An exposition of Hrushovski's New Strongly Minimal Set*

Martin Ziegler

Barcelona, July 2011

In [5] E. Hrushovski proved the following theorem:

Theorem 0.1 (Hrushovski's New Strongly Minimal Set)**.** *There is a strongly minimal theory which is not locally modular but does not interpret an infinite group.*

This refuted a conjecture of B. Zilber that a strongly minimal theory must either be locally modular or interpret an infinite field (see [7]). Hrushovski's method was extended and applied to many other questions, for example to the fusion of two strongly minimal theories ([4]) or recently to the construction of a bad field in [3].

There were also attempts to simplify Hrushovski's original constructions. For the fusion this was the content of [2]. I tried to give a short account of the New Strongly Minimal Set in a tutorial at the Barcelona Logic Colloquium 2011. The present article is a slightly expanded version of that talk.

1 Strongly minimal theories

An infinite L-structure M is *minimal* if every definable subset of M is either finite or cofinite. A complete L-theory T is *strongly minimal* if all its models are minimal. There are three typical examples:

- Infinite sets without structure
- Infinite vector spaces over a finite field
- Algebraically closed fields

The *algebraic closure* $\mathrm{acl}(A)$ of a subset A of M is the union of all finite A-definable subsets. In algebraically closed fields this coincides with the field-theoretic algebraic closure. In minimal structures acl has a special property:

Lemma 1.1. *In a minimal structure* acl *defines a pregeometry.*

*version2-4-g94cca80, Thu Aug 8 13:07:45 2013 +0200

A *pregeometry* (M, Cl) is a set M with an operator $Cl : \mathfrak{P}(M) \to \mathfrak{P}(M)$ such that for all $X, Y \subset M$ and $a, b \in M$

a) $X \subset Cl(X)$ (Reflexivity)

b) $X \subset Y \Rightarrow Cl(X) \subset Cl(Y)$ (Monotonicity)

c) $Cl(Cl(X)) = Cl(X)$ (Transitivity)

d) $a \in Cl(Xb) \setminus Cl(X) \Rightarrow b \in Cl(Xa)$ (Exchange)

e) $Cl(X)$ is the union of all $Cl(A)$, where A ranges over all finite subsets of X. (Finite character)

An operator with a), b) and c) is called a closure operator. Note that e) implies b).

Proof of 1.1. All properties except Exchange are true in general and do not need the minimality of M. To prove the exchange property, assume $a \in \mathrm{acl}(Ab)$ and $b \notin \mathrm{acl}(Aa)$. There is a formula $\phi(x, y)$ with parameters in A such that $\phi(M, b)$ contains a and is finite, say with m elements. We can choose ϕ in such a way that $\phi(M, b')$ has at most m elements for all b'. Since b is not algebraic over Aa, $\phi(a, M)$ must be infinite. But M is minimal, so the complement $\neg\phi(a, M)$ is finite, say with n elements. Assume that there are pairwise different elements $a_0, \ldots, a_m$ such that each $\neg\phi(a_i, M)$ has at most n elements. Then for some b', $\phi(M, b')$ contains all the a_i, which contradicts the choice of ϕ. So there are at most m many a' such that $\neg\phi(a', M)$ has n elements. This shows that a is algebraic over A. □

Let X be a subset of M. A *basis* of X is a subset X_0 which *generates* X in the sense that $X \subset Cl(X_0)$ and is *independent*, which means that no element x of X_0 is in the closure $X_0 \setminus \{x\}$.

Lemma 1.2. *Every set X has a basis. All these bases have the same cardinality, the* dimension *of X.*

Proof. See [6, Lemma C 1.6].y □

In the three examples given above the dimension is computed as follows: If M is an infinite set without structure, the dimension of X is its cardinality. If M is an infinite vector space over a finite field, the dimension of a subset is the linear dimension of the subspace it generates. If M is an algebraically closed field, $\dim(X)$ is the transcendence degree of the subfield generated by X.

The dimension function, restricted to finite sets, has the following properties:

1. $\dim(\emptyset) = 0$
2. $\dim(\{a\}) \leq 1$
3. $\dim(A \cup B) + \dim(A \cap B) \leq \dim(A) + \dim(B)$ (Submodularity)

4. $A \subset B \Rightarrow \dim(A) \leq \dim(B)$ (MONOTONICITY).

Any such function comes from a pregeometry, which is unique since $Cl(A) = \{b \in M \mid \dim(Ab) = \dim(A)\}$ (see e.g. [1, 6.14]).

Definition. A pregeometry is *modular* if for all Cl–closed X and Y

$$\dim(X \cup Y) + \dim(X \cap Y) = \dim(X) + \dim(Y) \qquad \text{(MODULARITY)}.$$

If the modular law is true whenever $X \cap Y$ has positive dimension, the pregeometry is *locally modular*.

Definition. A minimal structure M is (locally) modular if (M, acl) is (locally) modular. A strongly minimal theory is (locally) modular if all its models are (locally) modular.

Examples:

- Infinite sets and infinite vector spaces over a finite field are modular.
- Infinite affine spaces over a finite field are locally modular.
- An algebraically closed field K of at least transcendence degree 4 is not locally modular.

To see this, choose e, a, b, x algebraically independent over the prime field F of K. Let X be the algebraic closure of $F(e, a, b)$ and Y be the algebraic closure of $F(e, x, ax + b)$. Then the dimensions of $X \cup Y$, $X \cap Y$, X and Y are 4, 1, 3 and 3, respectively.

In the following we will present Hrushovki's example of a strongly minimal theory which is not locally modular but does not interpret an infinite group.

2 The setting

The theory we are going to construct will be an L-theory, where L consists of just a ternary relation symbol R.

We consider the class $\mathcal{C}$ of all L-structures $M = (M, R^M)$ where R^M is irreflexive and symmetric. So R^M can as well be given by a set $R(M)$ of three-element subsets of M. We also allow the empty structure $\emptyset$. For finite $A \in \mathcal{C}$ we define

$$\delta(A) = |A| - |R(A)|.$$

A finite subset A is *closed* in M, or M is a *strong extension* of A

$$A \leq M,$$

if $\delta(A) \leq \delta(B)$ for all $A \subset B \subset M$. We will work in the class

$$\mathcal{C}^0 = \{M \in \mathcal{C} \mid \emptyset \leq M\},$$

i.e. in the class of all $M \in \mathcal{C}$ with $\delta(A) \geq 0$ for all finite $A \subset M$.

Lemma 2.1. *$\mathcal{C}^0_{\text{fin}}$, the class of finite members of $\mathcal{C}^0$, has the amalgamation property for strong extensions (APS).*

Proof. If M_1 and M_2 are two extensions of N, we define their free amalgam $M_1 \otimes_N M_2$ as follows. We assume that M_1 and M_2 intersect in N and set $M_1 \otimes_N N_2 = M_1 \cup M_2$ and $R(M_2 \otimes_N M_2) = R(M_2) \cup R(M_2)$.

If B is closed in M and C a finite extension of B, then C is closed in $M \otimes_B C$. So if A and C are finite strong extensions of B, then $A \otimes_B C$ is a common strong extension of A and C. $\square$

Proposition 2.2 (Fraïssé)**.** *Let $\mathcal{K}$ be a non-empty subclass of $\mathcal{C}^0$, closed under taking closed substructures and direct unions. Assume further that $\mathcal{K}_{\text{fin}}$ has the APS. Then $\mathcal{K}$ contains a unique countable universal homogeneous structure M, i.e.*

a) All $A \in \mathcal{K}_{\text{fin}}$ can be strongly embedded in M.

b) Every isomorphism between two finite closed subsets of M extends to an automorphism of M.

Proof. By an easy adaption of the classical Fraïssé construction. See [6, Theorem 4.4.4]. For the existence of M one uses the fact that the composition of two strong extensions is again a strong extension. This follows from Corollary 3.2 below. Uniqueness uses that every finite subset of M is contained in a finite closed subset. This will be proved in Lemma 3.3. $\square$

For countable $M \in \mathcal{K}$ conditions a) and b) are equivalent to M being *rich*: If B is closed in M and $B \leq C \in \mathcal{K}_{\text{fin}}$, then C can be strongly embedded in M over B. Note that all rich structures are partially isomorphic (for a definition see e.g. [6, Exercise 1.3.5]) by the family of isomorphisms between finite closed subsets.

We call M the *strong Fraïssé-limit* of $\mathcal{K}_{\text{fin}}$. Hrushovski's example will be the strong Fraïssé-limit of a suitable chosen subclass of $\mathcal{C}^0_{\text{fin}}$.

3 Delta functions

The function δ which we have defined in the last section on finite elements of $\mathcal{C}$ has a lot of interesting properties. Surprisingly most of these properties follow from the fact that δ is a *δ-function* in the following sense:

Definition. Let M be a set. A function δ which associates an integer to any finite subset of M is a *δ-function* if the following axioms are satisfied:

1. $\delta(\emptyset) = 0$
2. $\delta(\{a\}) \leq 1$
3. $\delta(A \cup B) + \delta(A \cap B) \leq \delta(A) + \delta(B)$ (Submodularity)

Examples:

- The dimension function of a pregeometry on M.
- If M is in $\mathcal{C}$, the function $\delta(A) = |A| - |R(A)|$

For the rest of the section let δ be a δ-function on M.

A finite subset A of $Y \subset M$ is *closed* in Y if $\delta(A) \leq \delta(B)$ for all $A \subset B \subset Y$. We denote this by $A \leq Y$ and call Y a *strong extension* of A. If we define

$$\delta(A/B) = \delta(A \cup B) - \delta(B),$$

submodularity becomes

$$\delta(A/B) \leq \delta(A/A \cap B).$$

This allows us to define for infinite X

$$\delta(A/X) = \inf_{A \cap X \subset B \subset X} \delta(A/B) \ \in \ \{-\infty\} \cup \mathbb{Z}$$

and to call X closed in Y if $\delta(A/X) \geq 0$ for all $A \subset Y$.

The following lemma is only a reformulation of the definition.

Lemma 3.1. *Let X be a subset of Y. Then*

$$X \leq Y \ \Leftrightarrow \ \delta(A/A \cap X) \geq 0 \quad \textit{for all} \quad A \subset Y$$

□

Corollary 3.2.

1. *If $X \leq Y$, then $U \cap X \leq U \cap Y$ for all U.*
2. *$\leq$ is transitive.*
3. *If the X_i are closed in Y, then also their intersection.*

Proof.

1. follows immediately from the Lemma

2. Assume $X \leq Y \leq Z$ and let A be a finite subset of Z. Then by the lemma $\delta(A \cap X) \leq \delta(A \cap Y) \leq \delta(A)$. This implies $\delta(A \cap X) \leq \delta(A)$ and so $X \leq Z$ by the lemma again.

3. It is enough to consider finite intersections. But this follows from 1. and 2: If $X_2 \leq Y$, we have $X_1 \cap X_2 \leq X_1$, and if also $X_1 \leq Y$, we have $X_1 \cap X_2 \leq Y$. □

It follows that every X is contained in a smallest closed subset of M, the *closure* $cl(X)$. This defines a closure operator of finite character.

We assume now $\emptyset \leq M$, i.e. $\delta(A) \geq 0$ for all $A \subset M$.

Lemma 3.3. *The closure of a finite set is again finite.*

Proof. Let A be finite and $\delta(B)$ minimal for $A \subset B$. Then $B \leq M$. □

Definition. The dimension of A is defined as

$$\mathrm{d}(A) = \min\{\delta(B) \mid A \subset B\} = \delta(cl(A)).$$

Proposition 3.4. d *is the dimension function of a pregeometry* (M, Cl).

We call Cl the *geometric closure*. Note that $\mathrm{d}(A) \leq \delta(A)$ and $cl(X) \subset Cl(X)$.

Proof. We check that d satisfies the submodular law. The other properties of a dimension function are clear. Choose $A \subset A'$, $B \subset B'$ with $\mathrm{d}(A) = \delta(A')$ and $\mathrm{d}(B) = B'$. We have then

$$\begin{aligned} \mathrm{d}(A \cup B) + \mathrm{d}(A \cap B) &\leq \delta(A' \cup B') + \delta(A' \cap B') \\ &\leq \delta(A') + \delta(B') \\ &= \mathrm{d}(A) + \mathrm{d}(B). \end{aligned}$$

□

Remark 3.5. *If C is a subset of M, B closed in C and $\delta(C/B) = 0$, then C is contained in $Cl(B)$.*

Proof. Indeed, it follows that $\delta(C/cl(B)) = 0$ and whence $\mathrm{d}(C/B) = 0$. □

Lemma 3.6. *If (M, Cl) is modular, the union of two geometrically closed sets is closed in M.*

Proof. Let X and Y be geometrically closed. It is enough to show that every finite subset C of $X \cup Y$ is contained in a closed set of the form $A \cup B$, where $A \subset X$ and $B \subset Y$. Choose A and B closed with $C \subset A \cup B$ and so that $Cl(A) \cap Cl(B) = Cl(A \cap B)$. Modularity implies $\mathrm{d}(A \cup B) = \mathrm{d}(A) + \mathrm{d}(B) - \mathrm{d}(A \cap B)$. So

$$\mathrm{d}(A \cup B) \geq \delta(A) + \delta(B) - \delta(A \cap B) \geq \delta(A \cup B),$$

which means that $A \cup B$ is closed. □

4 The rank ω case

Before we construct Hrushovski's example we investigate the Fraïssé limit M^0 of $\mathcal{C}^0_{\text{fin}}$ itself. M^0 is not strongly minimal, but has Morley rank ω. Although this result will not be needed later, the notions and techniques of its proof will be used in the next section.

Remark 4.1. *(M^0, Cl) is not locally modular.*

Proof. Consider the structure $C_{\text{nm}} = \{a_1, a_2, b_1, b_2, c\}$ with $R(C_{\text{nm}})$ consisting of $\{a_1, b_1, c\}$ and $\{a_2, b_2, c\}$. C_{nm} belongs to $\mathcal{C}^0$, so we may assume that $C_{\text{nm}} \leq M^0$. The two sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ are geometrically closed in C_{nm}, but $A \cup B$ is not closed in C_{nm}. This implies that $Cl(A) \cup Cl(B)$ is not closed in M^0. So M^0 is not modular. To see that M^0 is not locally modular consider $C_{\text{nm}} \cup \{d\}$, $A \cup \{d\}$, $B \cup \{d\}$ for some d not connected to C_{nm}. $\square$

The following will be a complete axiomatisation of the theory of M^0.

Definition. M is a model of T^0 if the following holds:

a) M belongs to $\mathcal{C}^0$

b) Let B be a finite subset of M. Then M contains a copy of every strong extension C of B with $\delta(C/B) = 0$.

c) Let F_n denote the structure with n elements and no relations. Then F_n is strongly embeddable in an elementary extension of M.

The first two conditions are clearly expressible by a set of axioms. F_n is strongly embeddable in an elementary extension iff for all m the following is true: M contains a copy of F_n which is closed in every m-element subset of M which contains F_n. For each m this is an elementary statement.

Proposition 4.2. *An L-structure M is rich (with respect to $\mathcal{C}^0$) iff it is an ω-saturated model of T^0.*

So M^0 is a model of T^0, and every ω–saturated model of T^0 is partially isomorphic, and therefore elementarily equivalent, to M^0. This yields

Corollary 4.3. *T^0 axiomatises the complete theory of M^0.*

Proof of 4.2. Assume first that M is rich. M belongs to $\mathcal{C}^0$ by definition. F_n embeds strongly into M, since M is rich and $\emptyset$ is closed in M and F_n. Finally let B be a finite subset of M and C a strong extension with $\delta(C/B) = 0$. Choose $B' \leq M$ containing B and consider $C' = B' \otimes_B C$. As noted in the proof of Lemma 2.1, C' is a strong extension of B' and embeds therefore (strongly) in M. That M is ω-saturated will follow from the other direction.

Now assume that M is an ω-saturated model of T^0. Consider $B \leq M$ and an extension $B \leq C$. We may assume that the extension is *minimal*, i.e. B is a maximal proper closed subset of C. By Lemma 4.6 below there are two cases:

1. $\delta(C/B) = 0$. Then M contains a copy C' of C over B. Since B is closed in M and $\delta(C'/B) = 0$, it follows that C' is closed in M.

2. $C = B \cup \{c\}$ where $\delta(c/B) = 1$, which means that c is not connected with B. Since M strongly embeds every F_n and $\mathrm{d}(F_n) = n$, M has infinite geometric dimension. So M has an element c' which is not in the geometric closure of B. This means $\delta(c'/B) = 1$ and $B \cup \{c'\}$ is closed in M. $B \cup \{c'\}$ isomorphic to C over B.

It remains to show that a rich model M is ω-saturated. To see this choose an ω-saturated model M' of T^0. Then M' is rich and therefore partially isomorphic to M. This implies that also M is ω-saturated. $\square$

The following two lemmas hold inside any set M with a delta function:

Lemma 4.4. *A proper strong extension C of B is minimal iff $\delta(C/D) < 0$ for all C with $B \subsetneq D \subsetneq C$.*

Proof. Let $\delta(C/D)$ be maximal for D properly between B and C. If $\delta(C/D)$ is non-negative, D is closed in C, so the extension $B \leq C$ is not minimal. $\square$

Corollary 4.5. *If $B \leq C$ is minimal and C is neither contained in X nor disjoint from X, then $\delta(C/X \cup B) < 0$.* $\square$

Lemma 4.6. *If $B \leq C$ is minimal, there are two cases*

1. $\delta(C/B) = 1$ *and* $C = B \cup \{c\}$

2. $\delta(C/B) = 0$

Proof. If $\delta(C/B) > 0$, pick any $c \in C \setminus B$. Then $\delta(C/Bc) \geq \delta(C/B) - 1 \geq 0$ and it follows from the last lemma that $C = B \cup \{c\}$. $\square$

In M^0 two finite tuples $\bar{a}$ and $\bar{a}'$ have the same type iff their closures $cl(\bar{a})$ and $cl(\bar{a}')$ are isomorphic. This is true for all models of T^0:

Lemma 4.7. *Let M_1 and M_2 be two models of T^0. Then $\bar{a}_1 \in M_1$ and $\bar{a}_2 \in M_2$ have the same type iff $\bar{a}_1 \mapsto \bar{a}_2$ extends to an isomorphism $cl(\bar{a}_1) \to cl(\bar{a}_2)$.*

Proof. If $\bar{a}_1$ and $\bar{a}_2$ have the same type, they have the same geometric dimension. The closure C of a tuple $\bar{a}$ can be characterised as a minimal set C containing $\bar{a}$ with $\delta(C) = \mathrm{d}(\bar{a})$. So $\bar{a}_1$ and $\bar{a}_1$ have isomorphic closures.

If conversely $\bar{a}_1$ and $\bar{a}_2$ have isomorphic closures, we take ω-saturated extensions $M_i \prec M_i'$. In these extension $\bar{a}_1$ and $\bar{a}_2$ have the same closures. Since the M_i' are rich, this implies that $\bar{a}_1$ and $\bar{a}_2$ have the same type in M_1' and M_2'. $\square$

We work now in a big saturated model M of T^0.

Lemma 4.8. *Let $B \leq C$ be minimal, $\delta(C/B) = 0$ and C closed in M. Then* $\mathrm{tp}(C/B)$ *is isolated and strongly minimal.*

Note that we have to fix an enumeration of C when we speak of the type of C.

Proof. Let $\phi(\bar{x})$ be a quantifier free formula with parameters from B which describes the isomorphism type of C/B. If C' is any other realisation of $\phi(\bar{x})$ it follows from $B \leq M$ and $\delta(C'/B) = 0$ that C' is closed in M. So C' and C have the same type over B and we see that ϕ isolates $p = \mathrm{tp}(C/B)$. Since we can embed all $C \otimes_B C \otimes_B \ldots \otimes_B C$ in M, p has infinitely many realisation, i.e. p is not algebraic. In order to show that p is strongly minimal we have to show that p has only one non-algebraic extension to each B' extending B. For this we may assume that B' is closed in M.

Since $B' \otimes_B C$ is a strong extension of B', we find a closed isomorphic copy $B' \cup C'$ of it in M. We claim that $p' = \mathrm{tp}(C'/B')$ is the only non-algebraic extension of p to B'. Indeed, if C'' is any realisation of p in M, we have by minimality either $C'' \subset B'$, then C'' is algebraic over B', or $B' \cap C'' = B$. In the latter case $\delta(C''/B') = 0$ implies that $B' \cup C''$ is closed in M and isomorphic to $B' \cup C'$, so that C'' realises p'. □

Corollary 4.9. *If $B \leq C \leq M$ and $\delta(C/B) = 0$, then $\mathrm{tp}(C/B)$ has finite Morley rank. The rank is at least the length of a decomposition of C/B into minimal extensions.*

Proof. Strongly minimal types have Morley rank 1. So the corollary follows from two general facts about Morley rank: Assume that the type of b is isolated. Then the following holds:

1. If the Morley ranks of $\mathrm{tp}(a/b)$ and $\mathrm{tp}(b)$ are finite, then also the Morley rank of $\mathrm{tp}(ab)$ is finite.

2. The Morley rank of $\mathrm{tp}(ab)$ is not smaller than the (ordinal) sum of the Morley rank of $\mathrm{tp}(a/b)$ and the Morley rank of $\mathrm{tp}(b)$.

1. follows from the Erimbetov-Shelah inequality (see [6, Exercise 6.4.4]). 2 is easy to prove. □

Proposition 4.10. *T^0 has Morley rank ω.*

Proof. Let $B \leq M$ and c any element of M. If $\mathrm{d}(c/B) = 0$, the last corollary shows that $\mathrm{tp}(c/B)$ has finite Morley rank. Since there is only one other type over B, namely $\mathrm{tp}(c/B)$ with $\mathrm{d}(c/B) = 1$, this type has at most rank ω.

On the other hand it is easy to find $B \leq M$ and elements c_n such $\mathrm{d}(c_n/B) = 0$ and the extensions $cl(Bc_n)/B$ have decomposition length n. (Consider $B = \{c_{-1}, c_0\}$, $C = \{c_{-1}, \ldots, c_n\}$, and $R(C) = \{\{c_j, c_{j+1}, c_{j+2}\} \mid i = -1, 0, \ldots, n-2\}$.) Since $cl(Bc_n)$ is algebraic over Bc_n, $\mathrm{tp}(c_n/B)$ has at least Morley rank n. So there are 1–types of arbitrarily large finite rank. □

5 The collapse

We will now construct Hrushovski's example M^μ as the Fraïssé limit of a carefully chosen subclass $\mathcal{C}^\mu$ of $\mathcal{C}^0$. (Actually we will construct a family of structures, depending on a parameter μ.) M^μ will be strongly minimal and Cl will coincide with the algebraic closure operator. The structure C_{nm} constructed in Remark 4.1 will be a strong subset of M^μ. So M^μ will not be locally modular.

By Remark 3.5 if B is closed in a finite subset C of M^μ and $\delta(C/B) = 0$, we have to ensure that C will be algebraic (in M^μ) over B. We do this by imposing on a special class of such extensions $B \leq C$ a bound for the number of isomorphic copies of C over B in M^μ.

We call a pair A/X of disjoint sets *prealgebraic minimal* if

a) $X \cup A$ belongs to $\mathcal{C}^0$.

b) $X \leq X \cup A$ is a minimal extension.

c) $\delta(A/X) = 0$

We call a prealgebraic minimal pair A/B *good* if $\delta(A/B') > 0$ for every proper subset B' of B. For every prealgebraic minimal A/X there is a unique $B \subset X$ such that A/B is good: B is the set of all x which are connected with an element a of A (this means that for some $y \in X \cup A$ the triple xay belongs to R). We call B the *basis* of A/X. It is easy to see that

$$X \cup A = X \underset{B}{\otimes} (B \cup A).$$

We have also

$$|B| \leq 2 \cdot |A|,$$

which can be seen as follows: $\delta(A/B) = 0$ implies that $R' = R(B \cup A) \setminus R(B)$ has at most $|A|$ elements. Goodness implies that every element of B belongs to some set in R', but such a set contains at most 2 elements of B.

Note: The existence of a basis does not formally follow from the axioms of a delta function, cf. Remark 5.11.

Definition. A *code* α is the isomorphism type of a good pair (A_α/B_α). A *pseudo Morley sequence* of α over B is a pairwise disjoint sequence $A_0, A_1, \ldots$ such that all A_i/B are of type α.

Main Lemma 5.1. *Let $M \leq N$ be in $\mathcal{C}^0$. Assume that N contains a a pseudo Morley sequence (A_i) of α over B with more than $\delta(B)$ elements. Then one of the following occurs:*

1. *$B \subset M$*

2. *Some A_i lies in $N \setminus M$.*

Proof. Let $A_0, \ldots, A_{r-1}$ be in M and $A_r, \ldots A_{r+s-1}$ neither in M nor in $N \setminus M$. Assume that B is not contained in M. Then each of the A_i, $i < r$, adds a relation to B, so we have

$$\delta(B/M) \leq \delta(B/B \cap M) - r \leq \delta(B) - r.$$

The minimality of A_i/B implies $\delta(A_i/A_r \ldots A_{i-1}MB) < 0$ for all $i \in [r, r+s-1]$ (see Corollary 4.5). Whence $\delta(A_r \ldots A_{r+s-1}/MB) \leq -s$. This implies

$$0 \leq \delta(A_r \ldots A_{r+s-1}/M) \leq \delta(B) - (r+s)$$

and therefore $r + s \leq \delta(B)$. $\square$

We fix now for every code α a natural number $\mu(\alpha) \geq \delta(B_\alpha)$.

Definition. $\mathcal{C}^\mu$ is the class of all $M \in \mathcal{C}^0$ in which every pseudo Morley sequence of α has length most $\mu(\alpha)$.

We call a pseudo Morley sequence of length $> \mu(\alpha)$ a *long* pseudo Morley sequence.

Examples:

- If M is in $\mathcal{C}^\mu$ and we add a new unconnected point c to M, then $M \cup \{c\}$ is in $\mathcal{C}^\mu$.
- The structure C_{nm} is in $\mathcal{C}^\mu$. (Up to automorphisms of C_{nm} the only good pairs which occur are c/a_1b_1 and b_1/a_1c.)

Corollary 5.2. *$\mathcal{C}^\mu_{\mathrm{fin}}$ has the amalgamation property for strong extensions.*

Proof. Consider $B \leq M$ and $B \leq N$ in $\mathcal{C}^\mu$. We want to construct a common strong extension of M and N which belongs to $\mathcal{C}^\mu$. We may assume that N is a minimal extension of B and also that $M \otimes_B N$, which is a common strong extension of M and N (see Lemma 2.1), does not belong to $\mathcal{C}^\mu$. So $M \otimes_B N$ contains a long pseudo Morley sequence (A'_i) of some α over B'. By the Main Lemma there are two cases:

1. $B' \subset M$. Since $M \in \mathcal{C}^\mu$, there is an A'_i which lies not completely in M. So, since A'_i/B' is minimal, A'_i is contained in $A = N \setminus B$. Now the minimality of A/B implies that A/M is minimal. On the other hand, we have $\delta(A'/M) = 0$. So A'_i and A must be equal.

   A/B' is a good pair, whence B' must be contained in B. Since $N \in \mathcal{C}^\mu$, there is an A'_j which lies in $M \setminus B$. It follows that B' is the basis of A/B and of A'_j/B. Whence A'_j/B and A/B are isomorphic and we can amalgamate M and N by mapping N onto $B \cup A_j$.

2. $A'_i \subset N \setminus M$ for some i. Since A'_i/B' is good, we have $B' \subset N$. N belongs to C^μ and so some A'_j lies in $M \setminus B$. This gives again that $B' \subset B$ and we are back in Case 1.

□

Let M^μ be the Fraïssé limit of $\mathcal{C}^\mu_{\mathrm{fin}}$. The following will be a complete axiomatisation of the theory of M^μ:

Definition. M is a model of T^μ if the following holds:

a) M belongs to $\mathcal{C}^\mu$.

b) No prealgebraic minimal extension of M belongs to $\mathcal{C}^\mu$.

c) M is infinite.

We have to explain why the second axiom can be elementarily expressed. Let $M \in \mathcal{C}^\mu$ and A/M a prealgebraic minimal pair with basis B and α the type of A/B. We will show that depending on α there are are only a finite number of codes α' which can have a long pseudo Morley sequence in $N = M \cup A = M \otimes (B \cup A)$. This implies easily that b) can be expressed by a set of sentences.

So assume that (A'_i) is a long pseudo Morley sequence of α' over B' in N. We apply the Main Lemma: If $B' \subset M$, we conclude that some A'_i equals A as in the proof of the amalgamation property. Then also $B' = B$ and we have $\alpha' = \alpha$. If some A'_i lies in A, the size of B' can be bounded by $2|A|$. So there are only finitely many possibilities for α'.

Proposition 5.3. *A structure M is rich iff it is an ω-saturated model of T^μ.*

Corollary 5.4. *T^μ axiomatises the complete theory of M^μ.*

Proof of 5.3. Assume that M is rich. Since all F_n belong to $\mathcal{C}^\mu$, M is infinite. For the second axiom let A/M be a prealgebraic minimal extension with basis B and α the type of A/B. Assume that $M \cup A$ belongs to $\mathcal{C}^\mu$. Let C be any extension of B which is closed in M. Then M contains a copy A' of A over C. We choose $C' \leq M$ which contains $C \cup A'$ and continue. It results an infinite pseudo Morley sequence of α, a contradiction. That M is ω-saturated will follow from the other direction as in the proof of 4.2.

For the converse we need the following lemma.

Lemma 5.5. *In every ω-saturated structure $M \in \mathcal{C}^\mu$, the algebraic closure contains the geometric closure.*

Proof. Since $cl(B)$ can be described by a type over B, $cl(B)$ is algebraic over B. In order to show that $Cl(B)$ is algebraic over B we may therefore assume that B is closed in M. Then $Cl(B)$ is the union of all extensions C with $\delta(C/B) = 0$. So it is enough to show that every prealgebraic minimal extension A/B is algebraic. Let B_0 be the basis of A/B and α the type of A/B_0. Any sequence (A_i) of sets with the same type over B as A is a pseudo Morley sequence of α and therefore bounded in length by $\mu(\alpha)$. □

To finish the proof of the proposition we show that an ω-saturated model M of T^μ is rich. Consider $B \leq M$ and an extension $B \leq C \in \mathcal{C}^\mu$. We may assume that the extension is minimal. There are two cases:

1. $\delta(C/B) = 0$. By Corollary 5.2 (or its proof) since $M \otimes_B C$ is not in $\mathcal{C}^\mu$, C embeds over B into M.

2. $C = B \cup \{c\}$ where $\delta(c/B) = 1$, which means that c is not connected with B. In order to embed C strongly into M we have to find a c' outside $Cl(B)$. But this follows from the last lemma because ω-saturation implies that $\mathrm{acl}(B)$ is a proper subset of the infinite structure M.

$\square$

The next lemma has the same proof as in the T^0-case.

Lemma 5.6. *Let M_1 and M_2 be two models of T^μ. Then $\bar{a}_1 \in M_1$ and $\bar{a}_2 \in M_2$ have the same type iff $\bar{a}_1 \mapsto \bar{a}_2$ extends to an isomorphism $cl(\bar{a}_1) \to cl(\bar{a}_2)$.* $\square$

Theorem 5.7. *T^μ is strongly minimal.*

Proof. If $\mathrm{d}(c/B) = 0$, c is algebraic over B. There is only one type $\mathrm{tp}(c/B)$ with $\mathrm{d}(c/B) = 1$, namely the type which says that c is not connected to $cl(B)$ and $cl(B) \cup \{c\}$ is closed. $\square$

It follows also from the proof that acl and Cl coincide (and therefore that the relative dimension $\mathrm{d}(A/B)$ is the Morley rank of $\mathrm{tp}(A/B)$). Since C_{nm} belongs to $\mathcal{C}\mu$, we have therefore:

Corollary 5.8. *T^μ is not locally modular.* $\square$

Corollary 5.9. *T^μ is model complete.*

Proof. T^μ is $\forall\exists$-axiomatisable. Now use Lindström's theorem: A $\forall\exists$-theory which is categorical in some cardinal is model complete. $\square$

We note here that T^0 is not model complete.

In order to show that T^μ does not interpret an infinite group we need the following lemma:

Lemma 5.10. *In structures from $\mathcal{C}^0$,* d *is* flat *on Cl-closed finite dimensional sets $E_1, \ldots, E_n$:*

$$\sum_{\Delta \subset \{1,\ldots,n\}} (-1)^{|\Delta|}\, \mathrm{d}(E_\Delta) \leq 0$$

where $E_\emptyset = E_1 \cup \ldots \cup E_n$ and $E_\Delta = \bigcap_{i \in \Delta} E_i$ if $\Delta \neq \emptyset$.

Proof. Choose finite closed sets $A_i \leq E_i$ big enough so that $Cl(A_\Delta) = E_\Delta$ for all Δ. We have then to show that

$$\sum_{\Delta \subset \{1,\ldots,n\}} (-1)^{|\Delta|} \delta(A_\Delta) \leq 0.$$

But this is true for arbitrary sets A_i, since by the inclusion-exclusion principle the left hand side equals

$$|R(A_1) \cup \ldots \cup R(A_n)| - |R(A_1 \cup \ldots \cup A_n)|.$$

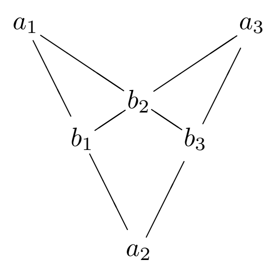
□

Remark 5.11. *Let δ be a flat δ-function. If A, B are disjoint and $\delta(A/B) = 0$, then there is a smallest $B_0 \subset B$ such that $\delta(A/B_0) = 0$.*

Proposition 5.12. *There is no infinite group interpretable in T^μ.*

Proof. Let G be a group interpreted in a model M of T^μ, i.e. definable in M^{eq}. First we consider the case where G is actually definable in M. To ease notation we also assume that G is 0-definable.

Let g be the Morley rank of G. Consider the group diagram: Choose independent elements a_1, a_2, a_3 of G of dimension g. Put $b_1 = a_1 \cdot a_2$, $b_3 = a_2 \cdot a_3$ and $b_2 = b_1 \cdot a_3 = a_1 \cdot b_3$. We consider these six elements as the points of a geometry with "lines" $L_1 = \{a_1, b_1, a_2\}, L_2 = \{a_2, b_3, a_3\}, L_3 = \{a_1, b_2, b_3\}$ and $L_4 = \{b_1, b_2, a_3\}$.



It is easy to see that each point on a line is algebraic over the other two points on the line, and any three non-collinear points are independent.

We apply flatness to the four sets $E_i = Cl(L_i)$. Any two of this sets intersects in the algebraic closure of a point, like $E_{14} = E_1 \cap E_4 = Cl(b_1)$, and the intersection of three equals $Cl(\emptyset)$. So we have

$$d(E_1 \cup E_2 \cup E_3 \cup E_4) = 3g$$

$$d(E_i) = 2g$$

$$d(E_{ij}) = g$$

$$d(E_{ijk}) = 0$$

$$d(E_{ijkl}) = 0$$

Flatness yields

$$g = 3g - 4 \cdot 2g + 6 \cdot g \leq 0.$$

So $g = 0$ and G is finite.

Now assume that G is definable in M^{eq}, say with parameters $A \subset M$. Since M is strongly minimal, we may assume that every element of G is over A interalgebraic with a tuple from M. So we can replace the group diagram of G by a group diagram in M with the same Morley rank (over A) and the proof above applies. □

References

[1] Martin Aigner. *Combinatorial theory.* Classics in Mathematics. Springer-Verlag, Berlin, 1997. Reprint of the 1979 original.

[2] A. Baudisch, A. Martin-Pizarro, and M. Ziegler. Hrushovski's Fusion. In F. Haug, B. Löwe, and T. Schatz, editors, *Festschrift für Ulrich Felgner zum 65. Geburtstag*, volume 4 of *Studies in Logic*, pages 15–31. College Publications, London, 2007.

[3] Andreas Baudisch, Martin Hils, Amador Martin-Pizarro, and Frank O. Wagner. Die böse Farbe. *J. Inst. Math. Jussieu*, 8(3):415–443, 2009.

[4] Ehud Hrushovski. Strongly minimal expansions of algebraically closed fields. *Israel J. Math.*, 79:129–151, 1992.

[5] Ehud Hrushovski. A new strongly minimal set. *Ann. Pure Appl. Logic*, 62:147–166, 1993.

[6] K. Tent and M. Ziegler. *A Course in Model Theory.* ASL Lecture Note Series. Cambridge University Press, 2012.

[7] B. Zilber. Strongly minimal countably categorical theories II-III. *Siberian Math. J.*, 25:396–412, 559–571, 1984.