Chernikov and Simon's proof of Shelah's theorem*

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Let T be a complete theory with monster model $\mathbb{C}$ and $M \prec \mathbb{C}$ a model of T. A relation $X \subset M$ is *externally definable* if for some formula $\phi(x, y)$ and parameter $c \in \mathbb{C}$ we have $X = \phi(M, c)$. Shelah has shown in [3] for dependent (i.e. NIP) T that the structure M together with all externally definable relation has quantifier elimination. Pillay gave a better proof in [2]. Recently a new proof was given by Chernikov and Simon in [1]. The purpose of this note is to present a variant of their proof which avoids adding new predicates to the language.

We will need the following notation: $a \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$ means that $\mathrm{tp}(a/Mc)$ is a coheir of $\mathrm{tp}(a/M)$. The following is easy to see:

Lemma 1. *(1)* $B \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c, \quad a \mathop{\smile\hspace{-0.9em}|}^{*}_{M} Bc \;\Rightarrow\; aB \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$

(2) If $A \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$*, then every type over* AM *is realised by some* b *with* $bA \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$*.*

The next proposition is a version of Proposition 1.1 of [1].

Proposition 2. *Let* M *be a model.* $\phi(x, y)$ *a formula without the independence property and* $c \in \mathbb{C}$*. Then there is a set* A *and an* $L(A)$*–formula* $\theta(x)$ *such that*

a) $A \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$

b) $\theta(M) = \phi(M, c)$

c) For all b *with* $bA \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$*, we have* $\models \theta(b) \to \phi(b, c)$*.*

Proof. Consider the set Q of all global types q which are coheirs of their restriction to M and contain $\phi(x, c)$. Clearly Q is a closed subset of $\mathrm{S}(\mathbb{C})$. Let q_α, $(\alpha < \lambda)$ be an enumeration of Q. We construct recursively an ascending sequence of sets A_α and of $L(A_\alpha)$ formulas $\theta_\alpha(x)$ as follows: Set $A' = \bigcup_{\beta<\alpha} A_\beta$. By assumption we have $A' \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$. Then try to choose an infinite sequence $a_0, b_0, a_1, \ldots$ such that

a') $\{a_0, b_0, a_1, \ldots\}A' \mathop{\smile\hspace{-0.9em}|}^{*}_{M} c$

b') $a_i \models q_\alpha \restriction a_0 b_0 \ldots b_{i-1} M A' c$

c') $b_i \models q_\alpha \restriction a_0 b_0 \ldots a_i M A' \cup \{\neg\phi(x, c)\}$

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Since q_α is a coheir of $q_\alpha \restriction M$, the sequence $a_0, b_0, \ldots$ is indiscernible over M. $\phi(x, y)$ is NIP and $\models \phi(a_i, c) \wedge \neg\phi(b_i, c)$, so the construction must fail. For b_i we can always choose an arbitrary realization of $q_\alpha \restriction a_0 b_0 \ldots b_{i-1} M A' c$ and have then $\{a_0, b_0, \ldots b_i\} A' \mathop{\smile\!\!\!\!\!\!\,|}^*_M c$ by Lemma 1 (1).

So the construction must fail at some stage where we cannot find b_i which satisfies a') and c'). Set $A_\alpha = \{a_0, b_0, \ldots, a_i\} \cup A'$ and choose a formula $\theta_\alpha \in q_\alpha \restriction M A_\alpha$ such that for all b with $b A_\alpha \mathop{\smile\!\!\!\!\!\!\,|}^*_M c$ we have $\models \theta(b) \to \phi(b, c)$.

We take for A the union of all the A_α. The formulas θ_α cover the space Q. So there are finitely many $\alpha_1, \ldots, \alpha_{n-1}$ such that $\theta = \bigvee_{i<n} \theta_{\alpha_i}$ is contained in all $q \in Q$. It is clear that the properties a) and c) are true. Since $mA \mathop{\smile\!\!\!\!\!\!\,|}^*_M c$ for all $m \in M$, this implies also $\theta(M) \subset \phi(M, c)$. If m belongs to $\phi(M, c)$, we have $q = \mathrm{tp}(m/\mathbb{C}) \in Q$. So $\theta \in q$ and we have $m \in \theta(M)$. $\square$

Let $R = \theta(M)$ be an externally defined relation on M. $\theta(x)$ is an *honest* definition of R if for all $L(M)$–formulas $\psi(x)$

$$\theta(M) \subset \psi(M) \ \Rightarrow\ \theta(\mathbb{C}) \subset \psi(\mathbb{C}).$$

Corollary 3 ([1])**.** *In a dependent theory every externally definable relation on M has an honest definition.*

Proof. If $R = \phi(M, c)$, choose A and θ as in the proposition. Then $R = \theta(M)$. We claim that $\theta(x)$ is an honest definition of R: Assume $\theta(M) \subset \psi(M)$. Then all $m \in M$ satisfy $\phi(m, c) \to \psi(m)$, which implies that $\models \phi(b, c) \to \psi(b)$ for all b with $b \mathop{\smile\!\!\!\!\!\!\,|}^*_M c$. So by the proposition we have $\models \theta(b) \to \psi(b)$ for all b with $bA \mathop{\smile\!\!\!\!\!\!\,|}^*_M c$. With Lemma 1 (2) this implies $\theta(\mathbb{C}) \subset \psi(\mathbb{C})$. $\square$

Shelah's theorem follows from this like in [1]: Let $R \subset M^2$ be a relation with an honest definition $\theta(x, y)$. Then $\exists y \theta(x, y)$ is an external definition of $\pi(R) = \{m \in M \mid (m, n) \in R \text{ for some } n \in M\}$. To see this consider $m_0 \in M$ and the formula $\psi(x, y) = \neg x \doteq m_0$. Then $m_0 \in \pi(R)$ iff $\theta(M) \not\subset \psi(M)$ iff $\theta(\mathbb{C}) \not\subset \psi(\mathbb{C})$ iff $\models \exists y \theta(m_0, y)$.

References

[1] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs. *arXiv:math.LO71007.4468*, 2010.

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[3] Saharon Shelah. Strongly dependent theories. *Israel Journal of Mathematics*, accepted. math.LO/0504197.