◦ Gromov-Hausdorff convergence

Def (Gromov-Hausdorff conv.)

(X_i, d_i) : metric spaces. $(i \in \mathbb{N} \cup \{\infty\})$

$(X_i, d_i) \longrightarrow (X_\infty, d_\infty)$ w.r.t GH-topology

$:\Leftrightarrow$ $\exists \varepsilon_i \searrow 0$, $\exists \phi_i : X_i \to X_\infty$ (not necessarily to be conti.)

s.t. $\begin{cases} \text{(i)} & |d_i(x,y) - d_\infty(\phi_i(x), \phi_i(y))| < \varepsilon_i \\ \text{(ii)} & X_\infty \subset B(\phi_i(X_i), \varepsilon_i) \end{cases}$

- Even if X_i = Riem. mfd of $\dim = n$,

  $\dim X_\infty$ may be less than n.

- how to show convergence

ex. (M, g_i), (N, h) : cpt Riem. mfds.
$(i \in \mathbb{N})$

$f_i : M \overset{\text{smooth}}{\longrightarrow} N$ Riem. submersion w.r.t. g_i and h.

(i.e. • f_i : surjective

• $\forall x \in M,\ T_x M = \underbrace{H_x}_{\text{horizontal subsp.}} \underset{\text{orth. decomp.}}{\oplus} \underline{\operatorname{Ker}(df_{\cancel{i}})_x}$.

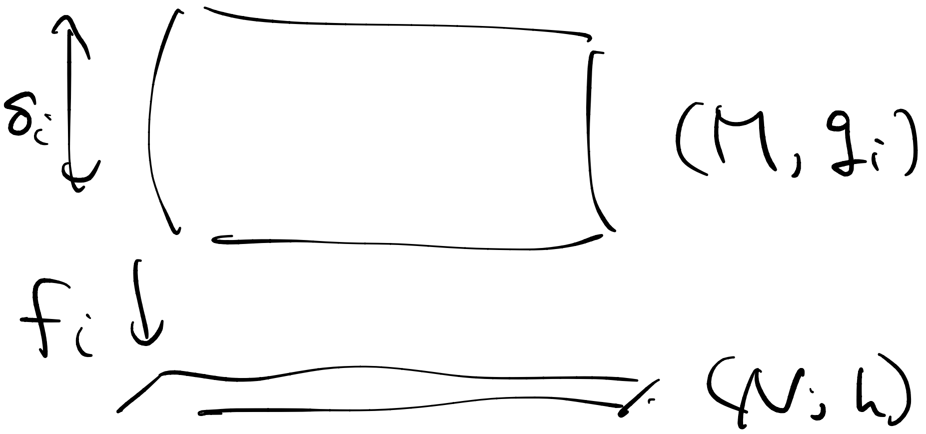
$\forall u, v \in H_x,\ h(df_{\cancel{i}}(u), df_{\cancel{i}}(v)) = g_{\cancel{i}}(u, v)$)

<u>Assumption</u>

$$\underbrace{\lim_{i \to \infty} \sup_{y \in N} \operatorname{diam}_{g_i}(f_{\cancel{i}}^{-1}(y))}_{\delta_i} = 0.$$

(for example $M = N \times F$, g_F : Riem. met. on F.

$g_i = h \oplus 2^{-i} g_F$.)



Then $\underline{(M, d_{g_i}) \xrightarrow{GH} (N, d_h)}$.

<u>proof.</u> Show that $\phi_i = f_{\cancel{i}}$ satisfies the def. of GH conv. (check (i)(ii)).

(ii) is obvious since $f_{\cancel{i}}$ are surjective

<u>(i).</u> Show.

① $d_i(x,y) < d_\infty(f_{\cancel{i}}(x), f_{\cancel{i}}(y)) + \varepsilon_i$.

② $d_\infty(f_{\cancel{i}}(x), f_{\cancel{i}}(y)) < d_i(x,y) + \varepsilon_i$

and $\lim_{i\to\infty} \varepsilon_i = 0$.

① Take $c : [a,b] \to N$: $f_{\cancel{i}}(x) \overset{c}{\frown} f_{\cancel{i}}(y)$

s.t. $L_h(c) < d_h(f_{\cancel{i}}(x), f_{\cancel{i}}(y)) + 2^{-i}$.

length.

$\tilde{c} : [a,b] \to M$: horizontal lift of c with $\tilde{c}(a) = x$.

$(f_{\cancel{i}} \circ \tilde{c} = c,\ \tilde{c}'(t) \in H_{\tilde{c}(t)})$.

Then $L_{g_i}(\tilde{c}) = L_h(c)$

$$\therefore\ d_{g_i}(x, y) \le \underbrace{d_{g_i}(x, \tilde{c}(b))}_{\le\, L_{g_i}(\tilde{c})} + \underbrace{d_{g_i}(\underline{\tilde{c}(b)}, \underline{y})}_{<\, \delta_i}$$

in the same fiber of $f_{\cancel{i}}$.

$$L_{g_i}(\tilde{c}) = L_h(c) \le d_h(f(x), f(y)) + 2^{-i}$$

$$< d_h(f(x), f(y)) + \underbrace{2^{-i} + \delta_i}_{\downarrow\ 0}$$

② Take $\hat{c} : [a, b] \longrightarrow M$.

s.t. $L_{g_i}(\hat{c}) < d_{g_i}(x, y) + 2^{-i}$

Then $L_{g_i}(\hat{c}) \ge L_h(f_{\cancel{i}} \circ \hat{c})$.

$$\underbrace{d_{g_i}(x,y) + 2^{-i}}_{\text{VI}}$$

$\therefore\ d_h(f_{\cancel{i}}(x), f_{\cancel{i}}(y)) \le L_h(f_{\cancel{i}} \circ \hat{c}) \le L_{g_i}(\hat{c})$