a which occur in the $q_k(a,y)$ and write $q_k(a,y) = \sum_{j=1}^N q_{j,k}(a'^p,m,y)\cdot a_j$, for some tuple m in M and a' in K. Let $\bar q_k(a'^p,m,y)$ be the vector $\big(q_{j,k}(a'^p,m,y)\big)_{1\leq j\leq N}$ and consider the formula

$$\varphi'(x,x';y',y) = p\text{-Dep}_{N,n}(\bar q_1(x'^p,y',y),\dots,\bar q_n(x'^p,y',y)) \;\vee \\ \big(\,\overline{\lambda}_N(\bar q_0(x'^p,y',y),\dots,\bar q_n(x'^p,y',y))\!\downarrow \,\wedge \\ \psi(x,y,\overline{\lambda}_{N,n}(\bar q_0(x'^p,y',y),\dots,\bar q_n(x'^p,y',y)))\,\big).$$

Clearly,

$$\mathsf{SCF}_{p,e} \models \forall y(\varphi'(a,a',m,b) \longrightarrow \varphi(a,y)).$$

By Corollary 4.3, the elements $a_1,\dots,a_N$ are linearly independent over the field $(K^p\cdot M)(b_i)$, so $\varphi'(a,a',m,b_i)$ holds in K, since $K\models\varphi(a,b_i)$ for $i<\omega$. By the previous claim, the λ-tame formula $\varphi'(x,x';y',y)$ is an equation. Since the sequence $(m,b_0),\dots(m,b_\omega)$ is indiscernible, we have that $\varphi'(a,a',m,b_\omega)$ holds in K, so $K\models\varphi(a,b_\omega)$, as desired. □

Together with Proposition 4.6, the above theorem yields the following:

Corollary 5.2. *The (incomplete) theory* SCF_p *of separably closed fields of characteristic* $p>0$ *is equational.*

Proof. Proposition 4.6 yields, that modulo SCF_p every formula is a Boolean combination of sentences (i.e. formulas without free variables) and λ-tame formulas. Sentences are equations by definition, λ-tame formulas are equations by Theorem 5.1. □

Lemma 2.6 and Theorem 5.1 yield a partial elimination of imaginaries for $\mathsf{SCF}_{p,e}$.

Corollary 5.3. *The theory* $\mathsf{SCF}_{p,e}$ *of separably closed fields of characteristic* $p>0$ *and imperfection degree* e *has elimination of imaginaries, after adding canonical parameters for all instances of* λ*-tame formulae.*

Question. Is there an explicit description of the canonical parameters of instances of λ-tame formulae, similar to the geometric sorts introduced in [12]?

6. Model Theory of differentially closed fields in positive characteristic

The model theory of existentially closed differential fields in positive characteristic has been thoroughly studied by Wood [19, 20]. In contrast to the characteristic 0 case, the corresponding theory is no longer ω-stable nor superstable: its universe is a separably closed field of infinite imperfection degree (see Section 4).
A differential field (K, δ) is *differentially closed* if it is existentially closed in the class of differential fields. That is, whenever a quantifier-free $\mathcal{L}_\delta = \mathcal{L}_{rings} \cup \{\delta\}$-formula $\varphi(x_1, \ldots, x_n)$, with parameters in K, has a realisation in a differential field extension (L, δ_L) of (K, δ), then there is a realisation of $\varphi(x_1, \ldots, x_n)$ in K.

A *differential polynomial* $p(x)$ is a polynomial in x and its higher order derivatives $\delta(x), \delta^2(x), \ldots$ The *order* of p is the order of the highest occurring derivative.

Fact 6.1. The class of differentially closed fields of positive characteristic p can be axiomatised by the complete theory DCF_p with following axioms:

- The universe is a differentially perfect differential field of characteristic p.
- Given two differential polynomials $g(x) \neq 0$ and $f(x)$ in one variable with $\mathrm{ord}(g) < \mathrm{ord}(f) = n$ such that the *separant* $s_f = \frac{\partial f}{\partial(\delta^n x)}$ of f is not identically 0, there exists an element a with $g(a) \neq 0$ and $f(a) = 0$.

The type of a differentially perfect differential subfield is determined by its quantifier-free type. The theory DCF_p is stable but not superstable, and has quantifier-elimination in the language $\mathcal{L}_{\delta,s} = \mathcal{L}_\delta \cup \{s\}$, where s is the following unary function:

$$s(a) = \begin{cases} b, & \text{with } a = b^p \text{ in case } \delta(a) = 0. \\ 0, & \text{otherwise.} \end{cases}$$

Note that every non-constant separable polynomial is a differential polynomial of order 0 whose separant is non-trivial (since $\delta^0(x) = x$). In particular, every model K of DCF_p is a separably closed field. Furthermore, the imperfection degree of K is infinite: Choose for every n in $\mathbb{N}$ an element a_n in K with $\delta^n(a_n) = 0$ but $\delta^{n-1}(a_n) \neq 0$. It is easy to see that the family $\{a_n\}_{n \in \mathbb{N}}$ is linearly independent over K^p.

Remark 6.2. The quotient field of any $\mathcal{L}_{\delta,s}$-substructure of a model of DCF_p is differentially perfect.

Proof. Let $\frac{a}{b}$ be an element in the quotient field with derivative 0. The element $ab^{p-1} = \frac{a}{b} b^p$ is also a constant, so $ab^{p-1} = s(ab^{p-1})^p$. Hence

$$\frac{a}{b} = \Big(\frac{s(ab^{p-1})}{b}\Big)^p.$$

□

From now on, we work inside a sufficiently saturated model K of DCF_p.

Corollary 6.3. *Consider two subfields A and B of K containing an elementary substructure M of K. Whenever*

$$A \mathop{\downarrow}_{M}^{\mathsf{DCF}_p} B,$$

the fields $K^p \cdot A$ and $K^p \cdot B$ are linearly disjoint over $K^p \cdot M$.

Proof. The quotient field A' of the $\mathcal{L}_{\delta,s}$-structure generated by A is K^p-special, by the Remarks 3.6 and 6.2. The result now follows from Lemma 3.2, as in the proof of Corollary 4.3. □

We will now present a relative quantifier elimination, by isolating the formulae which will be our candidates for the equationality of DCF_p.

Definition 6.4. Let x be a tuple of variables. A formula $\varphi(x)$ in the language $\mathcal{L}_\delta$ is *δ-tame* if there are differential polynomials $q_1, \ldots, q_m$, with q_i in the differential ring $\mathbb{Z}\{X, T_1, \ldots, T_{i-1}\}$, and a system of differential equations Σ in $\mathbb{Z}\{X, T_1, \ldots, T_n\}$ such that

$$\varphi(x) \quad = \quad \exists\, z_1 \ldots \exists z_n \Big(\bigwedge_{j=1}^{n} z_j^p \doteq q_j(x, z_1, \ldots, z_{j-1}) \wedge \; \Sigma(x, z_1, \ldots, z_n) \Big).$$

Proposition 6.5. *Every formula in DCF_p is a Boolean combination of δ-tame formulae.*

Proof. The proof is a direct adaptation of the proof of Proposition 4.6. We need only show that the equation $t(x) \doteq 0$ is a Boolean combination of δ-tame formulae, for every $\mathcal{L}_{\delta,s}$-term $t(x)$. Proceed by induction on the number of occurrences of s in t. Suppose that $t(x) = r(x, s(q(x)))$, for some $\mathcal{L}_{\delta,s}$-term r and a polynomial q, By induction, the equation $r(x, z) \doteq 0$ is equivalent to a Boolean combination $BK(\psi_1(x, z), \ldots)$ of δ-tame formulae. Thus $t(x) \doteq 0$ ist equivalent to

$$\big(\neg\delta(q(x)) \doteq 0 \,\wedge\, r(x, 0) \doteq 0\big) \;\vee\; \big(\delta(q(x)) \doteq 0 \,\wedge\, BK(\exists z\, z^p \doteq q(x) \wedge \psi_1(x, z), \ldots)\big),$$

which is, by induction, a Boolean combination of δ-tame formulae. □

We conclude this section with a homogenisation result for δ-tame formulae, as in Proposition 4.7.

Proposition 6.6. *Given a δ-tame formula $\varphi(x_1, \ldots, x_n)$ and natural numbers $k_1, \ldots, k_n$, there is a δ-tame formula $\varphi'(x_0, \ldots, x_n)$ such that*

$$\mathsf{DCF}_p \vdash \forall x_0 \ldots \forall x_n \Big(\varphi'(x_0, \ldots, x_n) \longleftrightarrow \Big(\varphi\Big(\frac{x_1}{x_0^{k_1}}, \ldots, \frac{x_n}{x_0^{k_n}} \Big) \vee x_0 \doteq 0 \Big) \Big).$$

Proof. We prove it by induction on the number of existential quantifiers iny φ. If φ is a system Σ of differential equations, rewrite

$$\Sigma(\frac{x_1}{x_0^{k_1}}, \ldots, \frac{x_n}{x_0^{k_n}}) \Longleftrightarrow \frac{\Sigma'(x_0, \ldots, x_n)}{x_0^N},$$

for some natural number N and a system of differential equations $\Sigma'(x_0, \ldots, x_n)$. Set

$$\varphi'(x_0, \ldots, x_n) = x_0 \cdot \Sigma'(x_0, \ldots, x_n).$$

For a general δ-tame formula, write

$$\varphi(x_1, \ldots, x_n) = \exists z \;\; (z^p \doteq q(x_1, \ldots, x_n) \wedge \psi(x_1, \ldots, x_n, z))\,,$$

for some polynomial q and a δ-tame formula ψ with one existential quantifier less. There is a polynomial $q'(x_0, \ldots, x_n)$ such that

$$q(\frac{x_1}{x_0^{k_1}}, \ldots, \frac{x_n}{x_0^{k_n}}) = \frac{q'(x_0, \ldots, x_n)}{x_0^{pN-1}},$$

for some natural number N. By induction, there is a δ-tame formula $\psi'(x_0, \ldots, x_n, z)$ such that

$$\mathsf{DCF}_p \vdash \forall x_0 \ldots \forall x_n \forall z \Big(\psi'(x_0, \ldots, x_n, z) \longleftrightarrow \Big(\psi\Big(\frac{x_1}{x_0^{k_1}}, \ldots, \frac{x_n}{x_0^{k_n}}, \frac{z}{x_0^N} \Big) \vee x_0 \doteq 0 \Big)\Big).$$

Set now

$$\varphi'(x_0, \ldots, x_n) = \exists z \left(z^p \doteq x_0 \cdot q'(x_1, \ldots, x_n) \wedge \psi'(x_0, x_1, \ldots, x_n, z) \right).$$

$\square$

7. Equationality of DCF_p

We have now all the ingredients to show that the theory DCF_p of existentially closed differential fields of positive characteristic p is equational. Working inside a sufficiently saturated model K of DCF_p, given a δ-tame formula in a fied partition of the variables x and y, one can show, similar to the proof of Theorem 5.1, that the set $\varphi(a, y)$ is indiscernibly closed. However, we will provide a proof, which resonates with Srour's approach [18], using Lemma 2.4. We would like to express our gratitude to Zoé Chatzidakis and Carol Wood for pointing out Srour's result.

Theorem 7.1 (Srour [18])**.** *In any partition of the variables, the δ-tame formula $\varphi(x; y)$ is an equation.*

Srour proved this for the equivalent notion of S-formulae, cf. Definition 7.4 and Lemma 7.5.

Proof. We prove it by induction on the number n of existential quantifiers. For $n = 0$, the formula $\varphi(x; y)$ is a system of differential equations, which is clearly an equation, by Lemma 3.4.

For $n > 0$, write $\varphi(x, y)$ as

$$\exists z \Big(z^p \doteq q(x, y) \ \wedge \ \psi(x, y, z) \Big),$$

where $\psi(x, y, z)$ is a δ-tame formula with $n - 1$ existential quantifiers.

Claim. *Suppose that every differential monomial in x occurs in $q(x, y)$ as a p^{th}-power. Then $\varphi(x; y)$ is an equation.*

Proof of Claim. It suffices to prove that $\varphi(x, b)$ is equivalent to a δ-tame formula $\psi'(x, b, b')$ with $n - 1$ existential quantifiers, for some tuple b'. Choose a K^p-basis $1 = b_0, \ldots, b_N$ of the differential monomials in b occurring in $q(x, b)$ and write $q(x, b) = \sum_{i=0}^N q_i(x, b')^p \cdot b_i$. Then $\varphi(x, b)$ is equivalent (in DCF_p) to

$$\exists z \Big(z \doteq q_0(x, b') \ \wedge \ \bigwedge_{i=1}^N q_i(x, b') \doteq 0 \wedge \ \psi(x, b, z) \Big),$$

which is equivalent to

$$\psi'(x, b, b') = \Big(\bigwedge_{i=1}^N q_i(x, b') \doteq 0 \ \wedge \ \psi(x, b, q_0(x, b')) \Big).$$

$\square$ Claim

In order to show that $\varphi(x;y)$ is an equation, we will apply Remark 2.5. Consider a tuple a of length $|x|$, and two elementary substructures $M \subset N$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_M N$. Choose now a $K^p \cdot M$-basis $a_0, \dots, a_M$ of the differential monomials in a which occur in $q(a,y)$ and write

$$q(a,y) = \sum_{i=0}^{N} q_i(a'^p, m, y) \cdot a_i,$$

for tuples a' in K and m in M, and differential polynomials $q_i(x', y', y)$ with integer coefficients and linear in x' and y'. Observe that we may assume that $a' \mathop{\smile\!\!\!\!\!\!\mid}_{Ma} N$, which implies $aa' \mathop{\smile\!\!\!\!\!\!\mid}_M N$.
By Corollary 6.3, the elements $a_0, \dots, a_M$ remain linearly independent over $K^p \cdot N$. Thus, for all b in N,

$$K \models \varphi(a,b) \longleftrightarrow \psi'(a, a', m, b),$$

where

$$\psi'(x, x', y', y) \quad = \quad \exists z \Big(z^p \doteq q_0(x'^p, y', y) \wedge \bigwedge_{i=1}^{N} q_i(x'^p, y', y) \doteq 0 \wedge \ \psi(x, y, z) \Big).$$

By the previous claim, the δ-tame formula $\psi'(x, x'; y', y)$ is an equation, so

$$\mathrm{tp}(a, a'/M) \vdash \mathrm{tp}^+_{\psi'}(a, a'/N).$$

In order to show that

$$\mathrm{tp}(a/M) \vdash \mathrm{tp}^+_{\varphi}(a/N),$$

consider a realisation $\tilde{a}$ of $\mathrm{tp}(a/M)$ and an instance $\varphi(x, b)$ in $\mathrm{tp}^+_\varphi(a/N)$. There is a tuple $\tilde{a}'$ such that $aa' \equiv_M \tilde{a}\tilde{a}'$. Since $K \models \psi'(a, a', m, b)$, we have $K \models \psi'(\tilde{a}, \tilde{a}', m, b)$. Observe that there are $\tilde{a}_0, \dots, \tilde{a}_N$ whith $q(\tilde{a}, y) = \sum_{i=0}^{N} q_i(\tilde{a}'^p, m, y) \cdot \tilde{a}_i$, so we have in particular that

$$q(\tilde{a}, b) = q_0(\tilde{a}', m, b),$$

whence $K \models \varphi(\tilde{a}, b)$, as desired.

□

Together with Proposition 6.5, we conclude the following result:

Corollary 7.2. *The theory DCF_p of existentially closed differential fields is equational.*

Similar to Corollary 5.3, there is a partial elimination of imaginaries for DCF_p, by Lemma 2.6 and Theorem 7.1. Unfortunately, we do not have either an explicit description of the canonical parameters of instances of δ-tame formulae.

Corollary 7.3. *The theory DCF_p of differentially closed fields of positive characteristic p has elimination of imaginaries, after adding canonical parameters for all instances of δ-tame formulae.*

Digression: On Srour's proof of the equationality of DCF_p.

Definition 7.4 (Srour [18])**.** An S-Formula φ is a conjunction of $\mathcal{L}_{\delta,s}$-equations such that, for every subterm $s(r)$ of a term occurring in φ, the equation $\delta(r) \doteq 0$ belongs to φ.

Srour's proof first shows that every formula is equivalent in $\mathsf{DCF}_p$to a Boolean combination of S-formulae. This follows from Proposition 6.5, as the next Lemma shows.

Lemma 7.5. *Every tame δ-formula is equivalent to an S-formula, and conversely.*

Proof. For every $\mathcal{L}_{\delta,s}$-formula $\psi(x,z)$ and every polynomial $q(x)$, observe that

$$\mathsf{DCF}_p \vdash \; \exists z \Big(z^p \doteq q(x) \wedge \psi(x,z)\Big) \longleftrightarrow \Big(\delta(q) \doteq 0 \wedge \psi\big(x, s(q(x))\big)\Big).$$

□

In order to show that S-formulae $\varphi(x;y)$ are equations, Srour uses the fact that, whenever A and B are elementary submodels of a model K of DCF_p which are linearly disjoint over their intersection M, then for every $a \in A$ and any S-formula $\varphi(x;y)$,

$$\mathrm{tp}(a/M) \vdash \mathrm{tp}^+_\varphi(a/B).$$

In order to do so, he observes that S-formulae are preserved under differential ring homomorphisms, as well as a striking result of Shelah (see the the proof of [16, Theorem 9]): the ring generated by A and B is differentially perfect, that is, it is closed under s. We would like to present a slightly simpler proof of Shelah's result.

Lemma 7.6. *(Shelah's Lemma* [16, Theorem 9]*) Let M be a common differential subfield of the the two differential fields A and B and $R = A \otimes_M B$. If M is existentially closed in B, then the ring of constants of R is generated by* C_A *and* C_B*.*

In particular, in characteristic p, the ring R is differentially perfect, whenever both A and B are.

Proof. **Claim 1.** The differential field A is existentially closed in R. In particular R is an integral domain.

Proof of Claim 1. Suppose $R \models \rho(a,r)$, for some quantifier-free δ-formula $\rho(x,y)$, and tuples a in A and r in R. Rewriting ρ, we may assume that $r = b$ and a occurs linearly in ρ and is an enumeration of a basis of A over M. In particular, there is a quantifier-free formula $\rho'(y)$ such that for all $b \in B$

$$R \models \forall y\Big(\rho(a,b) \longleftrightarrow \rho'(b)\Big).$$

Since M is existentially closed in B, and the validity of quantifier-free formulae is preserved under substructures, we conclude that there is some a' in M satisfying $\rho'(y)$ and thus $\rho(a,a')$ holds in R, and hence in A.

□ Claim 1

Let K be the quotient field of R.

Claim 2. The ring of constants of the ring R' generated by A and C_B is generated by C_A and C_B.

Proof of Claim 2. Let (a_i) be a basis of A over C_A, with $a_0 = 1$. Every x in R' can be written as $\sum_i a_i \cdot c_i$, for some c_i in the ring generated by C_A and C_B. By Fact 3.5, the a_i's are independent over C_K. If x is a constant in $R' \subset K$, then $x = c_0$. $\square$ Claim 2

Fix now a basis $(a_i)_{i \in I}$ of A over M and let $x = \sum_{i \in I} a_i \cdot b_i$ be a constant in R.

Claim 3. All $\delta(b_j)$ are in the M-span of $\{b_i \mid i \in I\}$.

Proof of Claim 3. Write $\delta(a_j) = \sum_{i \in I} m_{j,i} a_i$ for $m_{j,i} \in M$. Since $0 = \delta(x)$, we have

$$0 = \sum_i a_i \cdot \delta(b_i) + \sum_j (\sum_i m_{j,i} a_i) \cdot b_j = \sum_i a_i \cdot \delta(b_i) + \sum_i a_i \cdot (\sum_j m_{i,j} b_j),$$

whence $\delta(b_i) = -\sum_j m_{j,i} b_i$. $\square$ Claim 3

Claim 4. All b_i's lie in the ring generated by M and C_B.

Proof of Claim 4. Let $b \in B^n$ be the column vector of the non-zero elements of $\{b_i \mid i \in I\}$. By the last claim, there is an $n \times n$-matrix H with coefficients in M with $\delta(b) = H \cdot b$. Let $u^1, \ldots, u^m$ be a maximal linearly independent system of solutions of $\delta(y) = H \cdot y$ in M^n. Consider the $n \times m$-matrix U with columns $u^1, \ldots, u^m$. Since M is existentially closed in B, the column vectors $u^1, \ldots u^m, b$ must be linearly dependent, so $b = U \cdot z$ for some vector z in B^m. It follows that

$$H \cdot b = \delta(b) = \delta(U) \cdot z + U \cdot \delta(z) = H \cdot U \cdot z + U \cdot \delta(z) = H \cdot b + U \cdot \delta(z),$$

whence $\delta(z) = 0$. $\square$ Claim 4

In particular, the constant x lies in the ring R' from Claim 2, so x is in the ring generated by C_A and C_B. $\square$

Interlude: An alternative proof of the equationality of $\mathsf{SCF}_{p,\infty}$. As a by-product of Theorem 7.1, we obtain a different proof of the equationality of $\mathsf{SCF}_{p,\infty}$: We will show that every λ-tame formula is an equation, since it is equivalent in a particular model of $\mathsf{SCF}_{p,\infty}$, namely a differentially closed field of characteristic p, to a δ-tame formula. A similar method will appear again in Corollary 9.10.

Proposition 7.7. *Every λ-tame formula is equivalent in DCF_p to a δ-tame formula.*

Proof. Work inside a model (K, δ) of DCF_p. The proof goes by induction on the degree of the λ-tame formula $\varphi(x)$. If φ is a polynomial equation, there is nothing to prove. Since the result follows for conjunctions, we need only consider the particular case when φ is of the form:

$$\varphi(x) = p\text{-Dep}_n(q_1, \ldots, q_n) \ \vee \ (\overline{\lambda}(q_0, \ldots, q_n){\downarrow} \wedge \psi(x, \overline{\lambda}(q_0, \ldots, q_n))),$$

for some λ-tame formula $\psi(x, z_1, \ldots, z_n)$ of strictly smaller degree and polynomials $q_0, \ldots, q_n$ in $\mathbb{Z}[x]$.

Let $\mathrm{W}(x) = \mathrm{W}(q_1, \ldots, q_n)$ be the Wronskian of $q_1, \ldots, q_n$, that is, the determinant of the matrix

$$A(x) = \begin{pmatrix} q_1 & q_2 & \ldots & q_n \\ \delta(q_1) & \delta(q_2) & \ldots & \delta(q_n) \\ \vdots & & \vdots & \\ \delta^{n-1}(q_1) & \delta^{n-1}(q_2) & \ldots & \delta^{n-1}(q_n) \end{pmatrix}.$$

and $B(x)$ be the adjoint matrix of $A(x)$. Set

$$D(x) = \begin{pmatrix} q_0 \\ \delta(q_0) \\ \vdots \\ \delta^{n-1}(q_0) \end{pmatrix}.$$

Since K is differentially perfect, the elements $q_1(x), \ldots, q_n(x)$ are linearly independent over K^p if and only if $\mathrm{W}(x) \neq 0$. In that case, the functions $\overline{\lambda}(q_0, \ldots, q_n)$ are defined if and only if every coordinate of the vector $\mathrm{W}(x)^{-1} \cdot B(x) \cdot D(x)$ is a constant, in which case we have

$$\overline{\lambda}(q_0, \ldots, q_n)^p = \mathrm{W}(x)^{-1} \cdot B(x) \cdot D(x),$$

or equivalently,

$$(\mathrm{W}(x) \cdot \overline{\lambda}(q_0, \ldots, q_n))^p = \mathrm{W}(x)^{p-1} \cdot B(x) \cdot D(x)$$

By induction, the formula $\psi(x, z_1, \ldots, z_n)$ is equivalent to a δ-tame formula ψ_δ. Homogenising with respecto to $z_0, z_1, \ldots, z_n$, as in Proposition 6.6, there is a δ-tame formula $\psi'_\delta(x, z_0, z_1, \ldots, z_n)$ equivalent to

$$\psi_\delta(x, \frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}) \vee z_0 \doteq 0$$

Therefore, if $z = (z_1, \cdots, z_n)$, then

$$K \models \Big(\varphi(x) \longleftrightarrow \exists z \left(z^p \doteq \mathrm{W}(x)^{p-1} \cdot B(x) \cdot D(x) \ \wedge \ \psi'_\delta(x, \mathrm{W}(x), z)\right)\Big).$$

The right-hand side is a δ-tame formula, as desired. □

By Propositions 4.6 and 7.7, and Theorem 7.1, we obtain a different proof of Corollary 5.2:

Corollary 7.8. *The theory* $\mathsf{SCF}_{p,\infty}$ *of separably closed fields of characteristic* $p > 0$ *and infinite imperfection degree is equational.*

8. Model Theory of Pairs

The last theory of fields we will consider in this work is the (incomplete) theory ACFP of proper pairs of algebraically closed fields. Most of the results mentioned here appear in [8, 14, 1].

Work inside a sufficiently saturated model (K, E) of ACFP in the language $\mathcal{L}_P = \mathcal{L}_{rings} \cup \{P\}$, where $E = P(K)$ is the proper subfield. We will use the index P to refer to the expansion ACFP.

A subfield A of K is *tame* if A is algebraically independent from E over $E_A = E \cap A$, that is,

$$A \underset{E_A}{\overset{\mathsf{ACF}}{\downarrow}} E.$$

Tameness was called *P-independence* in [1], but in order to avoid a possible confusion, we have decided to use a different terminology.

Fact 8.1. The completions of the theory ACFP of proper pairs of algebraically closed fields are obtained once the characteristic is fixed. Each of these completions is ω-stable of Morley rank ω. The $\mathcal{L}_P$-type of a tame subfield of K is uniquely determined by its $\mathcal{L}_P$-quantifier-free type.
Every subfield of E is automatically tame, so the induced structure on E agrees with the field structure. The subfield E is a pure algebraically closed field and has Morley rank 1.
If A is a tame subfield, then its $\mathcal{L}_P$-definable closure coincides with the inseparable closure of A and its $\mathcal{L}_P$-algebraic closure is the field algebraic closure $\mathrm{acl}(A)$ of A, and $E_{\mathrm{acl}_P(A)} = \mathrm{acl}(E_A)$.

Based on the above fact, Delon [3] considered the following expansion of the language $\mathcal{L}_P$:

$$\mathcal{L}_D = \mathcal{L}_P \cup \{\mathrm{Dep}_n, \lambda_n^i\}_{1\leq i\leq n\in\mathbb{N}},$$

where the relation Dep_n is defined as follows:

$$K \models \mathrm{Dep}_n(a_1,\ldots,a_n) \Longleftrightarrow a_1,\ldots,a_n \text{ are } E\text{-linearly independent},$$

and the λ-functions take values in E and are defined by the equation

$$a_0 = \sum_{i=1}^{n} \lambda_n^i(a_0, a_1 \ldots, a_n)\, a_i,$$

if $K \models \mathrm{Dep}_n(a_1,\ldots,a_n) \wedge \neg\,\mathrm{Dep}_{n+1}(a_0,a_1,\ldots,a_n)$, and are 0 otherwise. Clearly, a field A is closed under the λ-functions if and only if it is linearly disjoint from E over E_A, that is, if it is P-special, as in Definition 3.1. Note that the fraction field of an $\mathcal{L}_D$-substructure is again closed under λ-functions and thus it is tame. The theory ACFP has therefore quantifier elimination [3] in the language $\mathcal{L}_D$. Note that the formula $P(x)$ is equivalent to $\mathrm{Dep}_2(1,x)$. Likewise, the predicate Dep_n is is equivalent to $\lambda_n^1(a_1, a_1 \ldots, a_n) = 1$.

Since the definable closure of a set is P-special, we conclude the following result by Lemma 3.2.

Corollary 8.2. *Given two subfields A and B of K containing an $\mathcal{L}_p$-elementary substructure M of K such that $A \mathop{\smile\!\!\!\!\!\!\perp}^P_M B$, then the fields $E\cdot A$ and $E\cdot B$ are linearly disjoint over $E\cdot M$.*

Our candidates for the equations in the theory ACFP will be called *tame* formulae.

Definition 8.3. Let x be a tuple of variables. A formula $\varphi(x)$ in the language $\mathcal{L}_P$ is *tame* if there are polynomials $q_1,\ldots,q_m$ in $\mathbb{Z}[X,Z]$, homogeneous in the variables Z, such that

$$\varphi(x) \quad = \quad \exists\,\zeta \in P^r\Big(\neg\zeta \doteq 0 \;\wedge\; \bigwedge_{j\leq m} q_j(x,\zeta) \doteq 0\Big).$$

Lemma 8.4. *Let $q_1, \ldots, q_m \in \mathbb{Z}[X, Y, Z]$ be polynomials, homogeneous in the variables Y and Z separately. The $\mathcal{L}_P$-formula*

$$\exists \upsilon \in P^r\, \exists \zeta \in P^s \Big(\neg \upsilon \doteq 0 \wedge \neg \zeta \doteq 0 \wedge \bigwedge_{k \leq m} q_k(x, \upsilon, \zeta) \doteq 0 \Big)$$

is equivalent in ACFP *to a tame formula.*

Proof. With the notation $\xi_{*,j} = \xi_{1,j}, \ldots, \xi_{r,j}$ and $\xi_{i,*} = \xi_{i,1}, \ldots, \xi_{i,s}$, the previous formula is equivalent in ACFP to the tame formula

$$\exists (\xi_{1,1}, \ldots, \xi_{rs}) \in P^{r,s} \setminus 0 \bigwedge_{i,j,k=1}^{r,s,m} q_k(x, \xi_{*,j}, \xi_{i,*}) \doteq 0.$$

□

Corollary 8.5. *The collection of tame formulae is closed under conjunctions and disjunctions.*

In order to prove that tame formulae determine the type in ACFP, we need a short observation regarding the E-annihilator of a (possibly infinite) tuple. Fix some enumeration $(M_i(x_1, \ldots, x_s))_{i=1,2,\ldots}$ of all monomials in s variables. Given a tuple a of length s, denote

$$\mathrm{Ann}_n(a) = \Big\{ (\lambda_1, \ldots, \lambda_n) \in E^n \;\Big|\; \sum_{i=1}^{n} \lambda_i \cdot M_i(a) = 0 \Big\}.$$

Notation. If we denote by $x \cdot y$ the scalar multiplication of two tuples x and y of length n, that is

$$x \cdot y = \sum_{i=1}^{n} x_i \cdot y_i,$$

then

$$\mathrm{Ann}_n(a) = \{ \lambda \in E^n \mid \lambda \cdot (M_1(a), \ldots, M_n(a)) = 0 \}.$$

Lemma 8.6. *Two tuples a and b of K have the same type if and only if*

$$\mathrm{ldim}_E \mathrm{Ann}_n(a) = \mathrm{ldim}_E \mathrm{Ann}_n(b)$$

and the type $\mathrm{tp}(\mathrm{Pk}(\mathrm{Ann}_n(a)))$ *equals* $\mathrm{tp}(\mathrm{Pk}(\mathrm{Ann}_n(a)))$ *(in the pure field language), for every n in $\mathbb{N}$.*

Proof. We need only prove the right-to-left implication. Since $\mathrm{Pk}(\mathrm{Ann}_i(a))$ is determined by $\mathrm{Pk}(\mathrm{Ann}_n(a))$, for $i \leq n$, we obtain an automorphism of E mapping $\mathrm{Pk}(\mathrm{Ann}_n(a))$ to $\mathrm{Pk}(\mathrm{Ann}_n(b))$ for all n. This automorphism maps $\mathrm{Ann}_n(a)$ to $\mathrm{Ann}_n(b)$ for all n and hence extends to an isomorphism of the rings $E[a]$ and $E[b]$. It clearly extends to a field isomorphism of the tame subfields $E(a)$ and $E(b)$ of K, which in turn can be extended to an automorphism of (K, E). So a and b have the same ACFP-type, as required. □

Proposition 8.7. *Two tuples a and b of K have the same* ACFP*-type if and only if they satisfy the same tame formulae.*

Proof. Let $q_1(Z), \ldots, q_m(Z)$ be homogeneous polynomials over $\mathbb{Z}$. By Lemma 8.6, it suffices to show that

« $\mathrm{Ann}_n(x)$ has a k-dimensional subspace V such that $\bigwedge_{j \leq m} q_j(\mathrm{Pk}(V)) = 0$ »

is expressible by a tame formula. Indeed, it suffices to guarantee that there is an element ζ in $\mathrm{Gr}_k(E^n)$ such that

$$(e \lrcorner \zeta) \cdot (M_1(x), \ldots, M_n(x)) = 0$$

for all e from a a fixed basis of $\bigwedge^{k-1}(E^n)^*$, and

$$\bigwedge_{j \leq m} q_j(\zeta) = 0.$$

In particular, the tuple ζ is not trivial, so we conclude that the above is a tame formula. □

By compactness, we conclude the following:

Corollary 8.8. *In the (incomplete) theory* ACFP *of proper pairs of algebraically closed fields, every formula is a Boolean combination of tame formulae.*

9. Equationality of belles paires of algebraically closed fields

In order to show that the stable theory ACFP of proper pairs of algebraically closed fields is equational, we need only consider tame formulae with respect to some partition of the variables, by Corollary 8.8. As before, work inside a sufficiently saturated model (K, E) of ACFP in the language $\mathcal{L}_P = \mathcal{L}_{rings} \cup \{P\}$, where $E = P(K)$ is the proper subfield.
Consider the following special case as an auxiliary result.

Lemma 9.1. *Let $\varphi(x; y)$ be a tame formula. The formula*

$$\varphi(x; y) \wedge x \in P$$

is an equation.

Proof. Let b be a tuple in K of length $|y|$, and suppose that the formula $\varphi(x, b)$ has the form

$$\varphi(x, b) \quad = \quad \exists \zeta \in P^r \left(\neg \zeta \doteq 0 \ \wedge \ \bigwedge_{j \leq m} q_j(x, b, \zeta) \doteq 0 \right).$$

for some polynomials $q_1, \ldots, q_m$ with integer coefficients and homogeneous in ζ. Express each of the monomials in b appearing in the above equation as a linear combination of a basis of K over E. We see that there are polynomials $r_1, \ldots, r_s$ with coefficients in E, homogeneous in ζ, such that the formula $\varphi(x, b) \wedge x \in P$ is equivalent to

$$\exists \zeta \in P^r \left(\neg \zeta \doteq 0 \ \wedge \ \bigwedge_{j \leq s} r_j(x, \zeta) \doteq 0 \right).$$

Working inside the algebraically closed subfield E, the expression inside the brackets is a projective variety, which is hence complete. By Remark 3.7, its projection is again Zariski-closed, as desired. □

Proposition 9.2. *Let $\varphi(x; y)$ be a tame formula. The formula $\varphi(x; y)$ is an equation.*

Proof. We need only show that every instance $\varphi(a,y)$ of a tame formula is indiscernibly closed. By Lemma 2.3, it suffices to consider a Morley sequence $(b_i)_{i\leq\omega}$ over an elementary substructure M of (K,E) with

$$a \mathop{\downarrow}_{M}^{P} b_i \text{ with } \models \varphi(a,b_i) \text{ for } i<\omega.$$

Suppose that the formula $\varphi(a,y)$ has the form

$$\varphi(a,y) \;=\; \exists\zeta\in P^r\Big(\neg\zeta\doteq 0 \;\wedge\; \bigwedge_{j\leq m} q_j(a,y,\zeta)\doteq 0\Big),$$

for polynomials $q_1,\ldots,q_n$ with integer coefficients and homogeneous in ζ.
By Corollary 8.2, the fields $E\cdot M(a)$ and $E\cdot M(b_i)$ are linearly disjoint over $E(M)$ for every $i<\omega$. A basis (c_ν) of $E\cdot M(a)$ over $E\cdot M$ remains thus linearly independent over $E\cdot M(b_i)$. By appropriately writing each monomial in a in terms of the basis (c_ν), and after multiplication with a common denominator, we have that

$$\varphi(a,y) \;=\; \exists\zeta\in P^r\Big(\neg\zeta\doteq 0 \;\wedge\; \bigwedge_{\nu} r_\nu(e,m,y,\zeta)\cdot c_\nu\doteq 0\Big),$$

where e a tuple from E and m is a tuple from M, and the polynomials $r_\nu(X,Y',Y,Z)$ are homogeneous in Z. Hence, linearly disjointness implies that

$$K\models\; \exists\zeta\in P^r\Big(\neg\zeta\doteq 0 \;\wedge\; \bigwedge_{\nu} r_\nu(e,m,b_i,\zeta)\doteq 0\Big) \text{ for } i<\omega.$$

By Lemma 9.1, the formula

$$\varphi'(e,y',y)=\exists\zeta\in P^r\Big(\neg\zeta\doteq 0 \;\wedge\; \bigwedge_{\nu} r_\nu(e,y',y,\zeta)\doteq 0\Big)$$

is indiscernibly closed. Since the sequence $(m,b_i)_{i\leq\omega}$ is indiscernible, we have $K\models \varphi'(e,m,b_\omega)$, so $K\models\varphi(a,b_\omega)$, as desired.

□

Corollary 8.8 and Proposition 9.2 yield now the equationality of ACFP.

Theorem 9.3. *The theory of proper pairs of algebraically closed fields of a fixed characteristic is equational.*

Encore! An alternative proof of equationality for pairs in characteristic 0. We will exhibit an alternative proof to the equationality of the theory T_p of belles paires of algebraically closed fields in characteristic 0, by means of differential algebra, based on an idea of Günaydın [5].

Definition 9.4. Consider an arbitrary field K with a subfield E. A subspace of the vector space K^n is *E-defined* if it is generated by vectors from E^n.
Since the intersection of two E-defined subspaces is again E-defined, every subset A of K is contained in a smallest E-defined subspace A^E, which we call the *E-hull* of A.

Notation. We write v^E to denote $\{v\}^E$. Clearly A^E is the sum of all v^E for v in A. The E-hull v^E can be computed as follows: Fix a basis $(c_\nu \mid \nu \in N)$ of K over E and write $v = \sum_{\nu \in N} c_\nu e_\nu$ for vectors $e_\nu \in E^n$. Then $\{e_\nu \mid \nu \in N\}$ is a generating set of v^E.

Similarly, every subset A of the ring of polynomials $K[X_1, \ldots, X_n]$ has an E-hull A^E, that is, the smallest E-defined subspace of $K[X_1, \ldots, X_n]$.

Lemma 9.5. *Let I be an ideal of $K[X_1, \ldots, X_n]$. Then I^E is the smallest ideal containing I and generated by elements of $E[X_1, \ldots, X_n]$.*

Proof. An ideal J generated by the polynomials f_i in $E[X_1, \ldots, X_n]$ is generated, as a vector space, by the products $X_j f_i$. Conversely, for each variable X_j, the vector space $\{f \mid X_j f \in I^E\}$ is E-defined and contains I. Thus it contains I^E, so I^E is an ideal. □

If the ideal I is generated by polynomials f_i, then the union of all f_i^E generates the ideal I^E. Note also that, if I is *homogeneous*, i.e. it is the sum of all

$$I_d = \{f \in I \mid h \text{ homogeneous of degree } d\},$$

then so is I^E, with $(I^E)_d = (I_d)^E$.

From now on, consider a sufficiently saturated algebraically closed differential field (K, δ), equipped with a non-trivial derivation δ. Denote its field of constants C_K by E. For example, we may choose (K, δ) to be a saturated model of DCF_0, the elementary theory of differential closed fields of characteristic zero.
Observe that the pair (K, E) is a model of the theory ACFP_0 of proper extensions of algebraically closed fields in characteristic 0. In order to show that this theory is equational, it suffices to show, by Proposition 8.7, that every instance of a tame formula determines a Kolchin-closed set in (K, δ). We first need some auxiliary lemmata on the differential ideal associated to a system of polynomial equations.

Lemma 9.6. *Let v be a vector in K^n. Then the E-hull of v is generated by $v, \delta(v), \ldots, \delta^{n-1}(v)$.*

Proof. Any E-defined subspace is clearly closed under δ. Thus, we need only show the the subspace V generated by $v, \delta(v), \ldots, \delta^{n-1}(v)$ is E-defined. Let $k \leq n$ be minimal such that v can be written as

$$v = a_1 e_1 + \cdots + a_k e_k$$

for some elements a_i in K and vectors e_i in E^n. Thus, the e_i's are linearly independent and generate v^E. Hence $V \subset v^E$. If the dimension of V is strictly smaller than k, then $v, \delta(v), \ldots, \delta^{k-1}(v)$ are linearly dependent over K. The rows of the matrix

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ \delta(a_1) & \delta(a_2) & \ldots & \delta(a_k) \\ \vdots & & \vdots & \\ \delta^{k-1}(a_1) & \delta^{k-1}(a_2) & \ldots & \delta^{k-1}(a_k) \end{pmatrix}$$

are thus linearly dependent over K. It follows from Fact 3.5 that $a_1, \ldots, a_k$ are linearly dependent over E. So there are ξ_i in E, not all zero, such that $\xi_1 a_1 + \cdots + \xi_k a_k = 0$. The vector space

$$\Big\{\sum_{i=1}^{k} b_i e_i \;\Big|\; \sum_{i=1}^{k} \xi_i b_i = 0\Big\},$$

which contains v, has a basis from E^n and dimension strictly smaller than k, contradicting the choice of the e_i's. □

In order to apply the previous result, consider the derivation D on the polynomial ring $K[X_1, \ldots, X_n]$ obtained by differentiating the coefficients of a polynomial in K (and setting $D(X_i) = 0$, for $1 \leq i \leq n$). We say that an ideal I of $K[X_1, \ldots, X_n]$ is *differential* if it is closed under D.

Corollary 9.7. *An ideal of $K[X]$ is differential if and only if it can be generated by elements from $E[X]$.*

Corollary 9.8. *Given homogeneous polynomials $h_0, \ldots, h_m$ in $K[X]$ of a fixed degree d, there exists an integer k in $\mathbb{N}$ (bounded only in terms of d and the length of X) such that the ideal generated by $\{D^j(h_i)\}_{\substack{i\leq m\\ j<k}}$ is a differential and homogeneous ideal.*

We now have all the ingredients in order to show that tame formulae are equations.

Proposition 9.9. *Let $\varphi(x, y)$ be a tame formula. The definable set $\varphi(x, b)$ is Kolchin-closed set in (K, δ).*

Proof. Suppose that

$$\varphi(x, b) \;\; = \;\; \exists \zeta \in P^r \Big(\neg \zeta \doteq 0 \;\wedge\; \bigwedge_{i\leq m} q_i(x, \zeta) \doteq 0\Big),$$

for polynomials $q_j(X, Z)$ over K homogeneous in Z of some fixed degree d. Let k be as in Corollary 9.8.
For a tuple a in K of length $|x|$, write

$$D^j(q_i(a, Z)) = q_{i,j}(a, \ldots, \delta^j(a), Z),$$

for polynomials $q_{i,j}(X_0, \ldots, X_k, Z)$ over K, homogeneous in Z. By Corollary 9.8, the ideal $I(a, Z)$ generated by

$$\{q_{i,j}(a, \ldots, \delta^j(a), Z)\}_{\substack{i<m\\ j<k}}$$

has a generating set consisting of homogeneous polynomials

$$g_1(Z), \ldots, g_s(Z)$$

with coefficients in $E[Z]$.

Now, since ζ ranges over the constant field, the tuple a realises $\varphi(x, b)$ if and only if

$$(K, E) \models \;\; \exists \zeta \in P^r \Big(\neg \zeta \doteq 0 \;\wedge\; I(a, \zeta) \doteq 0\Big),$$

which is equivalent to

$$(K,E) \models \ \exists \zeta \in P^r \Big(\neg\zeta \doteq 0 \ \wedge \ \bigwedge_{i \leq s} g_i(\zeta) \doteq 0 \Big),$$

The field E is an elementary substructure of K, so the above is equivalent to

$$K \models \ \exists \zeta \ \Big(\neg\zeta \doteq 0 \ \wedge \ \bigwedge_{i<s} g_i(\zeta) \doteq 0 \Big),$$

which is again equivalent to

$$K \models \ \exists \zeta \Big(\neg\zeta \doteq 0 \ \wedge \ I(a,\zeta) \doteq 0 \Big).$$

Since $I(a, Z)$ is homogeneous, the Zariski-closed set it determines is complete, hence its projection is given by a finite number of equations $X(a, \ldots, \delta^{k-1}(a))$. Thus, the tuple a realises $\varphi(x, b)$ holds if and only if

$$(K,\delta) \models X(a,\ldots,\delta^{k-1}(a)),$$

which clearly describe a Kolchin-closed set, as desired. □

By Corollary 8.8, we conclude the following:

Corollary 9.10. *The theory* ACFP_0 *of proper pairs of algebraically closed fields of characteristic* 0 *is equational.*

A definable set $\{a \in K^n \mid (K,E) \models \varphi(a,b)\}$ is *t-tame*, if φ is tame, for some b a tuple in K.

Corollary 9.11. *In models of* ACFP_0*, the family of t-tame sets has the DCC.*

Proof. The Kolchin topology is noetherian, by Ritt-Raudenbush's Theorem. □

Question. Do t-tame sets have the DCC in arbitrary characteristic?

10. Appendix: Linear Formulae

A stronger relative quantifier elimination was provided in [5, Theorem 1.1], , which yields a nicer description of the equations to consider in the theory ACFP_0. We will provide an alternative approach to Günaydın's result, valid in arbitrary characteristic. We work inside a sufficiently saturated model (K, E) of ACFP.

A tame formula $\varphi(x)$ (cf. Definition 8.3) is *linear* if the corresponding polynomials in φ are linear in Z, that is, if there is a matrix $(q_{i,j}(X))$ of polynomials with integer coefficients such that

$$\varphi(x) \quad = \quad \exists \zeta \in P^s \ \left(\neg\zeta \doteq 0 \ \wedge \ \bigwedge_{j=1}^{k} \zeta_1 q_{1,j}(x) + \cdots + \zeta_s q_{s,j}(x) \doteq 0 \right).$$

A linear formula is *simple* if $k = 1$, that is, if it has the form

$$\mathrm{Dep}_s(q_1(x), \ldots, q_s(x)),$$

for polynomials q_i in $Z[X_1, \ldots, X_n]$.

We will show that every tame formula is equivalent in ACFP to a conjunction of simple linear formulae. We first start with an easy observation.

Lemma 10.1. *Every tame formula is equivalent in* ACFP *to a linear tame formula.*

Proof. Consider a tame formula

$$\varphi(x) \quad = \quad \exists\,\zeta \in P^r\Big(\neg\zeta \doteq 0 \ \wedge \bigwedge_{j\le m} q_j(x,\zeta) \doteq 0\Big).$$

Denote by Z the tuple of variables $(Z_1,\dots,Z_{\mathrm{length}(\zeta)})$. For a tuple a in K of length $|x|$, denote by $I(a,Z)$ the ideal in $K[Z]$ generated by $q_1(a,Z),\dots q_m(a,Z)$. Recall the definition of the E-hull $I(a,Z)^E$ of $I(a,Z)$ (Definition 9.4). Since $I(a,Z) \subset I(a,Z)^E$, a zero of $I(a,Z)^E$ is a zero of $I(a,Z)$. A relative converse holds: If the tuple ζ in E^r is a zero of the ideal $I(a,Z)$, then $I(a,Z)$ is contained in the ideal generated by all $Z_i - \zeta_i$'s, which is E-defined, so ζ is a zero of $I(a,Z)^E$. As in the proof of Proposition 9.9, we conclude that $(K,E) \models \varphi(a)$ if an only if $I^E(a,Z)$ has a non-trivial zero in K^r.
The ideal $I(a,Z)^E$ is generated by polynomials from $q_j(a,Z)^E$. In particular, there is a degree d, independent from a, such that $I^E(a,Z)$ has a non-trivial zero if and only if the E-hull $(I(a,Z)^E)_d$ of $I(a,Z)_d$ is not all of $K[Z]_d$. As a vector space, the ideal $I(a,Z)_d$ is generated by all products $M\cdot q_j(a,Z)$, with M a monomial in Z such that $\deg(M) + \deg_Z(q_j(X,Z)) = d$. Given an enumeration $M_1,\dots,M_s$ of all monomials in Z of degree d, the vector space $I(a,Z)_d$ is generated by a sequence of polynomials $f_1,\dots,f_k$ of the form

$$f_j = M_1 r_{1,j}(a) + \cdots + M_s r_{s,j}(a),$$

for polynomials $r_{i,j}(X) \in \mathbb{Z}[X]$ which do not depend of a. Thus, the tuple a realises $\varphi(x)$ if and only if $(I(a,Z)^E)_d \neq K[Z]_d$, that is, if and only if there is a tuple $\xi \in E^s \setminus 0$ such that $\xi_1 r_{1,j}(a) + \cdots + \xi_s r_{s,j}(a) = 0$ for all $j = 1,\dots,k$. The latter is expressible by a linear formula. □

In order to show that every tame formula is equivalent to a conjunction of simple linear formulae, we need the following result:

Proposition 10.2. *For all natural numbers m and n, there is a natural number N and an $n\times N$-matrix $(r_{j,k})$ of polynomials from $\mathbb{Z}[x_{1,1},\dots,x_{m,n}]$ such that the linear formula*

$$\exists\,\zeta \in P^r\left(\neg\zeta \doteq 0 \ \wedge \bigwedge_{j=1}^{n} \zeta_1 x_{1,j} + \cdots + \zeta_m x_{m,j} \doteq 0\right). \tag{1}$$

is equivalent in ACFP *to the conjunction of*

$$\bigwedge_{j_1<\cdots<j_m} \det((x_{i,j_{i'}})) \doteq 0 \tag{2}$$

and

$$\bigwedge_{k=1}^{N} \mathrm{Dep}_m\big(\sum_{j=1}^{n} x_{1,j} r_{j,k}(\bar x),\dots,\sum_{j=1}^{n} x_{m,j} r_{j,k}(\bar x)\big). \tag{3}$$

Proof. The implication (1) $\Rightarrow$ ((2) $\wedge$ (3)) always holds, regardless of the choice of the polynomials $r_{j,k}$: Whenever a matrix $A = (a_{i,k})$ over K is such that there is a non-trivial vector ζ in E^m with

$$\bigwedge_{j=1}^{n}\sum_{i=1}^{m}\zeta_i a_{i,j} = 0,$$

then the rows of A are linearly dependent, so $\det((a_{i,j_{i'}}))=0$ for all $j_1<\cdots<j_m$. For all k, we have that

$$\sum_{i=1}^{m}\zeta_i\big(\sum_{j=1}^{n}a_{i,j}r_{j,k}(\bar{a})\big)=\sum_{j=1}^{n}\big(\sum_{i=1}^{m}\zeta_i a_{i,j}\big)r_{j,k}(\bar{a})=0.$$

For the converse, an easy compactness argument yields the existence of the polynomials $r_{j,k}$, once we show that (1) follows from (2) together with the infinite conjunction

$$\bigwedge_{r_1,\ldots r_n\in\mathbb{Z}[\bar{x}]}\mathrm{Dep}_m\big(\sum_{j=1}^{n}x_{1,j}r_j(\bar{x}),\ldots,\sum_{j=1}^{n}x_{m,j}r_j(\bar{x})\big). \tag{4}$$

Hence, let $A=(a_{i,k})$ be a matrix over K witnessing (2) and (4). The rows of A are K-linearly dependent, by (2). If the matrix were defined over E, its rows would then be E-linearly dependent, which yields (1). Thus, if we R is the subring of K generated by the entries of A, we may assume that the ring extension $E\subset E[R]$ is proper.

Claim 1. There is a non-zero element r in R which is not a unit in $E[R]$.

Proof of Claim 1. The field $E(R)$ has transcendence degree $\tau\geq 1$ over E. As in the proof of Noether's Normalisation Theorem [9, Theorem X 4.1], there is a transcendence basis $r_1,\ldots,r_\tau$ of R over E, such that $E[R]$ is an integral extension of $E[r_1,\ldots,r_\tau]$. If r_1 were a unit in $E[R]$, its inverse would u be a root of a polynomial with coefficients in $E[r_1,\ldots,r_\tau]$ and leading coefficient 1. Multiplying by a suitable power of r_1, we obtain a non-trivial polynomial relation among the $r_j's$, which is a contradiction. □ Claim 1

Claim 2. Given a sequence $V_1,\ldots V_n$ of finite dimensional E-subvector spaces of $E[R]$, there is a sequence $z_1,\ldots,z_n$ of non-zero elements of R such that the subspaces $V_1z_1,\ldots,V_nz_n$ are independent.

Proof of Claim 2. Assume that $z_1,\ldots,z_{k-1}$ have been already constructed. Let z be as in Claim 1. If we consider the sequence of ideals $z^kE[R]$, an easy case of Krull's Intersection Theorem ([9, Theorem VI 7.6]) applied to the noetherian integral domain $E[R]$ yields that

$$0=\bigcap_{k\in\mathbb{N}}z^kE[R].$$

Choose some natural number N_k large enough such that

$$(V_1z_1+\cdots+V_{k-1}z_{k-1})\cap z^{N_k}E[R]=0,$$

and set $z_k=z^{N_k}$. □ Claim 2

Let us now prove that the matrix A satisfies (1). Let V_j be the E-vector space generated by $a_{1,j},\ldots,a_{m,j}$, that is, by the j-th column of A. Choose $0\neq z_j$ in R as in Claim 2, and write each $z_j=r_j(\bar{a})$, for some polynomial $r_j(\bar{x})$ with integer coefficients. Since A satisfies (4), there is a non-trivial tuple ζ in E^m such that

$$\sum_{i=1}^{m}\zeta_i\big(\sum_{j=1}^{n}a_{i,j}z_j\big)=\sum_{j=1}^{n}\big(\sum_{i=1}^{m}\zeta_i a_{i,j}\big)z_j=0.$$

Observe that $\left(\sum_{i=1}^m \zeta_i a_{i,j}\right) z_j$ belongs to $V_j z_j$. The subspaces $V_i z_1, \ldots, V_n z_n$ are independent, so each $\left(\sum_{i=1}^m \zeta_i a_{i,j}\right) z_j$ must equal 0. Therefore so is

$$\sum_{i=1}^m \zeta_i a_{i,j} = 0,$$

as desired. □

Question. Can the integer N and the polynomials $r_{i,j}$ in Proposition 10.2 be explicitly computed?

Theorem 10.3. *Every tame formula is equivalent in* ACFP *to a conjunction of simple linear formulae.*

Proof. By Lemma 10.1, it suffices to show that every linear formula is is equivalent in ACFP to a conjunction of simple linear formulae. This follows immediately from Proposition 10.2, once we remark that the polynomial equation $q(x) \doteq 0$ is equivalent in ACFP to the simple linear formula $\mathrm{Dep}_1(q(x))$. □

Together with Corollary 8.8, we deduce another proof of [5, Theorem 1.1], valid in all characteristics:

Corollary 10.4. *In the theory* ACFP *of proper pairs of algebraically closed field, every formula is equivalent in to a boolean combination of simple tame formulae.*

In particular, we obtain another proof of the equationality of ACFP in characteristic 0, for every simple linear formula is an equation in a differential field: Indeed, the formula $\mathrm{Dep}_s(x_1, \ldots, x_s)$ is equivalent to the differential equation $\mathrm{W}(x_1, \ldots, x_s) \doteq 0$.

Corollary 10.4 implies, together with Corollary 8.5, that a finite conjunction of linear formulae is again linear. However, we do not think that the same holds for simple linear formulae.

A key point in the proof of [5, Theorem 1.1] is the fact that each $\mathcal{L}_D$-function λ_n^i defines, on its domain, a continuous function with respect to the topology generated by instances of simple linear formulae [5, Proposition 2.6]. We will conclude with an easy proof that all functions $\lambda_n^i \times \mathrm{id} \times \cdots \times \mathrm{id}$ are continuous with respect to this topology. For this, we need an auxiliary definition (cf. Definition 4.4):

Definition 10.5. The collection of λ_P-formulae is the smallest collection of formulae in the language $\mathcal{L}_D$, closed under conjunctions and containing all polynomial equations, such that, for any natural number n and polynomials $q_0, \ldots, q_n$ in $\mathbb{Z}[x]$, given a λ_P-formula $\psi(x, z_1, \ldots, z_n)$, the formula

$$\varphi(x) = \mathrm{Dep}_n(q_1(x), \ldots, q_n(x)) \ \vee \\ \left(\overline{\lambda}(q_0(x), \ldots, q_n(x))\!\downarrow \ \wedge \ \psi(x, \overline{\lambda}_n(q_0(x), \ldots, q_n(x))) \right)$$

is λ_P-tame, where $\overline{\lambda}(y_0, \ldots, y_n)\!\downarrow$ is an abbreviation for

$$\neg\, \mathrm{Dep}_n(y_1, \ldots, y_n) \wedge \mathrm{Dep}_{n+1}(y_0, \ldots, y_n).$$

Proposition 10.6. *Up to equivalence in* ACFP, *tame formulae and* λ_P*-formulae coincide.*

Proof. Notice that every simple linear formula is λ_P-tame, since

$$\mathrm{Dep}_n(y_1,\ldots,y_n) \ \Leftrightarrow\ \mathrm{Dep}_n(y_1,\ldots,y_n) \vee \big(\overline{\lambda}(0,y_1,\ldots,y_n)\!\downarrow\ \wedge (1 \doteq 0)\big)\,.$$

By Theorem 10.3, we conclude that all tame formulae are λ_P-tame.
We prove the other inclusion by induction on the degree of the λ_P-formula $\varphi(x)$. Polynomial equations are clearly tame. By Corollary 8.5, the conjunction of tame formulae is again tame. Thus, we need only show that $\varphi(x)$ is tame, whenever

$$\varphi(x) = \mathrm{Dep}_n(q_1,\ldots,q_n) \ \vee\ \ \big(\overline{\lambda}(q_0,\ldots,q_n)\!\downarrow\ \wedge\ \psi(x,\overline{\lambda}_n(q_0,\ldots,q_n))\big),$$

for some tame formula $\psi(x,z_1,\ldots,z_n)$. Write

$$\psi(x,z) \ \ = \ \ \exists\zeta\in P^s\Big(\neg\zeta\doteq 0\ \wedge\ \bigwedge_{k\le m} p_k(x,z,\zeta)\doteq 0\Big),$$

for some polynomials $p_1(x,z,u),\ldots,p_m(x,z,u)$ with integer coefficients and homogeneous in u.
Homogenising with respect to the variables $z_0,z_1,\ldots,z_n$, there is some natural number N such that, for each $k\le m$,

$$p_k(x,z_0^{-1}z,u)z_0^N = r_k(x,z_0,z,u),$$

where r_k is homogeneous in (z_0,z) and in u, separately. Thus,

$$\mathsf{ACFP}\models\Bigg(\varphi(x)\longleftrightarrow\Bigg(\exists(\zeta_0,\zeta)\in P^{n+1}\,\exists\upsilon\in P^s\Big(\neg(\zeta_0,\zeta)\doteq 0\wedge\neg\upsilon\doteq 0$$
$$\wedge\,\zeta_0q_0(x)+\cdots+\zeta_nq_n(x)\doteq 0\wedge\bigwedge_{k\le m} r_k(x,\zeta_0,\zeta,\upsilon)\doteq 0\Big)\Bigg)\Bigg).$$

The right-hand expression is a tame formula, by Lemma 8.4, and so is φ, as desired. □

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