Proof of Thm. 2 (Simplest case)

Show

$$\left(\mathcal{S}, d_{\hat{g}_{J_s}}, \frac{\mu_{\hat{g}_{J_s}}}{C\sqrt{s}^{\,n}}, u\right) \xrightarrow[s\to \text{pmGH}]{} \left(\mathcal{S}_{0,m}, \hat{d}_{0,m}, \hat{\mu}_{0,m}, u_{0,m}\right)$$

for $u \in \mathcal{S}|_{\mu^{-1}(b)}$ b : strict m-BS pt.

Assume $n=1$.

It is enough to see the nbd of $\mu^{-1}(b)$.

$$\text{\textbackslash\textbackslash}$$

$$U \subset \mathbb{R}\times S^1 \ni (x, e^{\sqrt{-1}\theta})$$

$$\omega|_U = dx \wedge d\theta$$

Put $J_s\left(\frac{\partial}{\partial x}\right) = \frac{1}{s}\frac{\partial}{\partial \theta}$, $J_s\left(\frac{\partial}{\partial \theta}\right) = -s\frac{\partial}{\partial x}$.

$\uparrow$ $\leadsto$ J_s : asymptotically semiflat fam.

(this is not general.)

$$g_{J_s} = \frac{1}{s}dx^2 + s\,d\theta^2$$

First of all, assume $m=1$. $(b \in BS^0_1)$

$\exists\, E \in \mathcal{P}(L|_{\mu^{-1}(b)})$: nontrivial flat section.

We can take an trivialization

$$\mathcal{S}|_U \cong U \times S^1$$
$$(x, e^{\sqrt{-1}\theta}, e^{\sqrt{-1}t})$$

s.t. $\hat{g}_{J_s} = (dt - x\,d\theta)^2 + \underbrace{g_{J_s}}_{= s\,d\theta^2 + s^{-1}dx^2}$.

$$= dt^2 - 2x\,dt\cdot d\theta + \underbrace{x^2 d\theta^2 + s\,d\theta^2}_{=(s+x^2)\,d\theta^2} + \frac{1}{s}dx^2$$

$$= \Bigl(\underbrace{\sqrt{s+x^2}\,d\theta - \frac{x\,dt}{\sqrt{s+x^2}}}_{=\alpha}\Bigr)^2 + \Bigl(1 - \frac{x^2}{s+x^2}\Bigr)dt^2 + \frac{1}{s}dx^2$$

$$= \alpha^2 + \frac{s}{s+x^2}\,dt^2 + \Bigl\{d\Bigl(\frac{x}{\sqrt{s}}\Bigr)\Bigr\}^2.$$

$\left(y = \frac{x}{\sqrt{s}}\right)$

$$= \alpha^2 + \underbrace{\frac{1}{1+y^2}\, dt^2 + (dy)^2}_{\substack{\Vert \\ g_{0,1}\ (\text{Thm. } 2)}}$$

Consider the following map.

$$\phi_s : (S|_U,\ \hat{g}_{J_s}) \longrightarrow (\mathbb{R}\times S^1,\ g_{0,1})$$

$$(x, \theta, e^{\sqrt{-1}t}) \longmapsto \left(\frac{x}{\sqrt{s}},\ e^{\sqrt{-1}t}\right)$$

Riemannian submersion.

We can also check that

$$\operatorname{diam}\left(\phi_s^{-1}(y, e^{\sqrt{-1}t})\right) \xrightarrow{s\to 0} 0$$

$(\because)$ $(x, \theta_0, e^{\sqrt{-1}t}), (x, \theta_1, e^{\sqrt{-1}t}) \in \phi_s^{-1}(y, e^{\sqrt{-1}t})$, $\quad y = \frac{x}{\sqrt{s}}$

$$\gamma(\tau) = (x, \tau\theta_1 + (1-\tau)\theta_0, e^{\sqrt{-1}t})$$

$$\gamma'(\tau) = (\theta_1 - \theta_0)\frac{\partial}{\partial\theta}.$$

$$\therefore\ |\gamma'(\tau)|_{\hat{g}_{J_s}} = |\theta_1 - \theta_0|\sqrt{s + x^2} = |\theta_1 - \theta_0|\sqrt{s}\sqrt{1+y^2} \xrightarrow{s\to 0} 0$$

$$(\$|_U, d_{\hat{g}_{J_s}}) \to (\mathbb{R}\times S^1, d_{g_{0,1}})$$

$(m=1)$

The case of $m>1$ $(b\in BS^o_m)$

$U\subset X$: nbd of $\mu^{-1}(b)$

$\exists\, p_m : \hat{U} \xrightarrow[m:1]{} U$

$\hat{U}$ connected.

s.t. $p_m^*(L|_{\mu^{-1}(b)}) \to p_m^{-1}(\mu^{-1}(b))$

has a nontrivial parallel section

$\rightsquigarrow$ trivialization

$$\begin{array}{ccc} p_m^*(\$|_U) \circlearrowleft \mathbb{Z}/m\mathbb{Z} & \to & \hat{U} \\ \downarrow m:1 & & \\ \$|_U & & \end{array}$$

$\mathbb{Z}/m\mathbb{Z}$: deck transformation.

$\rightsquigarrow$ We can describe $p_m^*\hat{g}_{J_s}$ by coord. x, θ, t.

$\mathfrak{F}|_{U}, \hat{g}_{J_s}$

$$\Big(\underbrace{p_m^{*}\mathfrak{F}|_{U}}_{\mathbb{Z}/m\mathbb{Z}} \curvearrowleft S^1 \times \mathbb{Z}/m\mathbb{Z},\ p_m^{*}\hat{g}_{J_s},\ \tilde{u}\Big)$$

$$\Big\downarrow s \to 0$$

$$\Big(\underbrace{\mathbb{R}\times S^1}_{\mathbb{Z}/m\mathbb{Z}} \curvearrowleft S^1,\ g_{0,m},\ (1,0)\Big) \qquad (S^1/\mathbb{Z}/m\mathbb{Z} = S^1)$$

$\mathbb{R}\times S^1 \curvearrowleft \mathbb{Z}/m\mathbb{Z}$

$$\frac{\mathbb{R}\times S^1}{\mathbb{Z}/m\mathbb{Z}} = \mathbb{R}\times S^1 \curvearrowleft S^1.$$