Higher inverse Limits

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Let I be a totally ordered set. A *projective system* is an I-indexed family (A_α) of abelian groups together with a commutative system of homomorphisms

$$\pi_{\alpha\beta} : A_\beta \to A_\alpha \ , \quad (\alpha < \beta \in I).$$

Projective systems forms an abelian category in a natural way. $\varprojlim$ is a left exact functor to the category of abelian groups. Since the category of projective systems has enough injectives $\varprojlim$ has right derived functors

$$\varprojlim = \varprojlim{}^0, \ \varprojlim{}^1, \ \varprojlim{}^2 \ldots$$

Fix a projective system $(A_\alpha, \pi_{\alpha\beta})_{\alpha<\beta\in I}$ and a number $n \geq 0$. We call a family

$$c = (c_{\alpha_0\ldots\alpha_n}),$$

indexed by ascending sequences $\alpha_0 < \ldots < \alpha_n$ of elements of I, an *n–cochain* if each $c_{\alpha_0\ldots\alpha_n}$ is an element of A_{α_0}. The set of n–chains form an abelian group C^n under component–wise addition. The coboundary homomorphisms

$$\delta : \mathrm{C}^n \to \mathrm{C}^{n+1},$$

defined by

$$(\delta c)_{\alpha_0\ldots\alpha_{n+1}} = \pi_{\alpha_0\alpha_1}(c_{\alpha_1\ldots\alpha_n}) + \sum_{i=1}^{n+1} (-1)^i c_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{n+1}},$$

make $\mathrm{C} = (\mathrm{C}^n)_{n\geq 0}$ into a cochain complex, which means that $\delta^2 = 0$.

As usual the cohomology groups of C are defined as the quotients

$$\mathrm{H}^n(\mathrm{C}) = \mathrm{Z}^n(\mathrm{C}) / \mathrm{B}^n(\mathrm{C})$$

of the groups

$$\mathrm{Z}^n(\mathrm{C}) = \{z \in \mathrm{C}^n \mid \delta z = 0\}$$

*Material related to a talk given in Hattingen, July 1999 of n–cocycles and the subgroups

$$\mathrm{B}^n(\mathrm{C}) = \{\delta c \mid c \in \mathrm{C}^{n-1}\}$$

of n–coboundaries.

Theorem 1 ([1, Théorème 4.1]).

$$\varprojlim_{\alpha\in I}{}^{n} A_\alpha = \mathrm{H}^n(\mathrm{C})$$

Readers who don't like derived functors can take $\mathrm{H}^n(\mathrm{C})$ as the definition of $\varprojlim_{\alpha\in I}{}^{n} A_\alpha$. The content of the last theorem is then that the $\varprojlim{}^{n}$ has the characterizing properties of the derived functors: They are trivial on injective projective systems and there is a natural long cohomology sequence.

Lemma 2 ([1, p.12]). *If J is cofinal in I, the natural restriction map*

$$\varprojlim_{\alpha\in I}{}^{n} A_\alpha \to \varprojlim_{\alpha\in J}{}^{n} A_\alpha$$

is an isomorphism for all n.

Proof. The $\varprojlim_{\alpha\in J}{}^{n} A_\alpha$ $(n = 0, 1, \dots)$ have the characterizing properties of the right derived functors of $\varprojlim_{\alpha\in I}{}^{0} A_\alpha = \varprojlim_{\alpha\in J}{}^{0} A_\alpha$. □

Lemma 3. *If I has a last element the projective system $(A_\alpha)_{\alpha\in I}$ is acyclic. That means that $\varprojlim_{\alpha\in I}{}^{n} A_\alpha = 0$ for all $n \geq 1$.*

Proof. We begin with a general observation, which will be useful later on. Fix an element $\lambda \in I$ and denote by C^n_λ the set of n–cochains over $I_\lambda = \{\alpha \in I \mid \alpha < \lambda\}$. Define two homomorphisms, the restriction

$$\mathrm{t} : \mathrm{C}^n \to \mathrm{C}^n_\lambda$$

and

$$\mathrm{h} : \mathrm{C}^n \to \mathrm{C}^{n-1}_\lambda$$

by $\mathrm{h}(c)_{\alpha_0\dots\alpha_{n-1}} = c_{\alpha_0\dots\alpha_{n-1}\lambda}$. h does not commute with δ, but we have for $c \in \mathrm{C}^n$

$$\mathrm{h}\delta(c) = (-1)^{n+1}\mathrm{t}(c) + \delta\mathrm{h}(c). \tag{1}$$

Now assume $n \geq 1$ and z a n–cocycle. Let λ be the last element of I. Define the $n-1$–cochain d by

$$d_{\alpha_0\dots\alpha_{n-1}} = \begin{cases} z_{\alpha_0\dots\alpha_{n-1}\lambda} & \text{if } \alpha_{n-1} < \lambda \\ 0 & \text{otherwise} \end{cases}$$

Then $\delta(d) = (-1)^n z$. This follows from (1) for indices in I_λ and

$$\delta(d)_{\alpha_0\ldots\alpha_{n-1}\lambda} = (-1)^n d_{\alpha_0\ldots\alpha_{n-1}} = (-1)^n z_{\alpha_0\ldots\alpha_{n-1}\lambda}.$$

□

Jensen proved in [1, Corollaire 3.2] that

$$\varprojlim_{\alpha\in I}{}^{n+2} A_\alpha = 0,$$

whenever $\mathrm{cf}(I) \leq \omega_n$. Furthermore he proved that the result is optimal: For every n there is a projective system $(A_\alpha)_{\alpha\in\omega_n}$ such that $\varprojlim_{\alpha\in\omega_n}{}^{n+1} A_\alpha \neq 0$ ([1, Proposition 6.2]).

If we look at *epimorphic* systems $(A_\alpha, \pi_{\alpha\beta})_{\alpha<\beta\in I}$, where all the $\pi_{\alpha\beta}$ are surjective, we have a better result:

Theorem 4 ([3, Theorem 3.3]). *For epimorphic systems with* $\mathrm{cf}(I) \leq \omega_n$ *we have*

$$\varprojlim_{\alpha\in I}{}^{n+1} A_\alpha = 0.$$

Proof. We use induction on n and begin with the case $n = 0$, where we can assume that $I = \mathbb{N}$. Let a 1–cocycle c be given. We choose recursively elements $d_i \in A_i$ such that $\pi_{i,i+1}(d_{i+1}) = d_i + c_{i,i+1}$. The relation $\delta c = 0$ entails now $\delta d = c$.

Now assume $n > 0$.

We may assume that I is isomorphic to ω_k for some $k \leq n$. Let c be an $(n+1)$–cocycle. We want to write c as the coboundary of an n–cochain d. We construct the components $d_{\alpha_0\ldots\alpha_n}$ by recursion on α_n.

Fix $\lambda \in I$ and assume that d is already constructed up to λ. This means that a $d' \in \mathrm{C}^n_\lambda$ is given such that $\delta(d') = \mathrm{t}(c)$. To extend d' to a suitable n–cochain d defined on $\{\alpha \in I \mid \alpha \leq \lambda\}$ means that $\mathrm{t}(d) = d'$ and that $\mathrm{t}\delta(d) = \mathrm{t}(c)$ and $\mathrm{h}\delta(d) = \mathrm{h}(c)$. But I_λ either has a last element or has a cofinality smaller that ω_n, which gives us $\varprojlim_{\alpha\in I_\lambda}{}^{n} A_\alpha = 0$. On the other hand $\delta(c) = 0$ implies $(-1)^n\mathrm{t}(c) + \delta\mathrm{h}(c) = 0$. Therefore $(-1)^n d' + \mathrm{h}(c)$ is a cocycle, which we may write as δe for some $(n-1)$–chain e on I_λ. Now extend d' to d such that $\mathrm{t}(d) = d'$ and $\mathrm{h}(d) = e$. Then $\mathrm{t}\delta(d) = \delta\mathrm{t}(d) = \delta(d') = \mathrm{t}(c)$ and

$$\begin{aligned}
\mathrm{h}\delta(d) &= (-1)^n\mathrm{t}(d) + \delta\mathrm{h}(d) \\
&= (-1)^{n+1}d' + \delta e \\
&= (-1)^{n+1}d' + (-1)^n d' + \mathrm{h}(c) \\
&= \mathrm{h}(c).
\end{aligned}$$

□

Lemma 5 (Todorcevic). *Let $(B_\xi)_{\xi\in\omega_1}$ be a family of infinite abelian groups. For the projective system $A_\alpha = \bigoplus_{\xi<\alpha} B_\xi \quad (\alpha \in \omega_1)$ with the the obvious projection maps we have*

$$\varprojlim^1_{\alpha\in\omega_1} A_\alpha \neq 0.$$

Proof. In ([2, p.70]) an Aronszajn tree is constructed from a sequence $(f_\alpha)_{\alpha<\omega_1}$ of injective functions $f_\alpha : \alpha \to \omega$ such that for all $\alpha < \beta$ the two functions f_α and $f_\beta \restriction \alpha$ differ only for finitely many arguments. In each B_ξ we choose a copy of ω. Then f_α defines an element of $A'_\alpha = \prod_{\xi<\alpha} B_\xi$. Define

$$c_{\alpha\beta} = f_\beta \restriction \alpha - f_\alpha \in A_\alpha.$$

Then c is a 1-cocycle, which is not a coboundary. For otherwise, there would be a sequence $d_\alpha \in A_\alpha \quad (\alpha \in \omega_1)$ such that $c_{\alpha\beta} = d_\beta \restriction \alpha - d_\alpha$. But then the functions $f_\alpha - d_\alpha$ form an ascending sequence and the union f of this sequence is a map defined on ω_1, which is finite to one since it is finite to one on every α. This is impossible. □

Theorem 6. *Let $(B_\xi)_{\xi\in\omega_n}$ be a family of countably infinite abelian groups and $A_\alpha = \bigoplus_{\xi<\alpha} B_\xi$. Assume $n \geq 1$ and that for each $1 < i \leq n$ $\diamondsuit_{\omega_i}(E_i)$ holds for $E_i = \{\alpha \in \omega_i \mid \mathrm{cf}(\alpha) = \omega_{i-1}\}$. Then*

$$\varprojlim^n_{\alpha<\omega_n} A_\alpha \neq 0.$$

Proof. [1] The proof proceeds by induction on n. The case $n = 1$ is a special case of Todorcevics lemma. So we assume $n \geq 2$. $\diamondsuit_{\omega_n}(E_n)$ gives us a sequence $(S^\lambda)_{\lambda\in E_n}$ such that

1. each S^λ is an $(n-1)$–cochain of $(A_\alpha)_{\alpha<\lambda}$
2. for each $(n-1)$–cochain d defined on ω_n the set

   $$\{\lambda \in E_n \mid d \restriction \lambda = S^\lambda\}$$

   is stationary in ω_n.

We define the components $c_{\alpha_0\ldots\alpha_n}$ of an n–cocycle c by induction on α_n. We can start the construction anywhere. For example with the zero n–cocycle defined on ω_0. Now assume that the $c_{\alpha_0\ldots\alpha_n}$ are already defined for all $\alpha_n < \lambda$, giving rise to a cocycle c' on C^n_λ.

<u>Claim</u> c' can be extended to a cocycle c defined on $\lambda + 1$.
Proof: If we let c extend the cocycle c' we have only to ensure that $\mathrm{h}\delta(c) = 0$. By (1) this is equivalent to

$$\delta\mathrm{h}(c) = (-1)^{n+1}c'.$$

[1] I thank Burban Veliskovic for a helpful discussion of this proof

By theorem 4 $\varprojlim^n_{\alpha<\lambda} A_\alpha = 0$. Whence there is an $e \in \mathrm{C}_\lambda^{n-1}$ with $\delta e = (-1)^{n+1} c'$ and we can extend c' by setting $\mathrm{h}(c) = e$. All other extension of c to cocycles on $\lambda + 1$ can be obtained by adding an $(n-1)$–cocycle (defined over λ) to $\mathrm{h}(c)$.

Now if λ is a successor or has cofinality smaller than ω_{n-1} we don't care and choose an arbitrary extension of c' to $\lambda + 1$.

If $\mathrm{cf}(\lambda) = \omega_{n-1}$ we choose c more carefully and ensure that the difference $\mathrm{h}(c) - (-1)^n S^\lambda$ is not the coboundary of an $(n-2)$–cycle over λ. If necessary we change $\mathrm{h}(c)$ by a cocycle e' which is not a coboundary over λ. Such an e' exists by the induction hypothesis.

To complete the proof we show that c is not a coboundary. For this look at an arbitrary $(n-1)$–cochain δd defined on ω_n. By the choice of the S^λ there is a $\lambda \in E_n$ such that $d \restriction \lambda = S^\lambda$. By (1) we have

$$\mathrm{h}\delta(d) - (-1)^n S^\lambda = \delta \mathrm{h}(d).$$

By our construction $\mathrm{h}\delta(d) \neq \mathrm{h}(c)$ and thererfore $\delta(d) \neq c$. □

Open Problem: Can one prove the last theorem without diamond?

References

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