4. Geometric quantization and S^1-pmGH conv.

- (X, ω) : closed sympl. mfd of dim $= 2n$
- (L, h, ∇) : prequantum line bdle
- $\mu : X \to B$ Lag. T^n-bundle.
- $\{J_s\}_{s>0}$: fam. of asymptotically semiflat ω-compatible cpx. str $\quad (J_s \xrightarrow{s\to 0} \mu)$

  $(\Rightarrow \mathrm{Ric}_{g_{J_s}} \geq^{\exists} \kappa\, g_{J_s})$

- $S = S(L, h)$, $\hat{g}_{J_s}$: conn. metric on S
- $b \in B$: m-BS point

  $:\Leftrightarrow$ $(L^m|_{\mu^{-1}(b)}, \nabla|_{\mu^{-1}(b)})$ has a nontrivial parallel section. (m-BS point $\Rightarrow$ lm-BS pt. $(\forall l > 0)$)

- $b \in B$: strict m-BS point

  $:\Leftrightarrow$ b : m-BS pt. & not l-BS pt. for $^\forall l < m$.

Thm. 2 (H).

b : strict m-BS pt, $u \in \mathcal{S}|_{\mu^{-1}(b)}$. Then $\exists\, C>0$

s.t. $\left(\mathcal{S}^{\curvearrowleft S^1}, d_{\hat{g}_{J_s}}, \frac{\mu_{\hat{g}_{J_s}}}{C\sqrt{s}^{\,n}}, u\right) \xrightarrow[S^1\text{-pmGH}]{s\to 0} \exists \left(\mathcal{S}_{0,m}^{\curvearrowleft S^1}, d_{0,m}, \hat{\mu}_{0,m}, u_{0,m}\right)$

Here,

$$\mathcal{S}_{0,m} = S^1 \times \mathbb{R}^n \curvearrowleft S^1$$

$$(e^{it}, x)\cdot e^{i\theta} := (e^{i(t+m\theta)}, x)$$

$$g_{0,m} = \frac{dt^2}{m^2(1+\|x\|^2)} + \sum_{i=1}^{n} (dx_i)^2 \rightsquigarrow d_{0,m}$$

$$\hat{\mu}_{0,m} = dt\, dx_1 \cdots dx_n$$

$$u_{0,m} = (\overset{S^1}{\underset{\in}{1}}, \overset{\mathbb{R}^n}{\underset{\in}{0}})$$

(proof → exercise class.)

Then, we have the following strong conv.

$$\left(\left(L^2\left(\mathcal{S}, \frac{\mu_{\hat{g}_{J_s}}}{C\sqrt{s}^{\,n}}\right)\otimes\mathbb{C}\right)^{\rho_k}, \Delta^{\rho_k}_{\hat{g}_{J_s}}\right) \xrightarrow{s\to 0} \left(\left(L^2(\mathcal{S}_{0,m})\otimes\mathbb{C}\right)^{\rho_k}, \Delta^{\rho_k}_{0,m}\right) \quad (*)$$

weighted Lap. given by $g_{0,m}, \hat{\mu}_{0,m}$

$$\left(L^2(X, L^{\ell}),\ 2\Delta^{\ell}_{\bar\partial_{J_s}} + \ell_2^2 + \ell_2 n\right)$$

$$\doteq \left(L^2(X, L^{\ell}),\ 2\Delta^{\ell}_{\bar\partial_{J_s}}\right) \xrightarrow{s\to 0} \left((L^2(S_{0,m})\otimes\mathbb{C})^{e_\ell},\ \Delta^{e_\ell}_{0,m} - \ell_2^2 - \ell_2 n\right)$$

Put.

$$BS_\ell := \{ b\in B \mid b : \ell\text{-BS pt.}\}$$

$$BS^0_\ell := \{ b\in B \mid b : \text{strict } \ell\text{-BS-pt.}\}$$

$$\Lambda_\ell := \{ m\in \mathbb{Z}_{>0} \mid \tfrac{\ell}{m}\in\mathbb{Z}\}$$

$$\Rightarrow \quad BS_\ell = \coprod_{m\in\Lambda_\ell} BS^0_m .$$

<u>Prop. 3.</u> (Yamashita-H.)

(1) $m\in\Lambda_\ell \Rightarrow$

$$\left((L^2(S_{0,m})\otimes\mathbb{C})^{e_\ell},\ \Delta^{e_\ell}_{0,m} - (\ell_2^2 + \ell_2 n)\right)$$

$$\cong \left(L^2(\mathbb{R}^n, e^{-\hbar\|x\|^2} dx) \otimes \mathbb{C},\ \Delta^{\hbar}_{\mathbb{R}^n} \right)$$

$$\Delta^{\hbar}_{\mathbb{R}^n} = \sum_i \left(-\frac{\partial^2}{\partial x_i^2} + 2\hbar x_i \frac{\partial}{\partial x_i} \right)$$

(2) $m \notin \Lambda_{\hbar}$

$$\Rightarrow \left(L^2(S_{\sigma,m}) \otimes \mathbb{C} \right)^{e_{\hbar}} = \{0\}.$$

$\therefore$ Applying (Thm 2. (*)) for every $b \in \coprod_{m \in \Lambda_{\hbar}} BS_m^{\circ}$,

we have

$$\left(L^2(X, L^{\hbar}), 2\Delta^{\hbar}_{\bar{\partial}_{J_s}} \right) \xrightarrow[\text{strong } (**)]{s \to 0} \bigoplus_{b \in BS_{\hbar}} \left(L^2(\mathbb{R}^n, e^{-\hbar\|x\|^2} dx) \otimes \mathbb{C},\ \Delta^{\hbar}_{\mathbb{R}^n} \right).$$

Thm 4. (Y-H).

The conv. (**) is compact convergence.

Rem. **Thm 4** holds also for the

following cases.

• (X, ω) : cpt smooth toric variety [Y-H].

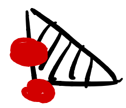
$\mu : X \longrightarrow \Delta$ moment map.

$\{J_s\}_{s>0}$: T^n-inv. cpx str. leading to the Large cpx structure limit.

$\updownarrow$

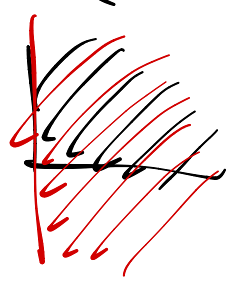
G_s : symmetric matrix func.
$= G_0 + s^{-1}A$

$X = \mathbb{CP}^2$



⇝ Thm ∃. (⇝ limit sp. depends on $b \in \mathring{\Delta}$ or $b \in \partial\Delta$).

$S^1 \times \mathbb{R}^n$ $\qquad$ $S^1 \times (\mathbb{R}_{\geq 0})^2$



• X : K3-surface [H].

$(\omega_1, \omega_2, \omega_3)$ [I_1, I_2, I_3] : hyperkähler on X (fixed).

(X, ω_1) : closed sympl. mfd. $\left[\frac{\omega_1}{2\pi}\right] \in H^2(X, \mathbb{Z})$.

$\mu : X_{I_3} \longrightarrow \mathbb{CP}^1$: elliptic fibration.

s.t. $\forall$ sing. fibers are of type I_1.

↗ focus-focus.

$(\rightsquigarrow \#\{\text{sing. fiber}\} = 24)$

$I_{1,s}$ $I_{2,s}$

$\exists \{(\omega_1, \omega_2, \omega_{3,s})\}_{s>0}$: family of hyperkähler str. on X

s.t. $\int_{\mu^{-1}(b)} \omega_{3,s} \xrightarrow[s\to 0]{} 0$ ($\Leftarrow$ such fam. always exists.)

$\rightsquigarrow$ Thm. 4.

(The limit sp. is indep. of whether b is critical or regular val.)

Thm 4 is shown by the next key Lemma.

Lem. $f_s \in (C^\infty(S^1) \otimes \mathbb{C})^{e_2}$ satisfies

$\|f_s\|_{L^2(S, \frac{\mu_{g_{I_s}}}{C_{I_s}})} = 1$. $\sup_{s>0} \|df_s\|_{L^2(\cdot)} < +\infty$.

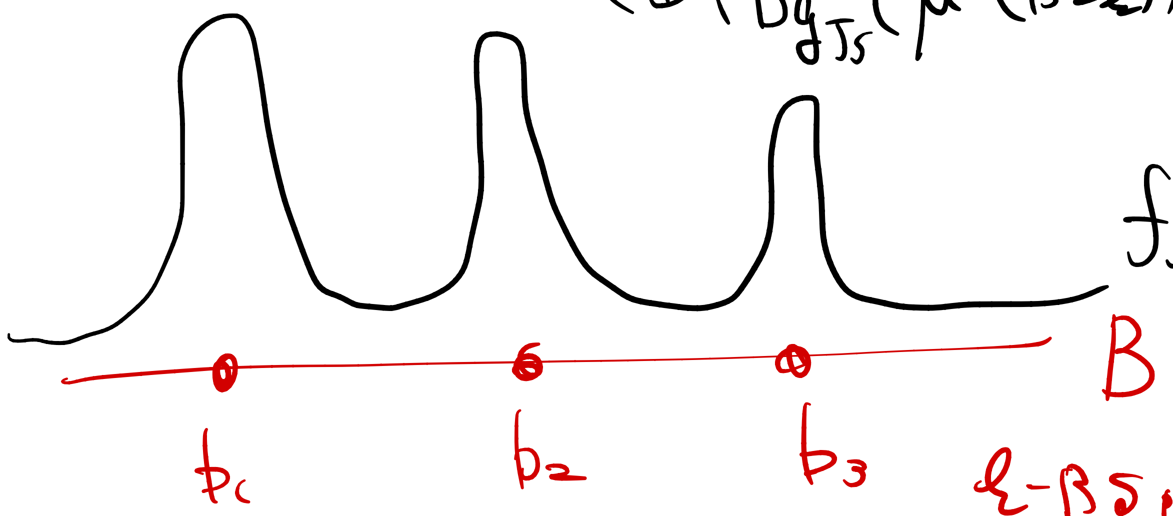
$\sup_s \lambda_s < +\infty$

$\Delta_{g_{I_s}} f_s = \lambda_s f_s$

Then, for $\forall \varepsilon > 0$ $\exists s_0, {}^{\exists}C > 0$ s.t.

$\forall s \in (0, s_0)$,

$$\| f_s \|_{L^2(\mathfrak{S}|_{B_{g_{J_s}}(\mu^{-1}(BS_\ell), C)})} \geq 1 - \varepsilon.$$



<u>proof.</u> (inspired by Fujita-Furuta-Yoshida)

$$\| df_s \|_{L^2}^2 = \int \hat{g}_{J_s}(df_s, df_s)\, d\mu_{\hat{g}_{J_s}}$$

$$\geq \int_{B(BS_\ell, C)} \left(\int \| df_s|_{\mathfrak{S}|_{\mu^{-1}(x)}} \|^2_{\hat{g}|_{\mathfrak{S}_{\mu^{-1}(x)}}} dt\, d\theta \right) dx.$$

$$\text{VI}$$

$$\int \Delta_{fib}(f_s|_{fiber}) \cdot f_s|_{fiber}\, dt\, d\theta.$$

estimate the lowest eigenvalue.

by dist. (x, BS_k)

• Conv. of eigenspaces

Let

$$P_s : L^2(X, L^k) \to \operatorname{Ker} \Delta^k_{\bar\partial_{J_s}} = H^0(L^k_{J_s})$$

$$P_\infty : L^2(\mathbb{R}^n, e^{-k|q|^2} dx) \to \operatorname{Ker} \Delta^k_{\mathbb{R}^n} = \{\text{const. fct.}\} \cong \mathbb{C}.$$

be orthogonal proj.

By Kuwae-Shioya, we can define

$$P_s \xrightarrow[s\to 0]{\text{strong}} P_0$$

Thm 5. (Y-H)

$\forall k > 0$, Then we have $P_s \xrightarrow[\text{strong}]{s\to\infty} P_0$.

$$\left(H^0(L^k_{J_s}) \to \bigoplus_{BS_k} \right)$$

$(\subset)$ given by Thm 4. and

we have to show

$\nexists \{\varphi_s\}$: eigen fct of $\Delta^{\xi}_{\bar{g}_{J_s}}$ s.t.

$$\Delta^{\xi}_{\bar{g}_{J_s}} \varphi_s = \lambda_s \varphi_s. \quad \|\varphi_s\|_{L^2} = 1.$$

$$\lambda_s \downarrow 0.$$

We can show it by spectral gap,

that is, $\mathrm{Spec}\left(\Delta^{\xi}_{g_{J_s}}\right) \subset \{0\} \cup [\frac{\xi}{2} + \kappa, \infty)$

$(\mathrm{Ric}_{g_{J_s}} \geq \kappa)$