Spectral convergence in geometric quantization

(based on joint work with Mayuko Yamashita.)

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1. Geometric quantization

Def. (X, ω) : symplectic mfd.

$:\Leftrightarrow \begin{cases} \cdot\ X \text{ is a } C^\infty\text{-mfd of dim} = 2n. \\ \cdot\ \omega \in \Omega^2(X) \text{ : non-deg. and closed.} \end{cases}$

Def. $(\underline{L, h, \nabla})$: prequantum bundle.

$:\Leftrightarrow \begin{cases} \cdot\ \pi : L \to X \text{ complex line bdle.} \\ \cdot\ h : \text{herm. met. on } L. \\ \cdot\ \nabla : \text{hermitian connection on } (L, h). \end{cases}$

- $F^{\nabla} = -\sqrt{-1}\,\omega$.

  curvature

  $(F^{\nabla}(u,v)s = \nabla_u \nabla_v s - \nabla_v \nabla_u s - \nabla_{[u,v]} s)$.

<u>Prop</u>. (X,ω) : symplectic mfd. $\left[\frac{\omega}{2\pi}\right] \in H^2(X,\mathbb{Z})$.

$\Rightarrow {}^{\exists}(L, h, \nabla)$ preq. bdle.

s.t. $\left[\frac{\omega}{2\pi}\right] = c_1(L)$.

- <u>Geometric quantization.</u>

… define the "good" subspace

$$V \subset \{ s : X \to L : \text{section} \}$$

by <u>the additional structure</u>

… Polarization. ① complex str.

② Lagrangian fibration.

① J : complex structure.

$(J \in \Gamma(\mathrm{End}(TX)),\ J^2 = -\mathrm{id}.\quad TX \otimes \mathbb{C} = \underbrace{T^{1,0}X}_{\sqrt{-1}} \oplus \underbrace{T^{0,1}X}_{-\sqrt{-1}}.$

$$[\Gamma(T^{1,0}X), \Gamma(T^{1,0}X)] \subset \Gamma(T^{1,0}X).\ )$$

Assume J: ω-compatible.

$$:\Longleftrightarrow \begin{cases} \cdot\ \omega(J\cdot, J\cdot) = \omega. \\ \cdot\ g_J := \omega(\cdot, J\cdot)\ \text{(symmetric)}. \\ \qquad\qquad > 0 \end{cases}$$

$(\rightsquigarrow g_J$: Riem. metric on $X)$.

$\Rightarrow$ $\underset{\Omega^{1,1}(X_J)}{\omega \atop \cap}$ is a Kähler form on $X_J = (X, J)$.

$F^\nabla \in \Omega^{1,1} \Rightarrow (L, \nabla)$: holomorphic line bdle.

$$\begin{aligned} \nabla_{\bar\partial_J} : \Gamma(L) &\to \Omega^{0,1}(L) \\ s &\longmapsto d\bar z^i \otimes \nabla_{\frac{\partial}{\partial \bar z^i}} s \end{aligned}$$

$\left(\begin{array}{l} z^1, \cdots, z^n \\ \text{: holomorphic coord. on } X_J \end{array}\right)$

$$\boxed{\begin{array}{c} H^0(X_J, L_J) \\ \shortparallel \\ V_J \end{array}} := \{ s \in \Gamma(L) \mid \nabla_{\bar\partial_J} s = 0 \}.$$

: space of holo. sections.

X : closed $\Rightarrow \dim V_J < +\infty$

can be computed by Riemann-Roch thm. if $L \otimes K_b^{-1} > 0$.

② $\mu : X \to B$. Lagrangian fiber bundle

$:\Leftrightarrow$
- B : C^∞-mfd of $\dim = n = \dim X / 2$.
- μ: smooth, surj. map s.t. $d\mu_x$: surj. $(\forall x \in X)$
- $\forall b \in B$ $\mu^{-1}(b)$ is Lagrangian submfd. i.e. $\omega|_{\mu^{-1}(b)} \equiv 0$.

(Moreover we assume. • μ : proper.

$\cdot\ {}^{\forall}\mu^{-1}(b)$: connected

$$\Rightarrow \mu^{-1}(b) \cong T^n = (S^1)^n \,)$$

Consider the sections $s : X \to L$. s.t.

$$\nabla|_{\mu^{-1}(b)} s \equiv 0. \qquad ({}^{\forall} b \in B)$$

<u>Def.</u> $\mu^{-1}(b)$ is a Bohr-Sommerfeld fiber.

(or. b : BS point.).

$:\Leftrightarrow \exists \underset{\neq 0}{s} \in \Gamma(L|_{\mu^{-1}(b)})$ s.t. $\nabla|_{\mu^{-1}(b)} s = 0$.

<u>Rem.</u> $\cdot$ $\omega|_{\mu^{-1}(b)} \equiv 0 \Rightarrow (L|_{\mu^{-1}(b)}, \nabla|_{\mu^{-1}(b)})$ is

$(F^\nabla = -\sqrt{-1}\,\omega)$ a flat line bdle.

$\cdot$ If $\underbrace{\{ s \in \Gamma(L|_{\mu^{-1}(b)}) \mid \nabla|_{\mu^{-1}(b)} s \equiv 0 \}}_{= H^0(L|_{\mu^{-1}(b)}, \nabla|_{\mu^{-1}(b)})}$

$\neq \{0\}$, then $\cong \mathbb{C}$.

$$V_\mu := \mathbb{C}^{\# BS} \left(= \bigoplus_{b \in BS} H^0(L|_{\mu^{-1}(b)}, \nabla|_{\mu^{-1}(b)}) \right)$$

(Rem $\{ b \in B \mid b : BS \text{ pt} \}$ is discrete in B.)

ex. 1.

$$X = T^2 \ni (e^{2\pi i x}, e^{iy}) \qquad J : \frac{\partial}{\partial x} \mapsto \frac{2\pi}{n} \frac{\partial}{\partial y}$$

$$\omega = dx \wedge dy \qquad \mathbb{R}^2 \times \mathbb{C} \qquad \frac{\partial}{\partial y} \mapsto -\frac{\partial}{\partial x}.$$

$$L = \frac{\mathbb{R}^2 \times \mathbb{C}}{\mathbb{Z}^2} \qquad \left(\begin{array}{l} (x, y, \xi) \cdot (m, n) \\ := (x+m, y+2\pi n, e^{imy} \xi) \end{array} \right)$$

$p \in \mathbb{R}^2$

$$\tilde{h}((p, \xi_1), (p, \xi_2)) := \xi_1 \overline{\xi_2}$$

$$\tilde{\nabla} = d - \sqrt{-1}\, x\, dy. \qquad \Rightarrow F^\nabla = -\sqrt{-1}\, \omega.$$

$\cdots$ descends to L.

$$\mu : X \longrightarrow T^1, \quad (e^{2\pi i x}, e^{iy}) \mapsto e^{2\pi i x}$$

$$\mu^{-1}(e^{2\pi i x}) : BS\text{-fiber} \Leftrightarrow x \in \mathbb{Z} \Leftrightarrow e^{2\pi i x} = 1.$$

- In many examples,

$$\dim V_J = \dim V_\mu.$$ can be observed.

(abelian variety, toric variety, ...

Frag manifold
(Guillemin - Sternberg.)

- Interpolation between V_J and V_μ.

In some examples, the following results are known.

$\exists \{J_s\}_{s>0}$: family of ω-compatible cpx str.

s.t. $J_s \longrightarrow \mu$ $^{(*)}$ $(s \to 0)$

and $V_{J_s} \longrightarrow V_\mu$ $^{(**)}$ $(s \to 0)$.

(*). Polarization ... subbundle $P \subset TX \otimes \mathbb{C}$

s.t. $\omega_x|_{P_x} = 0$, $\dim_{\mathbb{C}} P_x = n$.

$$[\Gamma(P), \Gamma(P)] \subset \Gamma(P).$$

J : ω-comp. complex str.

$$\Leftrightarrow P_J = T_J^{0,1}X \ (\subset TX \otimes \mathbb{C})$$

μ: Lag. fibration

$$\Leftrightarrow P_\mu := \operatorname{Ker} d\mu \otimes \mathbb{C}.$$

$$J_s \to \mu \ :\Leftrightarrow \ P_{J_s} \to P_\mu.$$

- $X = T^2 \ni (e^{2\pi i x}, e^{iy})$, $\omega = dx \wedge dy$.

$$J_s : \frac{\partial}{\partial x} \mapsto \frac{2\pi}{s}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y} \mapsto -\frac{s}{2\pi}\frac{\partial}{\partial x}$$

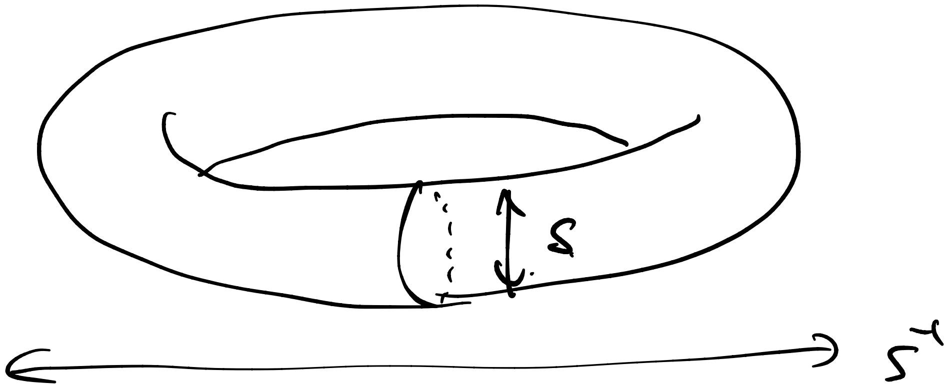
$(s > 0)$

$$\rightsquigarrow T_{J_s}^{0,1}X = \operatorname{span}\left\{-\frac{is}{2\pi}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\}.$$

$\downarrow s \to 0$

$$\operatorname{Ker} d\mu = \operatorname{span}\left\{\frac{\partial}{\partial y}\right\}.$$

$$\left(\rightsquigarrow g_{J_s} := \omega(\cdot, J_s \cdot) = \frac{2\pi}{s}dx^2 + \frac{s}{2\pi}dy^2\right)$$



$\underline{(\ast\ast)}$ $\exists\, \varphi_s \in \underset{\mathbb{C}}{\overset{\cong}{H^0}}(X_{J_s}, L_{J_s})$ s.t.

$$\varphi_s \xrightarrow{s \to 0} \delta \in (P(L))^*$$

distributional section supported by $\underline{\mu^{-1}(1)}$

BS fiber.

basis of V_μ.

$$\begin{array}{ccc} V_{J_s} & & V_\mu \\ \cup & & \cup \\ \varphi_{s,1} & \xrightarrow{s \to 0} & \delta_1 \\ \vdots & & \vdots \\ \varphi_{s,N} & \xrightarrow{s \to 0} & \delta_N \end{array}$$

$(N = \# BS)$

" $\exists J_s \to \mu$ s.t. $V_{J_s} \to V_\mu$ as s→0. "

holds in the case followings.

- Abelian variety ⋯ Baier-Mourão-Nunes, Yoshida.
- Toric variety ⋯ Baier-Florentino-Mourão-Nunes.
- Flag variety ⋯ Hamilton-Konno.
- <u>Aim of this talk.</u>

Give the conceptual explanation of

" $\exists J_s \to \mu$ s.t. $V_{J_s} \to V_\mu$ "

from the view point of

$\left\{ \begin{array}{l} \bullet \text{ measured Gromov-Hausdorff conv.} \\ \bullet \text{ Spectral convergence} \end{array} \right.$

Outline

- Show $V_{J_s} \cong$ Eigenspace of $\Delta_{\hat{g}_{J_s}}$.

  ($\hat{g}_{J_s}$: metric on $S(L,h)$: unit circle bundle.)

- Show $(S(L,h), \hat{g}_s) \xrightarrow{s\to 0} \exists$ met. meas. sp.

- Show the spectral convergence of $\Delta_{\hat{g}_s}$ as $s \to 0$

  and study the limit.

2. Holomorphic sections & eigenfcts.

Setting

(X^{2n}, ω) : closed symplectic mfd.

(L, h, ∇) : prequantum line bdle. $(F^\nabla = -\sqrt{-1}\omega)$