J : ω-compatible cpx str $\left(\Leftrightarrow \begin{array}{l} \omega(J\cdot, J\cdot) = \omega \\ \omega(\cdot, J\cdot) > 0 \end{array}\right)$

$g_J = \omega(\cdot, J\cdot) > 0$: Riem. met. on X.

$k > 0$: positive integer.

$L^k := L^{\otimes k}$

$S = S(L, h) = \{ u \in L \mid h(u,u) = 1 \}$.

$\left(\forall u \in S \quad \forall \lambda \in S^1 \; (= \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \}) \right.$

$\left. u \cdot \lambda \in S. \right)$

$\pi : S \longrightarrow X$: principal S^1-bundle on X.

$L^k = S \times_{\rho_k} \mathbb{C}$ (associate bundle)

$$\left(\begin{array}{rccc} \rho_k : & S^1 & \longrightarrow & \mathbb{C}^\times \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \lambda & \longmapsto & \lambda^k \end{array} \right)$$

$$= \mathcal{S}\times\mathbb{C}/S^1 \qquad (u,\xi)\in\mathcal{S}\times\mathbb{C}.\ \ \lambda\in S^1$$

$$(u,\xi)\cdot\lambda := (u\lambda,\ \lambda^{-\ell}\xi)$$

$$\left(u\times_{\rho_\ell}\xi := [(u,\xi)].\right).$$

$$\rightsquigarrow\ \Gamma(L^\ell) \cong \left(C^\infty(\mathcal{S})\otimes\mathbb{C}\right)^{\rho_\ell}$$

$$= \left\{ f:\mathcal{S}\xrightarrow{C^\infty}\mathbb{C} \;\middle|\; f(u\lambda)=\lambda^{-\ell}f(u)\ \ \forall u\in\mathcal{S}\ \forall\lambda\in S^1 \right\}$$

$$= \left\{ f \;\middle|\; \lambda^\ell(R_\lambda^* f) = f \right\}.$$

$$R_\lambda : \mathcal{S}\to\mathcal{S},\quad u\mapsto u\lambda$$

$$x\mapsto \underbrace{u\times_{\rho_\ell}f(u)}_{\in L^\ell_x.} \ \hookleftarrow\!\!\shortmid\ f.$$

$$(u\in\pi^{-1}(x)).$$

$$\nabla^\ell_{\bar\partial_J} : \Gamma(L^\ell)\longrightarrow\Omega^{0,1}_J(L^\ell) \quad : \bar\partial\text{-operator}.$$

(∇: conn. on L $\rightsquigarrow$ ∇ conn. on L^ℓ

$$\nabla^\ell_{\bar\partial_J} = d\bar z^i\otimes\nabla_{\frac{\partial}{\partial\bar z^i}} \qquad (z^1,\dots,z^n): \text{holo. coord on } X_J.)$$

$$\Delta^\ell_{\bar\partial_J} := \left(\nabla^\ell_{\bar\partial_J}\right)^*\left(\nabla^\ell_{\bar\partial_J}\right) : \Gamma(L^\ell)\longrightarrow\Gamma(L^\ell).$$

$\vdots$

formal adjoint of $\nabla^{\xi}_{\bar\partial_J}$.

Since X is closed,

$$\underbrace{H^0(X_J, L_J^{\xi})}_{\substack{\shortparallel \\ \mathrm{Ker}\, \nabla^{\xi}_{\bar\partial_J}}} = \mathrm{Ker}\, \Delta^{\xi}_{\bar\partial_J}.$$

($\because$ "$\subset$" is easy.
"$\supset$" $\varphi \in \mathrm{Ker}\, \Delta^{\xi}_{\bar\partial_J}$, $\overset{\displaystyle 0}{\shortparallel}\!\!\!\!\!\!\!\!\! \quad \langle \varphi, \Delta^{\xi}_{\bar\partial_J}\varphi \rangle$
$= |\nabla^{\xi}_{\bar\partial_J}\varphi|^2$.)

- Connection metric on $\$$.

∇ : herm conn. on L.

$\longleftrightarrow \sqrt{-1}A \in \Omega^1(\$, \sqrt{-1}\mathbb{R})$

connection form.

$(R_\lambda^* A = A, \ A(\xi^\#) = \xi)$

$(\xi^\#_u = \frac{d}{dt}\big|_{t=0} u \exp(t\xi))$

$$T_u \mathcal{S} = \underset{\substack{\| \\ \operatorname{Ker} A_u}}{H_u} \oplus \underset{\substack{\| \\ T_u(\text{fiber})}}{V_u}. \qquad \pi : \mathcal{S} \to X.$$

Define a Riem. met. $\hat{g}_J$ on $\mathcal{S}$ by.

$$\hat{g}_J := \begin{array}{c} \left(\begin{array}{c|c} g_J & 0 \\ \hline 0 & A \otimes A \end{array} \right) \\ H_u \oplus V_u. \end{array} \begin{array}{c} H_u \\ \oplus \\ V_u. \end{array}$$

$\oplus$ orthogonal.

$\rightsquigarrow (\mathcal{S}, \hat{g}_J) \curvearrowleft S^1$ isometrically.

$$\Delta_{\hat{g}_J} := d^* d : C^\infty(\mathcal{S}) \otimes \mathbb{C} \to C^\infty(\mathcal{S}) \otimes \mathbb{C}.$$

$$\Delta^{\rho_k}_{\hat{g}_J} : \underset{\|\!\wr}{(C^\infty(\mathcal{S}) \otimes \mathbb{C})^{\rho_k}} \to \underset{\|\!\wr.}{(C^\infty(\mathcal{S}) \otimes \mathbb{C})^{\rho_k}}.$$

$$\Delta^{q}_{\bar{\partial}_J} : \Gamma(L^q) \to \Gamma(L^q)$$

$$\hat{g}_J = \begin{pmatrix} g_J & 0 \\ 0 & A \otimes A \end{pmatrix}$$

<u>Prop.</u> $2\Delta^{q}_{\bar{\partial}_J} = \Delta^{\rho_q}_{\hat{g}_J} - (k^2 + kn)$.

$$\therefore H^0(L^q_J) = \operatorname{Ker} \Delta^{k}_{\bar{\partial}_J} \cong \left[\begin{array}{c} k^2 + kn\text{-eigen sp.} \\ \text{of } \Delta^{k}_{\hat{g}_J} \end{array} \right]$$

Convergence of $\{\varphi_s\}$ $\Leftrightarrow$ Conv. of eigenfcts. of $\hat{g}_{J_s}$.

$$\{\varphi_s\} \subset H^0(L^q_{J_s})$$

($\{J_s\}_s$: family of ω-comp. cpx str)

- Asymptotically semiflat family.

$\mu : X \to B$: Lagrangian T^n-fibration ((X, ω) : closed symp.)

$U \subset B$: open $\quad \omega|_{\mu^{-1}(U)} = \sum_i dx_i \wedge d\theta_i$

$(x_1, \cdots, x_n, \theta_1, \cdots, \theta_n$: action-angle coord.$)$

$(x_1, \cdots, x_n)$: coord. on U.

We describe $J|_{\mu^{-1}(U)}$ by $x_1, \cdots, x_n, \theta_1, \cdots, \theta_n$.

If J is "close to" μ, then

$$T^{0,1}_J X \xrightarrow{\text{proj.}} \operatorname{Ker} d\mu \otimes \mathbb{C}$$

$$\left(TX \otimes \mathbb{C} = \underbrace{\operatorname{span}\left\{\frac{\partial}{\partial \theta}\right\}}_{\operatorname{Ker} d\mu} \oplus \operatorname{span}\left\{\frac{\partial}{\partial x}\right\} \right)$$

is surjective.

$\therefore$ We can take the basis $(i=1, \cdots, n)$

$$\frac{\partial}{\partial \theta^i} + \sum_{j=1}^{n} \bar{A}_{ij} \frac{\partial}{\partial x_j} \in T^{0,1}_J X$$

: local frame.

$$J \leftrightarrow A_{ij}(x, \theta) \in M_n(\mathbb{C}).$$

Then, J : ω-compatible almost cpx str.

$:\Leftrightarrow$ $A_{ij} = A_{ji}$, $(\mathrm{Im}\, A_{ij})_{i,j}$ is positive definite.

(J : integrable $\Leftrightarrow$ some P.D.E. of A_{ij}.)

Def. $\{J_s\}_{s\geq 0}$: asymptotically semiflat family

$:\Leftrightarrow$ $B = {}^{\exists}\bigcup_{\alpha} U_\alpha$: open cover

$(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$: action-angle coord on $\mu^{-1}(U_\alpha)$

- $A_{ij}(s, x, \theta)$: C^∞ in s, x, θ (on the nbd of $s=0$)
- $A_{ij}(0, x, \theta) = 0$
- $\mathrm{Im}\,(A_{ij}(s, x, \theta)) = s \cdot Q_{ij}(x) + O(s^2)$ $\quad Q_{ij}(x, \theta) \geq 0$

$(Q_{ij})_{i,j} > 0.$

3. Spectral convergence.

(X_i, g_i) : closed Riemannian mfd of $\dim = N$.
$(i \in \mathbb{N})$

d_{g_i} : Riem. distance

μ_{g_i} : Riemannian measure $\left(= \sqrt{\det g_{ij}}\, dx_1 \cdots dx_N\right)$

$\Delta_{g_i} := d^* d$.

<u>Thm</u> (Spectral convergence. Rough statement)
(by Fukaya, Cheeger-Colding, Kuwae-Shioya.)

$$(X_i, d_{g_i}, \mu_{g_i}) \xrightarrow{i \to \infty} \underline{(X_\infty, d_\infty, \mu_\infty)}$$

(measured Gromov-Hausdorff top.)

(+ assumption)

$$\Rightarrow (L^2(X_i, \mu_{g_i}), \Delta_{g_i}) \xrightarrow{i \to \infty} (L^2(X_\infty, \mu_\infty), \Delta_{d_\infty, \mu_\infty})$$

Def. (X_i, d_i, μ_i) : metric measure space.
$(i \in \mathbb{N} \cup \{\infty\})$

$$(X_i, d_i, \mu_i) \xrightarrow{i\to\infty} (X_\infty, d_\infty, \mu_\infty) \ (\text{mGH-top.})$$

$:\Longleftrightarrow$ ${}^\exists \varepsilon_i \searrow 0$

${}^\exists \phi_i : X_i \xrightarrow{\text{Borel}} X_\infty$ s.t.

$$\begin{cases} \cdot \ |d_\infty(\phi_i(x), \phi_i(y)) - d_i(x,y)| < \varepsilon_i \cdots ① \\ \cdot \ X_\infty \subset B(\mathrm{Im}\,\phi_i, \varepsilon_i) \\ \cdot \ \displaystyle\int_{X_i} f \circ \phi_i \, d\mu_i \xrightarrow{i\to\infty} \int_{X_\infty} f \, d\mu_\infty \cdots ② \end{cases}$$

for ${}^\forall f \in C_{cpt}(X_\infty)$

Def. $(X_i^{\ni p_i}, d_i, \mu_i, p_i) \to (X_\infty^{\ni p_\infty}, d_\infty, \mu_\infty, p_\infty)$
(pointed mGH top.)

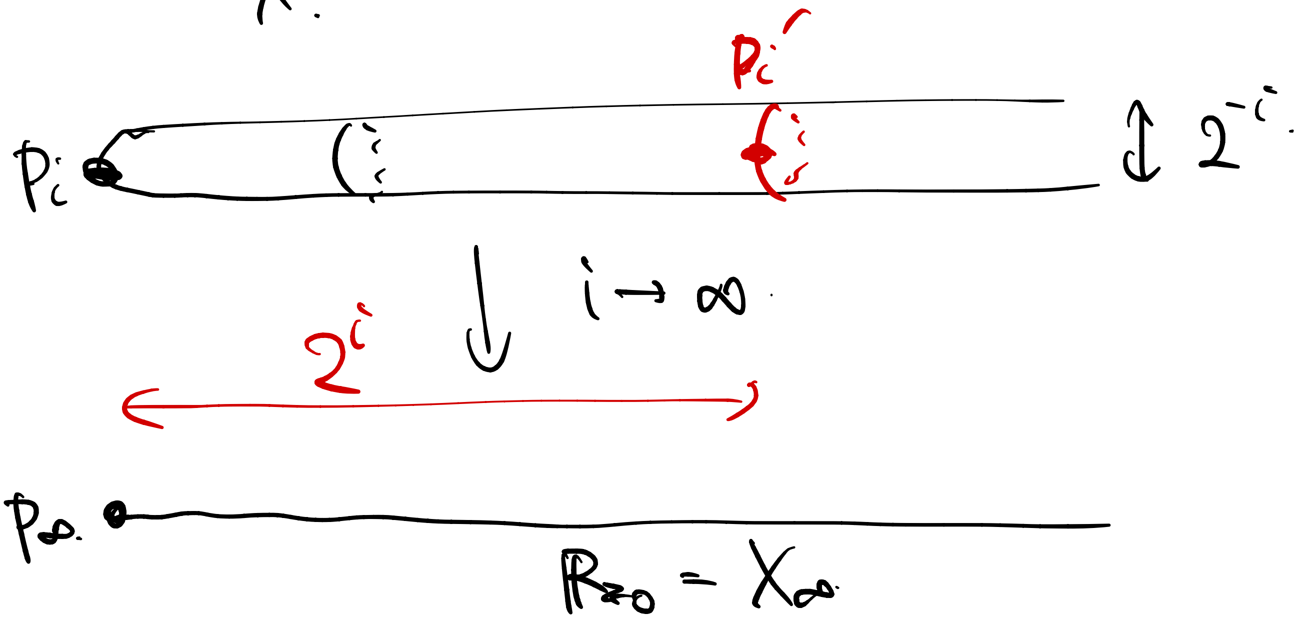
$:\Longleftrightarrow$ ${}^\exists \varepsilon_i \searrow 0$, ${}^\exists R_i, R_i' \nearrow +\infty$.

$\exists\ \phi_i : B(p_i, R_i) \longrightarrow X_\infty$ s.t.

$$\begin{cases} \bullet\ \phi_i(p_i) = p_\infty \\ \bullet\ ①\ ② \\ \bullet\ B(p_\infty, R_i - \varepsilon_i) \subset B(\mathrm{Im}\,\phi_i, \varepsilon_i) \end{cases}$$

<u>ex</u>. (X_i, d_i, p_i)

$\overset{\shortparallel}{X}$.



$(X, d_i, p_i') \longrightarrow$