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1 Session 1. Introduction to Symplectic Geometry, integrable systems and action-angle systems. Non-degenerate singularities of integrable systems

1.1 Symplectic manifolds

Definition 1.1 (Symplectic manifold)**.** A symplectic manifold is a pair (M, ω), where M is a manifold and $\omega \in \Lambda^2 T^*M$ is a nondegenerate closed 2-form. That is,

- $\mathrm{d}\omega = 0$,
- For each $X \in \mathfrak{X}(M)$, if $\omega(X, Y) = 0$ for all $Y \in \mathfrak{X}(M)$, then $X = 0$.

In other words, a symplectic manifold is a smooth manifold equipped with a closed 2-form ω such that at each $p \in M$, (T_pM, ω_p) is a symplectic vector space.

It is easy to see that all symplectic manifolds have even dimension. We will usually denote $\dim M = 2n$. Similarly, every symplectic manifold is orientable. However, and opposed to the case of Riemannian geometry, there are strong topological constraints to which manifolds admit a symplectic structure. This is known as a topological obstruction.

Proposition 1.2. *Let (M, ω) be a symplectic manifold without boundary. If M is compact, then ω cannot be exact. Equivalently, $H^2_{dR}(M) \neq 0$.*

Proof. Assume for contradiction that $\omega = \mathrm{d}\beta$ for some $\beta \in \Omega^1(M)$. Since ω is nondegenerate, $\omega^n \neq 0$. In particular, we may assume $\omega^n > 0$. Notice that

$$\mathrm{d}\big(\beta \wedge \omega^{n-1}\big) = \mathrm{d}\beta \wedge \omega^{n-1} - \beta \wedge \mathrm{d}\omega^{n-1} = \omega^n - \beta \wedge \mathrm{d}\,(\mathrm{d}\beta)^{n-1} = \omega^n. \tag{1}$$

Therefore, ω^n is also exact. Then, by Stokes theorem,

$$0 < \int_M \omega^n = \int_M \mathrm{d}\big(\beta \wedge \omega^{n-1}\big) = \int_{\partial M} \beta \wedge \omega^{n-1} = 0 \tag{2}$$

because $\partial M = \emptyset$. Hence, we have a contradiction. □

Notice that this implies, amongst other things, that S^{2k} does not admit a symplectic structure for any $k > 1$.

In contrast, some manifolds are naturally endowed with a symplectic structure. It is natural to seek a canonical symplectic structure on the cotangent bundle of a manifold. Indeed, notice that for any manifold M, $T_{(p,v)}(T^*M) \cong T_pM \oplus T_p^*M$.

Theorem 1.3 (Liouville form)**.** *There is a unique 1-form on T^*M, $\lambda \in \Omega^1(T^*M)$, called the Liouville form, such that for every smooth section $\sigma : M \to T^*M$ (that is, for every 1-form on M)*

$$\sigma^* \lambda = \sigma. \tag{3}$$

Moreover, if $q = (q_1, \dots, q_n)$ are local coordinates on M and $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ are the corresponding coordinates on T^*M, then λ can be expressed locally as

$$\lambda = \sum_{i=1}^{n} p_i \mathrm{d}q_i. \tag{4}$$

Proof. Let us prove uniqueness first. Let us fix $x \in M$, and suppose that λ and λ' are two different Liouville forms. Let us denote $\lambda - \lambda' = \theta = \sum_i \theta_{q,i} \mathrm{d}q_i + \theta_{p,i} \mathrm{d}p_i$. Then, for every $\sigma \in \Omega^1(M)$,

$$\sigma^*\left(\lambda - \lambda'\right) = \sigma^*(\theta) = 0 \iff \theta \circ \sigma_* = 0. \tag{5}$$

Denoting $\sigma(q) = (q_1, \dots, q_n, \sigma_1(q), \dots, \sigma_n(q))$, let us compute the pushforward of the basis vectors of T_qM:

$$\sigma_* \left(\frac{\partial}{\partial q_i}\Big|_x \right) = \frac{\partial}{\partial q_i}\Big|_x + \sum_{j=1}^{n} \frac{\partial \sigma_j}{\partial q_i}\Big|_x \frac{\partial}{\partial p_i}\Big|_{\sigma(x)}. \tag{6}$$

Applying this to (5) yields

$$\theta \circ \sigma_* \left(\frac{\partial}{\partial q_i}\right) = \theta\left(\frac{\partial}{\partial q_i}\right) + \sum_{j=1}^{n} \frac{\partial \sigma_j}{\partial q_i}\theta\left(\frac{\partial}{\partial p_i}\right) = \theta_{q,i}(x,y) + \sum_{j=1}^{n} \frac{\partial \sigma_j}{\partial q_i}\Big|_x \theta_{p,j}(x,y) = 0, \tag{7}$$

where $(x,y) = \sigma(x)$. In particular, choosing $\sigma = \sum_i c_i \mathrm{d}q_i$ for some constants c_i yields $\theta_{q,i} = 0$. Similarly, choosing $\sigma = \sum_i c_i q_i \mathrm{d}q_i$ yields $\theta_{p,i} = 0$. Therefore $\theta = 0$ at $(x,y) = \sigma(x)$ for any σ, and thus $\lambda = \lambda'$, because x was arbitrary.

To prove existence, consider $\pi : T^*M \to M$ the natural projection. Let $(q,p) \in T*M$ be a point on the cotangent bundle, and consider the differential $\mathrm{d}\pi_{(q,p)} : T_{(q,p)}(T^*M) \to T_qM$. Notice that p defines a map $p : T_qM \to \mathbb{R}$. We define

$$\lambda_{(q,p)} := p \circ \mathrm{d}\pi_{(q,p)} : T_{(q,p)}(T^*M) \to \mathbb{R}. \tag{8}$$

By construction $\lambda \in \Omega^1(T^*M)$. Let us see that it satisfies the desired property. Indeed, let $\sigma \in \Omega^1(M)$, fix $x \in M$ and denote $(x,y) = \sigma(x)$. Then,

$$\sigma^*\left(\lambda_{\sigma(x)}\right) = y \circ \mathrm{d}\pi_{\sigma(x)} \circ \mathrm{d}\sigma_x = y \circ \mathrm{d}\left(\pi \circ \sigma\right)_x = y \circ \mathrm{d}\left(\mathrm{id}_M\right)_x = y = \sigma_x. \tag{9}$$

The expression of λ in local coordinates follows immediately from its definition. □

Corollary 1.4. *The cotangent bundle of any manifold admits a symplectic structure, where the symplectic form is given by*

$$\omega = \mathrm{d}\lambda = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i. \tag{10}$$

Proof. It is a simple computation to check that

$$\omega^n = (\mathrm{d}\lambda)^n = \left(\sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i\right)^n = \mathrm{d}p_1 \wedge \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q^n \neq 0 \tag{11}$$

because $\{\mathrm{d}q_i, \mathrm{d}p_i\}$ are linearly independent. It is obviously closed because it is exact. □

Because of this fact, the Liouville form is sometimes also refered to as the 'symplectic potential'. We have also obtained, 'for free', another interesting result:

Corollary 1.5. *The cotangent bundle of any manifold is orientable.*

Definition 1.6 (Symplectomorphism)**.** Let (M,ω) and $(N,\widetilde{\omega})$ be two symplectic manifolds, and let $F : M \to N$ be a diffeomorphism. We say that F is a symplectomorphism if $\omega = F^*\widetilde{\omega}$. We then say that M and N are symplectomorphic.

1.2 Darboux theorem

Darboux theorem implies that, locally, all symplectic manifolds look the same.

Theorem 1.7 (Darboux)**.** *Let M be a symplectic manifold. For every $p \in M$, there exist local coordinates around p, $(U, \varphi = (x_1, \ldots, x_n, y_1, \ldots, y_n))$, in which $\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$.*

This theorem has a very important consequence: there are no local symplectic invariants that allow us to classify symplectic manifolds, and so problems in symplectic geometry are inherently global. This is a fundamental difference between symplectic and Riemannian geometry.

To prove this theorem we will need some additional tools.

Definition 1.8 (Time dependent vector field)**.** Let $I \subset \mathbb{R}$ be an interval and M be a manifold. A (smooth) time dependent vector field is a smooth map $X : I \times M \to TM$ such that for each t, the map $X(\cdot, t) = X_t(\cdot)$ is a smooth vector field. In a similar manner one may define arbitrary time dependent tensor fields.

Time dependent vector fields have associated flows, in the same way as autonomous (i.e., time independent) vector fields do. An important difference between the flows of autonomous and nonautonomous vector fields is that in the latter case, the composition property for the flow map, ψ, is

$$\psi(t,t_0,p)=\psi(t,t_1,\psi(t_1,t_0,p)) \tag{12}$$

whenever the composition is defined. Most of the other properties for the flows of autonomous vector fields also hold in the nonautonomous case.

Proposition 1.9. *Let X be a time dependent vector field, and let $\psi(t,t_0,p)=\psi_{t,t_0}(p)$ be its flow. For any covariant tensor field $A\in\Gamma(T^{(0,k)}(TM))$ and any (p,t_1,t_0) in the domain of definition of ψ,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1}\left(\psi_{t,t_0}^*A\right)_p=\left(\psi_{t_1,t_0}^*\left(\mathcal{L}_{V_{t_1}}A\right)\right)_p. \tag{13}$$

Proof. Consider first the special case $t_1=t_0$. Then $\psi(\cdot,t_1,t_0)$ is the identity and thus the equation in the statement is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0}\left(\psi_{t,t_0}^*A\right)_p=\left(\mathcal{L}_{X_{t_0}}A\right)_p. \tag{14}$$

Suppose that A is a $(0,0)$-tensor field (i.e., a smooth map). Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0}\left(\psi_{t,t_0}^*f\right)_p=\frac{\partial}{\partial t}\Big|_{t=t_0}f\left(\psi(t,t_0,p)\right)=X_{t_0}\left(\psi(t_0,t_0,p)\right)(f)=\left(\mathcal{L}_{X_{t_0}}f\right)_p. \tag{15}$$

Now let $A=\mathrm{d}f$ for some function f. Note that for each fixed t_0, the map $(\psi_{t,t_0}^*f)(x)=f(\psi(x,t,t_0))$ is a smooth function of $(t,x)\in I\times M$. Hence, in any system of coordinates, the operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_i}$ commute when applied to ψ_{t,t_0}^*f. Therefore, the exterior derivative operator commutes with $\frac{\partial}{\partial t}$. Hence, by the properties of the Lie derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0}\left(\psi_{t,t_0}^*\mathrm{d}f\right)_p=\frac{\partial}{\partial t}\Big|_{t=t_0}\mathrm{d}\left(\psi_{t,t_0}^*f\right)_p=\mathrm{d}\left(\frac{\partial}{\partial t}\Big|_{t=t_0}\left(\psi_{t,t_0}^*f\right)\right)_p=\mathrm{d}\left(\mathcal{L}_{X_{t_0}}f\right)_p=\left(\mathcal{L}_{X_{t_0}}\mathrm{d}f\right)_p. \tag{16}$$

We have proved the result in the case $t_1=t_0$ for 0-forms and exact 1-forms. The claim follows for arbitrary $(0,k)$-tensors by the product rule for the Lie derivative, which the left hand side of (14) also satisfies (indeed, 0-forms and exact 1-forms generate the whole covariant tensor algebra).

Now we consider the case $t_1\neq t_0$. Note that for any map $F:M\to M$,

$$(F^*A)_p=(\mathrm{d}F)_p^*A_{F(p)}. \tag{17}$$

Consider the pullback map of the differential of the flow,

$$\mathrm{d}(\psi_{t_1,t_0})_p^*:T^{(0,k)}(T_{\psi_{t_1,t_0}(p)}M)\to T^{(0,k)}(T_pM), \tag{18}$$

which does not depend on t. By (12) and (17) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1}\left(\psi_{t,t_0}^*A\right)_p&=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1}\mathrm{d}\left(\psi_{t_1,t_0}\right)_p^*\circ\mathrm{d}\left(\psi_{t,t_1}\right)_{\psi_{t_1,t_0}(p)}^*\left(A_{\psi_{t,t_0}(p)}\right) &(19)\\
&=\mathrm{d}\left(\psi_{t_1,t_0}\right)_p^*\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1}\mathrm{d}\left(\psi_{t,t_1}\right)_{\psi_{t_1,t_0}(p)}^*\left(A_{\psi_{t,t_1}(\psi_{t_1,t_0}(p))}\right) &(20)\\
&=\mathrm{d}\left(\psi_{t_1,t_0}\right)_p^*\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1}\left(\psi_{t,t_1}^*A\right)_{\psi_{t_1,t_0}(p)} &(21)\\
&=\mathrm{d}\left(\psi_{t_1,t_0}\right)_p^*\left(\mathcal{L}_{X_{t_1}}A\right)_{\psi_{t_1,t_0}(p)} &(22)\\
&=\left(\psi_{t_1,t_0}^*\left(\mathcal{L}_{X_{t_1}}A\right)\right)_p, &(23)
\end{aligned}$$

as we wanted to prove. ☐

Corollary 1.10. *Suppose that A is a time dependent $(0,k)$-tensor field on M. The following identity holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1}\left(\psi_{t,t_0}^*A_t\right)_p=\left(\psi_{t_1,t_0}^*\left(\mathcal{L}_{X_{t_1}}A_{t_1}+\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1}A_t\right)\right)_p. \tag{24}$$

Proof. Consider the map

$$F(u,v) = (\psi^*_{u,t_0} A_v)_p = \mathrm{d}(\psi_{u,t_0})^*_p \left(A_v\big|_{\psi_{u,t_0}(p)}\right). \tag{25}$$

By the chain rule and the previous proposition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1} F = \frac{\partial F}{\partial u}(t_1,t_1) + \frac{\partial F}{\partial v}(t_1,t_1) \tag{26}$$

$$= \left(\psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}} A_{t_1}\right)\right)_p + \frac{\partial}{\partial v}\Big|_{v=t_1} \mathrm{d}\left(\psi_{t_1,t_0}\right)^* \left(A_v\big|_{\psi_{t_1,t_0}}(p)\right). \tag{27}$$

Since $\mathrm{d}\left(\psi_{t_1,t_0}\right)^*$ does not depend on v, it commutes with $\frac{\partial}{\partial v}$ and thus the claim follows. □

Proof of Theorem 1.7. Notice that for each $p \in M$ there exists a neighbourhood U containing p such that U is diffeomorphic to T_pM. Indeed, let (U,φ) be a chart around p. Then,

$$U \overset{\varphi}{\cong} \mathbb{R}^n \overset{\mathrm{d}\varphi_p}{\cong} T_pM. \tag{28}$$

Let us call $\psi : U \to T_pM$ the diffeomorphism between U and T_pM. There exists a change of coordinates in which ω_p can be written as $\Omega_0 = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$ (we remark that, for now, this expression is only valid at p!). Let us define $\omega_0 = \psi^*\Omega_0$ and $\omega_1 = \omega_p$, and consider the time dependent form defined by

$$\omega_t = (1-t)\omega_0 + t\omega_1, \quad t \in [0,1]. \tag{29}$$

Note that $\omega_t = \omega_1 = \omega_0$ at p. Let us check that ω_t is a symplectic form at each t. It is obviously closed, because by linearity of the exterior derivative,

$$\mathrm{d}\omega_t = (1-t)\mathrm{d}\omega_0 + t\mathrm{d}\omega_1 = 0. \tag{30}$$

Moreover, it is nondegenerate. Indeed, for each $t \in [0,1]$ there is a neighbourhood of (p,t) in $M \times [0,1]$ in which ω_t is nondegenerate. Since $[0,1]$ is compact, finitely many of these neighbourhoods cover t. By taking the intersection of these and renaming it U we see that ω_t is indeed nondegenerate in U.

By Poincaré lemma, since $\omega_0 - \omega_1$ is closed, we may assume that $\omega_0 - \omega_1 = \mathrm{d}\beta$ in U, for some 1-form β. Then, there exists a time dependent vector field X_t that satisfies

$$\iota_{X_t}\omega_t = -\beta. \tag{31}$$

This is well defined by the nondegeneracy of ω_t (we will delve further into this matter in the next section). Let us denote by $\varphi_{t,t_0}(p)$ the flow of X_t. By Corollary 1.10 and Cartan's magic formula we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi^*_{t,0}\omega_t\right) = \varphi^*_{t,0}\left(\mathcal{L}_{X_t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right) \tag{32}$$

$$= \varphi^*_{t,0}\left(\mathrm{d}\iota_{X_t}\omega_t + \iota_{X_t}\mathrm{d}\omega_t + \omega_1 - \omega_0\right) \tag{33}$$

$$= \varphi^*_{t,0}\left(-\mathrm{d}\beta + \mathrm{d}\beta\right) = 0. \tag{34}$$

Therefore, $\varphi^*_{t,0}\omega_t$ is constant in t, and because $\varphi^*_{0,0}\omega_0 = \omega_0$ we deduce that $\varphi^*_{t,0}\omega_t = \omega_0$. Since the map $\varphi_{1,0}(\cdot)$ is a diffeomorphism onto its image and $\varphi_{1,0}(p) = p$, $\varphi_{1,0}$ defines a change of coordinates around p in which ω takes the canonical form,

$$\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i. \tag{35}$$

□

1.3 Hamiltonian vector fields

As we suggested earlier, the fact that the symplectic form ω on a symplectic manifold is nondegenerate implies that ω defines an isomorphism between TM and T^*M given by

$$\omega : TM \to T^*M \tag{36}$$

$$X_p \mapsto \iota_{X_p}\omega_p. \tag{37}$$

As in the case of gradients of functions, we will be interested in the vector fields defined by differential forms. This concept is precisely that of Hamiltonian vector fields:

Definition 1.11 (Hamiltonian vector field)**.** Let $f : M \to \mathbb{R}$ be a function (which we will call the Hamiltonian). The Hamiltonian vector field associated to f is the unique vector field X_f defined by

$$\iota_{X_f}\omega = -\mathrm{d}f. \tag{38}$$

Remark 1.12. It is important to notice that we have included a negative sign in the definition of Hamiltonian vector field. This choice is arbitrary but convenient, as we will see.

One may show that if f is smooth, and so is ω, then the Hamiltonian vector field X_f is smooth too. We will skip the details here.

The following properties follow directly from the definition:

Proposition 1.13. *Let $f \in \mathcal{F}(M)$ be a Hamiltonian on a symplectic manifold M, and let X_f be its Hamiltonian vector field. The following hold:*

1. *f is constant along the integral curves of X_f.*
2. *X_f is tangent to the submanifolds defined as the preimages of regular values of f.*

That is to say, the integral curves of X_f are contained in the level sets of f.

Another important property of Hamiltonian vector fields is that they preserve the symplectic form.

Definition 1.14 (Symplectic vector field)**.** A vector field $X \in \mathfrak{X}(M)$ on a symplectic manifold M is called symplectic if $((\Phi_t^X)^*\omega)_p = \omega_p$, where Φ_t^X denotes the flow of X. Equivalently, if

$$\mathcal{L}_X\omega = 0. \tag{39}$$

Proposition 1.15. *Every Hamiltonian vector field is symplectic. Moreover, if $H^1_{dR}(M) = 0$ then every symplectic vector field is Hamiltonian.*

Proof. Let X_f be a Hamiltonian vector field with Hamiltonian f. By Cartan's formula, we need to see that

$$\iota_{X_f}\mathrm{d}\omega + \mathrm{d}\iota_{X_f}\omega = 0. \tag{40}$$

The first term is 0 because ω is closed. For the second one, note that

$$\mathrm{d}\iota_{X_f}\omega = \mathrm{d}\left(-\mathrm{d}f\right) = -\mathrm{d}^2 f = 0. \tag{41}$$

Now, assume that $H^1_{dR}(M) = 0$ and suppose that $X \in \mathfrak{X}(M)$ satisfies

$$\mathcal{L}_X\omega = \mathrm{d}\iota_X\omega = 0. \tag{42}$$

This means that the 1-form $\iota_X\omega$ is closed, and since $H^1_{dR}(M) = 0$, there exists $f \in \mathcal{F}(M)$ such that

$$\iota_X\omega = -\mathrm{d}f, \tag{43}$$

so X is Hamiltonian. □

Remark 1.16. If $H^1_{dR}(M) \neq 0$, then one can find symplectic vector fields that are not Hamiltonian, but note that by Poincaré lemma they will always be *locally* Hamiltonian.

Given two functions on a symplectic manifold, there is an important operation that one can do with them:

Definition 1.17 (Poisson bracket)**.** Given $f, g \in \mathcal{F}(M)$, the Poisson bracket of f and g is defined as

$$\{f, g\} = \omega(X_f, X_g), \tag{44}$$

where X_f and X_g are the Hamiltonian vector fields associated to f and g.

We will sometimes refer to this operation on a symplectic manifold as the *symplectic* Poisson bracket. The Poisson bracket endows $\mathcal{F}(M)$ with the structure of a Lie algebra.

Lemma 1.18. *Given $f, g \in \mathcal{F}(M)$, the following identity holds:*

$$X_{\{f,g\}} = \left[X_f, X_g\right]. \tag{45}$$

Proof. By 1.15, we have

$$\begin{aligned}\iota_{[X_f,X_g]}\omega &= \mathcal{L}_{X_f}\iota_{X_g}\omega - \iota_{X_g}\mathcal{L}_{X_f}\omega && (46)\\ &= \mathrm{d}\iota_{X_f}\iota_{X_g}\omega + \iota_{X_f}\mathrm{d}\iota_{X_g}\omega && (47)\\ &= \mathrm{d}\iota_{X_f}\iota_{X_g}\omega - \iota_{X_f}\mathrm{d}^2 g && (48)\\ &= \mathrm{d}\iota_{X_f}\iota_{X_g}\omega = -\mathrm{d}\omega\left(X_f, X_g\right) = -\mathrm{d}\{f,g\}. && (49)\end{aligned}$$

□

Proposition 1.19. *The following properties hold:*

1. *Antisymmetry:* $\{f,g\} = -\{g,f\}$ *for all* $f,g \in \mathcal{F}(M)$.
2. *Jacobi identity:* $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ *for all* $f,g,h \in \mathcal{F}(M)$.
3. *Leibniz rule:* $\{f,gh\} = h\{f,g\} + g\{f,h\}$ *for all* $f,g,h \in \mathcal{F}(M)$.

As a consequence, $(\mathcal{F}(M), \{\cdot,\cdot\})$ *has the structure of a Lie algebra.*

Proof.

1. We have
$$\{f,g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = -\{g,f\}. \tag{50}$$
2. By the properties of the Lie bracket,
$$\begin{aligned}0 = \mathrm{d}\omega(X_f,X_g,X_h) &= X_f\left(\omega(X_g,X_h)\right) - X_g\left(\omega(X_f,X_h)\right) + X_h\left(\omega(X_f,X_g)\right) && (51)\\ &\quad - \omega\left([X_f,X_g], X_h\right) + \omega\left([X_f,X_h], X_g\right) - \omega\left([X_g,X_h], X_f\right) && (52)\\ &= \{f,\{g,h\}\} - \{g,\{f,h\}\} + \{h,\{f,g\}\} && (53)\\ &\quad - \{\{f,g\},h\} + \{\{f,h\},g\} - \{\{g,h\},f\} && (54)\\ &= 2\left(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}\right). && (55)\end{aligned}$$
3. We have $X_f(g) = \mathrm{d}g(X_f) = -\iota_{X_f}\iota_{X_g}\omega = \omega(X_f,X_g) = \{f,g\}$. Therefore,
$$\{f,gh\} = X_f(gh) = \mathrm{d}(gh)\left(X_f\right) = \left(g\mathrm{d}h + h\mathrm{d}g\right)\left(X_f\right) = g\{f,h\} + h\{f,g\}. \tag{56}$$

□

Corollary 1.20. *The map*

$$\begin{aligned}\Phi : (\mathcal{F}(M), \{\cdot,\cdot\}) &\to (\mathfrak{X}(M), [\cdot,\cdot]) && (57)\\ f &\mapsto X_f && (58)\end{aligned}$$

is a Lie algebra morphism.

Since the motivation for symplectic geometry is found in classical mechanics, it is often the case that one wishes to study a particular mechanical system which is described by a particular function. This distinguished function $H \in \mathcal{F}(M)$ is called the Hamiltonian of the system. The flow of the Hamiltonian vector field X_H is the time evolution of the mechanical system.

Definition 1.21 (Hamiltonian system)**.** A Hamiltonian system is a triple (M,ω,H), where (M,ω) is a symplectic manifold and $H \in \mathcal{F}(M)$ is the Hamiltonian function.

The introduction of the Poisson bracket provides a powerful formalism to study Hamiltonian systems. The most trivial application is the following:

Proposition 1.22. *Let* (M,ω,H) *be a Hamiltonian system and let* $f \in \mathcal{F}(M)$. *Then,* f *is constant along the flowlines of* X_H *if, and only if,* $\{f,H\} = 0$. *We then say that* f *is an integral of motion.*

Proof. We have

$$\{f,H\} = -X_H(f) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^X \circ f = 0 \tag{59}$$

if, and only if, $\Phi_t^X \circ f$ is constant, where Φ_t^X denotes the flow of X_H. □

The following simple but important result is a reformulation of Propositions 1.15 and 1.22. It is due to Emmy Noether.

Definition 1.23 (Infinitessimal symmetries)**.** A vector field X on M is an infinitessimal symmetry of (M, ω, H) if $X(H) = 0$ and $\mathcal{L}_X \omega = 0$.

Theorem 1.24 (Noether)**.** *Let (M, ω, H) be a Hamiltonian system. If f is an integral of motion, its Hamiltonian vector field is an infinitessimal symmetry. Conversely, if $H^1_{dR}(M) = 0$, then each infinitessimal symmetry is the Hamiltonian vector field of some integral of motion which is unique up to an additive constant on each connected component of M.*

In general it is difficult to obtain integrals of motion. In some cases one may produce new integrals of motion from already known ones:

Lemma 1.25. *If f and g are integrals of motion of a Hamiltonian system (M, ω, H), then $\{f, g\}$ is also an integral of motion.*

Proof. It is a direct consequence of the Jacobi identity which $\{\cdot, \cdot\}$ satisfies. □

1.4 Integrable systems

Systems with many integrals of motion are interesting because they are very well understood.

Definition 1.26. Let (M, ω, H) be a Hamiltonian system, and let $\dim M = 2n$. We say that the system is completely integrable if it has n independent integrals of motion. That is, if there exist $f_1, \dots, f_n \in \mathcal{F}(M)$ such that

- They are in involution: $\{f_i, f_j\} = 0$ for all $i, j = 1, \dots, n$,
- They are linearly independent: $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$ almost everywhere in M.

The term *integrable* is due to historical reasons, as we will see. In the theory of differential equations and dynamical systems, one says that a system is integrable by quadratures if the flow map can be written in terms of integrals of elementary functions and its inverses. The interest in completely integrable systems lies in the theorem that we now state.

Definition 1.27 (Lagrangian submanifold)**.** Given a symplectic manifold (M, ω), a submanifold $S \subset M$ is called Lagrangian if the tangent space $T_pS \subset T_pM$ is a Lagrangian subpace. In other words, the pullback of ω to S is the zero form.

Theorem 1.28 (Liouville-Arnold)**.** *Let (M, ω, H) be a completely integrable Hamiltonian system with integrals of motion $f_1, \dots, f_n$, and assume that $f_1 = H$. We define the map $f : M \to \mathbb{R}^n$ by $f = (f_1, \dots, f_n)$. Let $c \in \mathbb{R}^n$ be a regular value of f. The following hold:*

1. *$f^{-1}(c)$ is a Lagrangian submanifold of (M, ω).*
2. *If the flows of the associated Hamiltonian vector fields, X_{f_i}, are complete on a connected component $L \subseteq f^{-1}(c)$, then L is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $k \in \{0, 1, \dots, n\}$. In particular, if L is compact then it is an n-dimensional torus. Moreover, L is equipped with n coordinates, $(\theta_1, \dots, \theta_n)$, called angle coordinates*
3. *In angle coordinates, the flow of X_H on L is linear. That is,*
$$\Phi_t^{X_H}(\theta_0) = (v_1 t + \theta_1^0, \dots, v_n t + \theta_n^0), \quad \text{with } v_i \in \mathbb{R}. \tag{60}$$
4. *There exist coordinates $(p_1, \dots, p_n)$ defined on a neighbourhood of L, called action coordinates, which are integrals of motion for H and such that together with the angle coordinates they form a Darboux chart.*

To prove this theorem we will need a few preliminary results.

Lemma 1.29. *Let $\Gamma \subset \mathbb{R}^n$ be a discrete subgroup of $\mathbb{R}^n$. There exists $k \in \{1, \dots, n\}$ and k linearly independent vectors $t_1, \dots, t_k \in \mathbb{R}^n$ such that*
$$\Gamma = \bigoplus_{i=1}^{k} \mathbb{Z} t_i. \tag{61}$$

Proof. Exercise (or check [Arnold, Mathematical Methods of Classical Mechanics, Lemma 3, Chap. 10]). □

Theorem 1.30. *Let M be an n-dimensional manifold, and let $X_1, \ldots, X_n \in \mathfrak{X}(M)$ be linearly independent vector fields such that*

$$[X_i, X_j] = 0 \quad i, j = 1, \ldots, n. \tag{62}$$

Then, there exists $k \in \{0, 1, \ldots, n\}$ such that M is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

Proof. Let $\Phi_t^{X_i}$ denote the flow of X_i. Since the vector fields commute, so do their flows:

$$\Phi_t^{X_i} \circ \Phi_s^{X_j} = \Phi_s^{X_j} \circ \Phi_t^{X_i} \quad \forall i, j = 1, \ldots, n, \quad \forall s, t \in \mathbb{R}. \tag{63}$$

Let us define an action of $\mathbb{R}^n$ on M as

$$\Phi(t_1, \ldots, t_n; p) = \Phi_{t_1}^{X_1} \circ \cdots \circ \Phi_{t_n}^{X_n}(p). \tag{64}$$

For each $p \in M$, we denote

$$\phi_p(t) := \Phi(t; p). \tag{65}$$

Note that the columns of $\mathrm{d}\phi_p(0)$ are the components of the vectors $X_1, \ldots, X_n$, so $\mathrm{d}\phi_p(0)$ is a local diffeomorphism around $0 \in \mathbb{R}^n$ and $p \in M$ for each $p \in M$.

First we claim that the action defined by Φ is transitive. For this, fix $p \in M$ and let ϕ_p be as before. Take $y \in M$ with $y \neq p$ and let $\gamma : [0, 1] \to M$ be a path from p to y. By the previous remark, for each $q \in \gamma([0, 1])$ there are neighbourhoods $U \subseteq \mathbb{R}^n$ around 0 and $V \subseteq M$ around q such that $\phi_q : U \to V$ is a diffeomorphism. Since $[0, 1]$ is compact, there are finitely many such neighbourhoods $U_1, \ldots, U_m$ of 0 and $V_1, \ldots, V_m$ of $q_1, \ldots, q_m \in \gamma([0, 1])$ such that for each i the restriction

$$\phi_{q_i} : U_i \to V_i \tag{66}$$

is a diffeomorphism and

$$\gamma([0, 1]) \subseteq \bigcup_{i=1}^{m} V_i. \tag{67}$$

Since $[0, 1]$ is connected, there exist points $x_1, \ldots, x_m$ such that $x_i \in V_{i-1} \cap V_i$ for each $i = 2, \ldots, m$. Then, by the properties of flows,

$$\Phi\left(\phi_{q_i}^{-1}(x_{i+1}) - \phi_{q_i}^{-1}(x_i); x_i\right) = \Phi\left(\phi_{q_i}^{-1}(x_{i+1}); q_i\right) = x_{i+1} \tag{68}$$

for $i = 1, \ldots, m-1$. Therefore we have

$$\Phi\left(\phi_{q_1}(x_1) + \sum_{i=1}^{m-1} \left(\phi_{q_i}^{-1}(x_{i+1}) - \phi_{q_i}^{-1}(x_i)\right) - \phi_{q_m}^{-1}(x_m) + \phi_{q_m}^{-1}(y);\ p\right) = \tag{69}$$

$$= \Phi\left(\sum_{i=1}^{m-1} \left(\phi_{q_i}^{-1}(x_{i+1}) - \phi_{q_i}^{-1}(x_i)\right) - \phi_{q_m}^{-1}(x_m) + \phi_{q_m}^{-1}(y);\ x_1\right) \tag{70}$$

$$= \cdots \tag{71}$$

$$= \Phi\left(-\phi_{q_m}^{-1}(x_m) + \phi_{q_m}^{-1}(y);\ x_m\right) = y, \tag{72}$$

so y is in the orbit of p, as we wanted to see.

Now consider the stabilizer of $p \in M$, which we denote by Γ:

$$\Gamma = \{t \in \mathbb{R}^n : \phi_p(t) = p\}. \tag{73}$$

We claim that Γ is a discrete subgroup of $\mathbb{R}^n$. We already know that it is a group, so we only need to check that it is discrete. That is, for each $t \in \Gamma$ there exists a neighbourhood U of t in $\mathbb{R}^n$ such that $U \cap \Gamma = \{t\}$. Indeed, this follows from the fact that the vector fields $X_1, \ldots, X_n$ are independent in the same way that we argued it for the case $t = 0$.

By the previous lemma, we know that there are $k \in \{0, \ldots, n\}$ linearly independent vectors $t_1, \ldots, t_k \in \mathbb{R}^n$ such that

$$\Gamma = \bigoplus_{i=1}^{k} \mathbb{Z} t_i. \tag{74}$$

Extend the collection of $\{t_i\}$ to form a basis of $\mathbb{R}^n$, and denote by A the linear isomorphism which sends the standard basis of $\mathbb{R}^n$ to the basis $\{t_i\}$. We observe that

$$\mathbb{R}^n/\Gamma \cong \mathbb{T}^k \times \mathbb{R}^{n-k}. \tag{75}$$

We deduce that M is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$. In conclusion, we have the following diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\ A\ } & \mathbb{R}^n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \phi_p} \\ \mathbb{T}^k \times \mathbb{R}^{n-k} & \xrightarrow[\widetilde{\phi}]{} & M, \end{array} \tag{76}$$

where π is the quotient map and $\widetilde{\phi}$ is a diffeomorphism. □

Proof of Theorem 1.28.

1. We note that by hypothesis, $\mathrm{d}f_i(X_{f_j}) = \{f_j, f_i\} = 0$ for all $i, j = 1, \ldots, n$. Therefore, for all $p \in f^{-1}(c)$ we have

$$X_{f_j}(p) \in \bigcap_{i=1}^{n} T_p(f_i^{-1}(c_i)) = T_p\left(f^{-1}(c)\right), \tag{77}$$

where we have denoted $c = (c_1, \ldots, c_n)$. Since $f^{-1}(c)$ has dimension n, the vector fields $X_{f_1}, \ldots, X_{f_n}$ are a basis of $T_p(f^{-1}(c))$. Note that

$$\omega\left(X_{f_i}, X_{f_j}\right) = \{f_i, f_j\} = 0 \tag{78}$$

for all $i, j = 1, \ldots, n$, so ω vanishes on $f^{-1}(c)$, and so $f^{-1}(c)$ is Lagrangian.

2. Let $L \subseteq f^{-1}(c)$ be a connected component and assume that the flows of $X_{f_1}, \ldots, X_{f_n}$ are complete on L. Since the f_i are in involution, so are their associated vector fields:

$$\left[X_{f_i}, X_{f_j}\right] = \{f_i, f_j\} = 0 \quad i, j = 1, \ldots, n, \tag{79}$$

so by the previous theorem we deduce that L is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$. The angle coordinates, $(\theta^1, \ldots, \theta^n)$, are induced by the diffeomorphism

$$\widetilde{\phi} : \mathbb{T}^k \times \mathbb{R}^{n-k} \to L. \tag{80}$$

3. Fix $p \in L$. In the notation of the proof of Theorem 1.30, the flow of X_H is given by the map $\Phi\left((t, 0, \ldots, 0); p\right) =: \Phi_t^{X_H}(p)$. Fix $p \in L$. Since Φ acts transitively on L, for any $x \in L$, there exists $\theta \in \mathbb{T}^k \times \mathbb{R}^n$ such that (identifying θ with an element of $\mathbb{R}^n$)

$$\widetilde{\phi}(\theta) = \Phi\left(A^{-1}(\theta); p\right) = x. \tag{81}$$

Therefore,

$$\Phi\left(A^{-1}(\theta) + (t, 0, \ldots, 0); p\right) = \Phi((t, 0, \ldots, 0); x) = \Phi_t^{X_H}(x). \tag{82}$$

In other words,

$$\Phi_t^{X_H}(x) = \widetilde{\phi}(\theta + vt), \tag{83}$$

where $v = A^{-1}(e_1)$ and $e_1 = (1, 0, \ldots, 0)$, and so the flow is indeed linear.

4. The proof of this part requires more advanced tools which we currently do not have (namely, Weinstein tubular neighbourhood theorem), so we will skip it.

□

1.5 Non-degenerate singularities in integrable Hamiltonian systems

A Hamiltonian system is completely integrable if it is defined by n first integrals in involution with respect to the Poisson bracket. Completely integrable Hamiltonian systems are closely related to Lagrangian foliations through the following result.

Proposition 1.31. *Let $f_1, \dots, f_n$ be n functions such that $\{f_i, f_j\} = 0, \forall i, j$. Suppose that $d_p f_1 \wedge \dots \wedge d_p f_n \neq 0$ at a point $p \in M$. Then, the distribution generated by the Hamiltonian vector fields $\mathcal{D} = \langle X_{f_1}, \dots, X_{f_n} \rangle$ is involutive and the leaf through p is a Lagrangian submanifold.*

The dynamics of an integrable system $F = (f_1, \dots, f_n)$ is explained by the Arnold-Liouville-Mineur Theorem at the regular points, namely, at the points of the manifold where the differential $dF = (df_1, \dots, df_n)$ is not singular. This theorem was restated by Kiesenhofer and Miranda in [KM17] revealing that at a semilocal level the regular leaves are equivalent to a completely toric cotangent lift model.

Theorem 1.32. *Let $F = (f_1, \dots, f_n)$ be an integrable system on a symplectic manifold (M, ω). Then, semilocally around a regular Liouville torus, the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{can}$ restricted to a neighbourhood of the zero section $(T^*\mathbb{T}^n)_0$ of $T^*\mathbb{T}^n$.*

At the singular points, the degeneracy of dF determines in general how difficult is to understand the dynamics, and for the case of non-degenerate singular points there are powerful results. The following definitions give the precise details of these concepts.

Definition 1.33. A point $p \in M^{2n}$ is a *singular point* of an integrable Hamiltonian system given by $F = (f_1, \dots, f_n)$ if the rank of $dF = (df_1, \dots, df_n)$ at p is less than n. The singular point p has *rank* k and *corank* of $n - k$ if $\text{rank}(dF)_p = \text{rank}\left((df_1)_p, \dots, (df_n)_p\right) = k$.

Definition 1.34. Let $\mathfrak{g}$ be a Lie algebra. A *Cartan subalgebra* $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is self-normalizing, i.e., if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, then $Y \in \mathfrak{h}$. If $\mathfrak{g}$ is finite-dimensional and semisimple over an algebraically closed field of characteristic zero, a Cartan subalgebra is a maximal abelian subalgebra (a subalgebra consisting of semisimple elements).

Definition 1.35. Let (M^{2n}, ω) be a symplectic manifold with an integrable Hamiltonian system of n independent and commuting first integrals $f_1, \dots, f_n$. Consider a singular point $p \in M$ of rank 0, i.e. $(df_i)_p = 0$ for all i. It is called a *non-degenerate singular point* if the operators $\omega^{-1} d^2 f_1, \dots, \omega^{-1} d^2 f_n$ form a Cartan subalgebra in the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{sp}(T_p M, \omega)$.

Remark 1.36. The operators $\omega^{-1} d^2 f_i$, where df_i is the Hessian of f_i, associate a function to the Hessian by visualizing the Hessian as a quadratic form $H(u, v)$ and taking the symplectic dual of the function obtained. A good reference for details of the algebraic construction of the Cartan subalgebra is [BF04].

The classification of non-degenerate critical points of the moment map in the real case was obtained by Williamson [Wil36]. In the complex case, all the Cartan subalgebras are conjugate and hence there is only one model for non-degenerate critical points of the moment map.

Theorem 1.37 (Williamson)**.** *For any Cartan subalgebra $\mathcal{C}$ of $\mathfrak{sp}(2n, \mathbb{R})$, there exists a symplectic system of coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ in $\mathbb{R}^{2n}$ and a basis $f_1, \dots, f_n$ of $\mathcal{C}$ such that each of the quadratic polynomials f_i is one of the following:*

$$
\begin{aligned}
& f_i = x_i^2 + y_i^2 && \text{for } \ 1 \leq i \leq k_e \\
& f_i = x_i y_i && \text{for } \ k_e + 1 \leq i \leq k_e + k_h \\
& \begin{cases} f_i = x_i y_{i+1} - x_{i+1} y_i \\ f_{i+1} = x_i y_i + x_{i+1} y_{i+1} \end{cases} && \text{for } \ i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f
\end{aligned}
$$

The three types are called elliptic, hyperbolic and focus-focus, respectively.

Remark 1.38. Notice that the focus-focus components always go by pairs. Because of theorem 1.37, the triple (k_e, k_h, k_f) at a singular point it is an invariant. It is also an invariant of the orbit of the integrable system through the point [Zun96].

If p is a non-degenerate singularity of the moment map F, it is characterized by four integer numbers, the rank k of the singularity and the triple (k_e, k_h, k_f). By the way they are defined, they satisfy $k + k_e + k_h + 2k_f = n$, where n is the number of degrees of freedom of the integrable system.

The following is a result of Eliasson [Eli90] and Miranda and Zung ([Mir03], [MZ04]).

Theorem 1.39 (Smooth local linearization)**.** *Given an smooth integrable Hamiltonian system with n degrees of freedom on a symplectic manifold (M^{2n}, ω), the Liouville foliation in a neighborhood of a non-degenerate singular point of rank k and Williamson type (k_e, k_h, k_f) is locally symplectomorphic to the model Liouville foliation, which is the foliation defined by the basis functions of Theorem 1.37 plus "coordinate functions" $f_i = x_i$ for $i = k_e + k_h + 2j + 1$ to n.*

Remark 1.40. The theorem states the existence of a semilocal symplectomorphism between foliations with a non degenerate singularity of rank k and the same parameters (k_e, k_h, k_f). One could think that functions are also preserved via a symplectomorphism, but it is not possible to guarantee this statement when $h_k \neq 0$ as one can add up analytically flat terms on different connected components (see counterexample in [Mir03]). In general one needs more information about the topology of the leaf to conclude.

Remark 1.41. Because of Theorem 1.39, if one considers the Taylor expansions of $F = (f_1, \ldots, f_n)$ at the non-degenerate singular point in a canonical coordinate system and removes all terms except for linear and quadratic, the functions obtained remain commuting and define a Liouville foliation that can be considered as the *linearization* of the initial foliation $\mathcal{F}$ given by $f_1, \ldots, f_n$, to which it is symplectomorphic.

The description of non-degenerate singularities at the semilocal level is completed with the following two results.

Theorem 1.42 (Model in a covering)**.** *The manifold can be represented, locally at a non-degenerate singularity of rank k and Williamson type (k_e, k_h, k_f), as the direct product*

$$M^{reg} \times \overset{k}{\cdots} \times M^{reg} \times M^{ell} \times \overset{k_e}{\cdots} \times M^{ell} \times M^{hyp} \times \overset{k_h}{\cdots} \times M^{hyp} \times M^{foc} \times \overset{k_f}{\cdots} \times M^{foc}$$

Where:

- M^{reg} *is a "regular block", given by*
$$f = x,$$
- M^{ell} *is an "elliptic block", representing the elliptic singularity given by*
$$f = x^2 + y^2,$$
- M^{hyp} *is an "hyperbolic block", representing the hyperbolic singularity given by*
$$f = xy,$$
- M^{foc} *is a "focus-focus block", representing the focus-focus singularity given by*
$$\begin{cases} f_1 = x_1 y_2 - x_2 y_1 \\ f_2 = x_1 y_1 + x_2 y_2 \end{cases}.$$

For the first three types of blocks the symplectic form is $\omega = dx \wedge dy$, while for the focus-focus block it is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

In the case of a smooth system (defined by a smooth moment map), a similar result was proved and described by Miranda and Zung in [MZ04]. It summarizes some previously results proved independently and fixes the case where there are hyperbolic components ($k_h \neq 0$), because in this case the result is slightly different and it has to be taken the semidirect product in the decomposition. As opposite to the case where there are only elliptic and focus-focus singularities, in which the base of the fibration of the neighbourhood is an open disk, if there are hyperbolic components the topology of the fiber can become complicated. The reason is essentially that for the smooth case a level set of the form $\{x_i y_i = \epsilon\}$ is not connected but consists of two components.

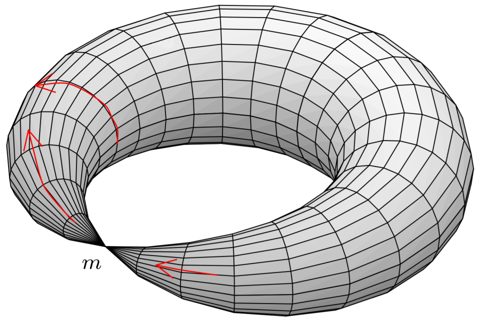

Figure 1: A simple singular leaf of focus-focus type. The topology of this fiber is a pinched torus

Theorem 1.43 (Miranda-Zung). *Let $V = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ with coordinates $(p_1, ..., p_k)$ for D^k, $(q_1(mod1), ..., q_k(mod1))$ for $\mathbb{T}^k$, and $(x_1, y_1, ..., x_{n-k}, y_{n-k})$ for $D^{2(n-k)}$ be a symplectic manifold with the standard symplectic form $\sum dp_i \wedge dq_i + \sum dx_j \wedge dy_j$. Let F be the moment map corresponding to a singularity of rank k with Williamson type (k_e, k_h, k_f). There exists a finite group Γ, a linear system on the symplectic manifold V/Γ and a smooth Lagrangian-fibration-preserving symplectomorphism ϕ from a neighborhood of O into V/Γ, which sends O to the torus $\{p_i = x_i = y_i = 0\}$. The smooth symplectomorphism ϕ can be chosen so that via ϕ, the system-preserving action of a compact group G near O becomes a linear system-preserving action of G on V/Γ. If the moment map F is real analytic and the action of G near O is analytic, then the symplectomorphism ϕ can also be chosen to be real analytic. If the system depends smoothly (resp., analytically) on a local parameter (i.e. we have a local family of systems), then ϕ can also be chosen to depend smoothly (resp., analytically) on that parameter.*

In this case, the so-called *twisted hyperbolic* component can arise (see Figure 2), and the group of all linear moment maps preserving symplectomorphisms of the linear direct model of Williamson type (k_e, k_h, k_f) is isomorphic to

$$\mathbb{T}^k \times \mathbb{T}^{k_e} \times (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})^{k_h} \times (\mathbb{R} \times \mathbb{T}^1)^{k_f}.$$

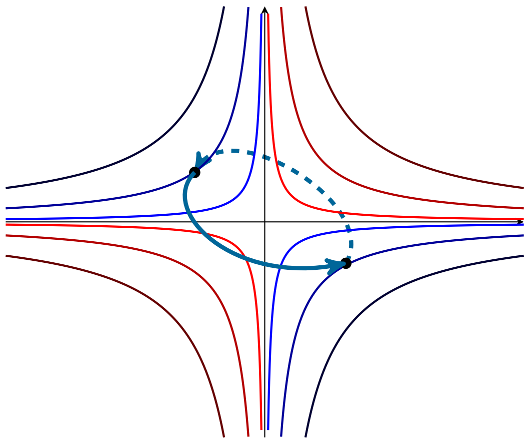

Figure 2: In the neighbourhood of an orbit of rank 1 and Williamson type $(0, 1, 0)$, the return map corresponding to the flow of circle action can give rise to two different behaviours. After one turn, the point can return to itself or it can return to its "opposite" branch (twisted hyperbolic case), and this defines a $\mathbb{Z}/2\mathbb{Z}$ action. The twisted hyperbolic case is described in this picture.

To end this section, we recall a related result which highlights the importance of considering the symplectomorphism at the level of the Lagrangian fibration induced by the Hamiltonian vector fields of the integrable system. Assume that (M, ω) is a symplectic manifold with a non-degenerate singularity of Williamson type (k_e, k_h, k_f). Assume that the foliation $\mathcal{F}$ at the singularity is the linear foliation defined by $\mathcal{F} = \langle X_1, \dots, X_n \rangle$, where the vector fields X_i are the linear Hamiltonian vector fields corresponding to the basis functions of Theorem 1.37. Namely, X_i are the vector fields induced by $\iota_{X_i}\omega = -df_i$, that is:

- $X_i = -y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}$ for elliptic components,

- $X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$ for hyperbolic components,
- $X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_{i+1}}$ and
  $X_{i+1} = x_{i+1} \frac{\partial}{\partial x_i} + y_{i+1} \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial x_{i+1}} - y_i \frac{\partial}{\partial y_{i+1}}$ for focus-focus components.

Then, the following theorem holds.

Theorem 1.44. *[Mir03] Let ω be a symplectic form defined in a neighbourhood of the singularity at p for which the foliation $\mathcal{F}$ is Lagrangian. Then, there exists a local diffeomorphism $\phi : (U, p) \longrightarrow (\phi(U), p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i dx_i \wedge dy_i) = \omega$, where x_i, y_i are local coordinates on $(\phi(U), p)$.*

For completely elliptic singularities (of rank 0 and Williamson type $(k_e, 0, 0)$) Theorem 1.44 was proved by Eliasson [Eli90]. When $h_e \neq 0$, the foliation given by the hyperbolic components is preserved but the components of the moment map are not necessarily preserved (for more details see [Mir03]).

2 Session 2. Integrable systems in physics. (Problem session)

Integrable Hamiltonian systems with non-degenerate singularities are really common in Mechanics problems, and one does not have to go to complicated models to already find the three basic types of non-degenerate singularities (in the Williamson sense, see Theorem 1.37). In the classical examples of the harmonic oscillator, the simple pendulum and the spherical pendulum, there appear the elliptic singularity, the hyperbolic singularity and the focus-focus singularity, respectively.

On the other hand, the three basic singularities can be formulated (in the elliptic case only formally) as the cotangent lift of a Lie group action, which shows how cotangent models are a useful tool when dealing with integrable systems.

In this session, we give the formulation of the elliptic, hyperbolic and focus-focus singularities as cotangent lifts. Then, we give the mathematical description of the physical models illustrating the three types of non-degenerate singularities.

2.1 The cotangent lift

The cotangent bundle of a smooth manifold can be naturally equipped with a symplectic structure in the following way. Let M be a differential manifold and consider its cotangent bundle T^*M. There is an intrinsic canonical linear form λ on T^*M defined pointwise by

$$\langle \lambda_p, v \rangle = \langle p, d\pi_p v \rangle, \qquad p = (m, \xi) \in T^*M, v \in T_p(T^*M),$$

where $d\pi_p : T_p(T^*M) \longrightarrow T_m M$ is the differential of the canonical projection at p. In local coordinates (q_i, p_i), the form is written as $\lambda = \sum_i p_i \, dq_i$ and is called the *Liouville 1-form*. Its differential $\omega = d\lambda = \sum_i dp_i \wedge dq_i$ is a symplectic form on T^*M.

Definition 2.1. Let $\rho : G \times M \longrightarrow M$ be a group action of a Lie group G on a smooth manifold M. For each $g \in G$, there is an induced diffeomorphism $\rho_g : M \longrightarrow M$. The *cotangent lift of* ρ_g, denoted by $\hat{\rho}_g$, is the diffeomorphism on T^*M given by

$$\hat{\rho}_g(q, p) := (\rho_g(q), ((d\rho_g)_q^*)^{-1}(p)), \qquad (q, p) \in T^*M$$

which makes the following diagram commute:

$$\begin{array}{ccc} T^*M & \xrightarrow{\hat{\rho}_g} & T^*M \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\rho_g} & M \end{array}$$

Given a difeomorphism $\rho : M \longrightarrow M$, its cotangent lift preserves the canonical form λ. Then, the canonical 1-form is preserved by $\hat{\rho}$.

As a consequence:

$$\hat{\rho}^*(\omega) = \hat{\rho}^*(d\lambda) = d(\hat{\rho}^*\lambda) = d\lambda = \omega.$$

Meaning that the cotangent lift $\hat{\rho}_g$ preserves the Liouville form and the symplectic form of T^*M.

Example 2.2. Let $\rho : (\mathbb{R}^3, +) \times \mathbb{R}^3 \to \mathbb{R}^3$ be the Lie group action corresponding to a space translation defined by $\rho_x(q) = q + x$. Write (q, p) for an element of the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$.

By definition, $\hat{\rho}_x$, the cotangent lift of ρ_x is

$$\hat{\rho}_x(q,p) = (\rho_x(q), ((d\rho_x)_q^*)^{-1}(p)) = (q + x, ((Id^*)^{-1}(p)) = (q + x, p). \tag{84}$$

Example 2.3. Let $\rho : SO(3, \mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Lie group action defined by $\rho_A(q) = Aq$. Write (q, p) for an element of $T_q^*\mathbb{R}^3$. By definition, $\hat{\rho}_A$, the cotangent lift of ρ_A is

$$\hat{\rho}_A(q,p) = (\rho_A(q), ((d\rho_A)_q^*)^{-1}(p)) = (Aq, ((A^*)^{-1}(p)) = (Aq, Ap),$$

where the last equality holds because A is orthogonal. Like any cotangent lift, since the induced action in the cotangent bundle is Hamiltonian, it has an associated momentum map which, in this case, corresponds to the classical quantity $q \wedge p$.

2.2 The hyperbolic singularity as a cotangent lift

Exercise 2.4. Compute the infinitesimal generator of the cotangent lift of the action given by:

$$\begin{array}{rccc} \rho : & \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (t, x) & \longmapsto & e^{-t}x \end{array},$$

and see that it coincides with the vector field associated to the normal form of the hyperbolic singularity.

Take coordinates (x, y) on $T^*\mathbb{R}$ such that the symplectic form is $\omega = dx \wedge dy$ and the moment map is $f = xy$.

If we compute the Hamiltonian vector field associated to f, we obtain

$$X = -\frac{\partial f}{\partial y}\left(\frac{\partial}{\partial x}\right) + \frac{\partial f}{\partial x}\left(\frac{\partial}{\partial y}\right) = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = (-x, y). \tag{85}$$

Consider the action of $(\mathbb{R}, +)$ on $\mathbb{R}$ given by:

$$\begin{array}{rccc} \rho : & \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (t, x) & \longmapsto & e^{-t}x \end{array},$$

and the induced an action $\rho_t : \mathbb{R} \longrightarrow \mathbb{R}$. The differential of ρ_t at a point $x \in \mathbb{R}$ is:

$$\begin{array}{rccc} (d\rho_t)_x : & T_x\mathbb{R} & \longrightarrow & T_x\mathbb{R} \\ & y & \longmapsto & e^{-t}y \end{array},$$

Then, $((d\rho_t)_x^*)^{-1}$ acts as $y \longmapsto e^t y$, and the cotangent lift $\hat{\rho}_t$ associated to the group action ρ_t, in coordinates (x, y) of $T^*\mathbb{R}$ is exactly:

$$\begin{array}{rccc} \hat{\rho} : & T^*\mathbb{R} & \longrightarrow & T^*\mathbb{R} \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{-t}x \\ e^t y \end{pmatrix} \end{array}.$$

Deriving the last vector with respect to t and evaluating at $t = 0$, we obtain exactly $X = (-x, y)$, the vector field associated to the hyperbolic singularity.

2.3 The elliptic singularity as a cotangent lift

Exercise 2.5. Compute the infinitesimal generator of the cotangent lift of the action given by:

$$\begin{array}{rccc} \rho : & \mathbb{R} \times \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (t, z) & \longmapsto & e^{it}z \end{array},$$

and see that it coincides with the vector field associated to the normal form of the elliptic singularity.

The cotangent lift in the elliptic case is computed in $\mathbb{C}$ but is not holomorphic. It is a formal development and by no means holomorphicity is assumed.

Take complex coordinates $(z,\bar{z}) = (x+iy, x-iy)$ such that the symplectic form is $\omega = \frac{i}{2} dz \wedge d\bar{z}$. The moment map corresponding to the elliptic singularity is $f = \frac{1}{2}\left(x^2+y^2\right) = \frac{1}{2} z\bar{z}$.

The Hamilton's equations in this complex setting are:

$$\begin{aligned}
\iota_X \omega = -df \iff \\
\iff \iota_{\left(\alpha \frac{\partial}{\partial z} + \beta \frac{\partial}{\partial \bar{z}}\right)} \left(\frac{i}{2} dz \wedge d\bar{z}\right) = -\frac{\partial f}{\partial z} dz - \frac{\partial f}{\partial \bar{z}} d\bar{z} \iff \\
\iff \frac{i\alpha}{2} d\bar{z} - \frac{i\beta}{2} dz = -\frac{1}{2}\bar{z} dz - \frac{1}{2} z d\bar{z} \iff \\
\iff \begin{cases} \alpha = iz \\ \beta = -i\bar{z} \end{cases}
\end{aligned}$$

Then, the Hamiltonian vector field associated to f is

$$X = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial \bar{z}} = (iz, -i\bar{z}). \tag{86}$$

Now, consider the following action of $\mathbb{R}$ on $\mathbb{C}$, which corresponds to a rotation of z of angle t:

$$\begin{array}{cccc} \rho: & \mathbb{R}\times\mathbb{C} & \longrightarrow & \mathbb{C} \\ & (t,z) & \longmapsto & e^{it}z \end{array},$$

And consider the induced an action $\rho_t : \mathbb{C} \longrightarrow \mathbb{C}$. The differential of ρ_t at a point $z \in \mathbb{C}$ is (notice that we take we could take any general coordinate w for the tangent space $T\mathbb{C}$ but we take $\bar{z}$ on purpose):

$$\begin{array}{cccc} (d\rho_t)_z: & T_z\mathbb{C} & \longrightarrow & T_z\mathbb{C} \\ & \bar{z} & \longmapsto & e^{it}\bar{z} \end{array}.$$

Then, $((d\rho_t)_{\bar{z}}^*)^{-1}$ acts as $\bar{z} \longmapsto e^{-it}\bar{z}$, and the cotangent lift $\hat{\rho}_t$ associated to the group action ρ_t, in coordinates $(z,\bar{z})$ of $T^*\mathbb{C}$ is:

$$\begin{array}{cccc} \hat{\rho}: & T^*\mathbb{C} & \longrightarrow & T^*\mathbb{C} \\ & \begin{pmatrix} z \\ \bar{z} \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{it}z \\ e^{-it}\bar{z} \end{pmatrix} \end{array}.$$

Deriving the last vector with respect to t and evaluating at $t=0$ we obtain $X = (iz, -i\bar{z})$, the vector field associated to the elliptic singularity.

2.4 The focus-focus singularity as a cotangent lift

Exercise 2.6. Compute the infinitesimal generator of the cotangent lift of the action given by:

$$\begin{array}{cccc} \rho: & (S^1\times\mathbb{R})\times\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & ((\theta,t), \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) & \longmapsto & \rho_{\theta,t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{array},$$

and see that it coincides with the vector field associated to the normal form of the focus-focus singularity.

Exercise 2.7. Compute the cotangent lift of ρ_A

To describe the basic singularity of focus-focus type in a manifold of dimension 4 we take coordinates (x_1,x_2,y_1,y_2). The symplectic form is $\omega = dx_1\wedge dy_1 + dx_2 \wedge dy_2$ and the moment map associated to this singularity is $F = (f_1,f_2) = (x_1y_2 - x_2y_1, x_1y_1 + x_2y_2)$.

If we compute the Hamiltonian vector field associated to f_1 and f_2, we obtain

$$X_1 = -\frac{\partial f_1}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_1}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_1}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_1}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{87}$$

$$= x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} = (x_2, -x_1, y_2, -y_1), \tag{88}$$

and

$$X_2 = -\frac{\partial f_2}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_2}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_2}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{89}$$

$$= - x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} = (-x_1, -x_2, y_1, y_2). \tag{90}$$

Now consider the action of a rotation and a radial dilation on $\mathbb{R}^2$ given by:

$$\begin{array}{rccccl} \rho: & (S^1 \times \mathbb{R}) \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 & & \\ & ((\theta, t), \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) & \longmapsto & \rho_{\theta,t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & = e^{-t}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & . \end{array}$$

The differential of the induced action $\rho_{\theta,t}$ at a point $x = (x_1, x_2)$ is the following linear map:

$$\begin{array}{rccc} d\rho_{\theta,t}: & T_x\mathbb{R}^2 & \longrightarrow & T_x\mathbb{R}^2 \\ & \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} & \longmapsto & e^{-t}\begin{pmatrix} y_1\cos\theta + y_2\sin\theta \\ -y_1\sin\theta + y_2\cos\theta \end{pmatrix} \end{array}.$$

Then, $((d\rho_{\theta,t})^*)^{-1}$ acts as:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto e^{t}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

And the cotangent lift $\hat{\rho}_{\theta,t}$ associated to the group action $\rho_{\theta,t}$ is:

$$\begin{array}{rccc} \hat{\rho}_{\theta,t}: & T^*\mathbb{R}^2 & \longrightarrow & T^*\mathbb{R}^2 \\ & \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{-t}(x_1\cos\theta + x_2\sin\theta) \\ e^{-t}(-x_1\sin\theta + x_2\cos\theta) \\ e^{t}(y_1\cos\theta + y_2\sin\theta) \\ e^{t}(-y_1\sin\theta + y_2\cos\theta) \end{pmatrix} \end{array}.$$

Finally, deriving the last vector with respect to θ and evaluating at 0 and deriving the vector with respect to t and evaluating at 0 we obtain, respectively, $X_1 = (x_2, -x_1, y_2, -y_1)$ and $X_2 = (-x_1, -x_2, y_1, y_2)$, the vector fields associated with f_1 and f_2, the components of the moment map of the focus-focus singularity.

2.5 The elliptic singularity in the harmonic oscillator

Exercise 2.8. Prove that the singularity of the harmonic oscillator is of elliptic type.

Consider an ideal one-dimensional oscillating system consisting of a mass m connected to a rigid foundation by way of a spring of stiffness constant k, as in Figure 3, with no friction of any kind and, hence, with no loss of mechanical energy. The Hamiltonian of the system is the sum of the kinetic and the elastic potential energies. In terms of the natural coordinates of the phase space of the system $(\mathbb{R}^2, \omega = dx \wedge dv)$, which are the position x and the velocity v of the mass, it writes as:

$$\hat{H}(x, v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \tag{91}$$

Applying the following symplectic transformation:

$$\begin{cases} x = q \cdot \frac{1}{\sqrt[4]{k/m}} \\ v = p \cdot \sqrt[4]{k/m} \end{cases}, \tag{92}$$

the symplectic manifold is now $(\mathbb{R}^2, \omega = dq \wedge dp)$ and the Hamiltonian becomes:

$$H(p, q) = \frac{1}{2}\sqrt{mk}\left(p^2 + q^2\right). \tag{93}$$

Dropping the physical constants m and k, this Hamiltonian is exactly the normal form of the moment map of a one-dimensional system with an elliptic singularity at the origin, the unique equilibrium point of the system.

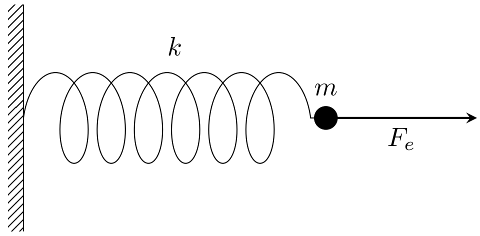

Figure 3: The harmonic oscillator.

2.6 The hyperbolic singularity in the simple pendulum

Exercise 2.9. Prove that the singularity at the top of the simple pendulum is of hyperbolic type. Hint: expand the Hamiltonian locally at the singularity.

The simple pendulum is another of the basic models in classical mechanics. The most natural approximation to its formulation is the Newtonian setting, where we consider the forces and acting in the system formed by a mass m attached to an end of a rigid massless rod of length l which has the other end fixed, as in Figure 4. It is assumed that the mass moves in the vertical plane formed by the vertical direction and the initial position and, since the rod has fixed length, the natural coordinate is the angle $\theta \in [0, 2\pi)$ with respect to the lower vertical equilibrium position.

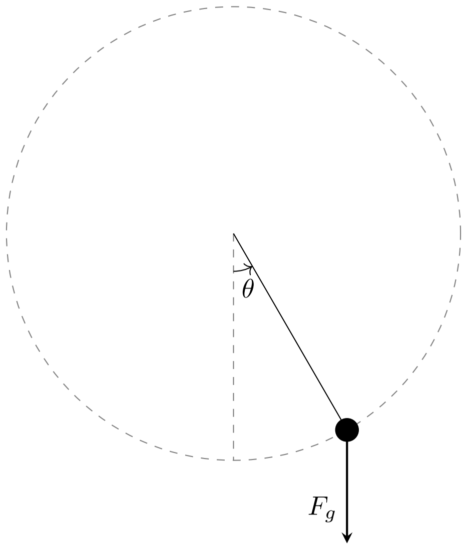

Figure 4: The simple pendulum.

Newton's law states that the acceleration of the mass in the direction of motion, which is always perpendicular to the direction of the rod, is proportional to the total force in this direction of motion. Since the only force in this direction is the component of the gravity force, Newton's law reduces to:

$$ma_\perp = F_\perp. \tag{94}$$

Taking into account that the acceleration is related to the angular coordinate through $a_\perp(\theta) = l\frac{\partial^2\theta}{\partial t^2}$ and that the force is also function of the angle through $F_\perp(\theta) = -mg\sin\theta$, where g is the gravity acceleration, the equation rewrites as the following 2nd order ODE:

$$\frac{\partial^2\theta}{\partial t^2} = -\frac{g}{l}\sin\theta. \tag{95}$$

If we define $\rho := \frac{\partial\theta}{\partial t}$ and consider the symplectic structure $(S^1 \times \mathbb{R}, \omega = d\theta \wedge d\rho)$ of the phase space, Equation (95) is equivalent to the Hamiltonian first order system of ODE's:

$$\begin{cases} \frac{\partial\theta}{\partial t} = \rho \\ \frac{\partial\rho}{\partial t} = -\frac{g}{l}\sin\theta \end{cases}, \tag{96}$$

whose Hamiltonian is

$$\hat{H}(\theta, \rho) = \frac{\rho^2}{2} - \frac{g}{l}\cos\theta. \tag{97}$$

The first equilibrium point of (96) is found at $\theta = \rho = 0$ and it is an stable point. Dropping out the physical constants, the Hamiltonian there has the normal form $\bar{H} = \frac{1}{2}(\rho^2 + \theta^2)$, which corresponds to an elliptic singularity like in the harmonic oscillator. We are more interested in the second equilibrium point, found at $\theta = \pi, \rho = 0$.

The Hamiltonian there can be locally expanded as:

$$H(\theta, \rho) = \frac{1}{2}\left(\rho^2 - \frac{g}{l}\theta^2\right). \tag{98}$$

Dropping the physical constants g and l, this Hamiltonian corresponds to the normal form of a one-dimensional system with a hyperbolic singularity at the origin.

2.7 The focus-focus singularity in the spherical pendulum

Exercise 2.10. Prove that the singularity at the top of the spherical pendulum is of focus-focus type. Hint: use local coordinates $(x, y, z) = (x, y, \sqrt{l^2 - x^2 - y^2})$.

The most basic physical example of a singularity of focus-focus type comes from the spherical pendulum. Consider a point of mass m attached to an end of a rigid massless rod of length l and assume that the other end of the rod is fixed at the origin and that the mass can move freely as long as it remains attached to the rod, as in Figure 5. The mass can move, then, on a sphere of radius l.

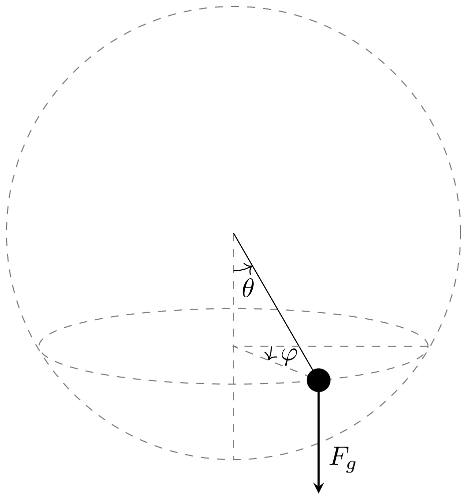

Figure 5: The spherical pendulum.

The natural phase space is the cotangent bundle T^*S^2 and, while spherical coordinates are the optimal setting to study the dynamics of the spherical pendulum, Cartesian coordinates are more appropriated to analyze the singularities of the system. In Cartesian, the position of the point of mass will be given by $\vec{r} = (x, y, z)$, with $\|\vec{r}\| = l$. The conjugate variable to $\vec{r}$ is the linear momentum of the point, $\vec{p} = (p_x, p_y, p_z) = m\dot{\vec{r}}$, which has to satisfy $\vec{r} \cdot \vec{p} = 0$ in order to be contained in the tangent space of the sphere.

The Hamiltonian of the system is the sum of kinetic and potential energies and in the symplectic setting $(\mathbb{R}^6, \omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z)$ writes as:

$$H(\vec{r}, \vec{p}) = \frac{\|\vec{p}\|^2}{2m} + mgl\frac{\vec{r} \cdot \hat{z}}{\|\vec{r}\|}, \tag{99}$$

where g accounts for the gravity acceleration and $\hat{z}$ is the unit vector in the z direction. There is another conserved quantity, the angular momentum in the z direction: $L := L_z = xp_y - yp_x$. H and L satisfy $\{H, L\} = 0$ and are independent almost everywhere. Hence, they form the Liouville integrable system corresponding the spherical pendulum.

There are two singularities in the pendulum system, one corresponding to $z = -l$ (or to $\vec{r}_- = (0,0,-l)$) and the other one to $z = l$ (or to $\vec{r}_+ = (0,0,l)$). We are interested in $\vec{r}_+$, the unstable equilibrium, where we are going to identify the focus-focus singularity.

To study the system near $z = l$, we use that $z = \sqrt{l^2 - x^2 - y^2}$ and take local coordinates $(x, y, z) = (x, y, \sqrt{l^2 - x^2 - y^2})$. The conjugate momentum $\vec{p} = (p_x, p_y, p_z)$ satisfies locally that $p_z = 0$. In these symplectic coordinates the symplectic form is $\omega = dx \wedge dp_x + dy \wedge dp_y$ and the Hamiltonian of the system writes as:

$$H = \frac{1}{2ml^2}\left(p_x^2(l^2 - x^2) + p_y^2(l^2 - y^2) - 2xyp_xp_y\right) + mg(\sqrt{l^2 - x^2 - y^2} - l). \tag{100}$$

At this point, it is convenient to apply a symplectic scaling in order to adimensionalize the Hamiltonian. We apply the following symplectic transformation:

$$\begin{cases} x = \frac{\xi}{\sqrt{m\nu}} \\ p_x = p_\xi \sqrt{m\nu} \\ y = \frac{\eta}{\sqrt{m\nu}} \\ p_y = p_\eta \sqrt{m\nu} \end{cases}, \tag{101}$$

where $\nu = \sqrt{g/l}$. In these local symplectic coordinates near the unstable equilibrium of the spherical pendulum, the symplecit form is rewritten as $\omega = d\xi \wedge dp_\xi + d\eta \wedge dp_\eta$ and the Hamiltonian becomes:

$$H = \nu\left(\frac{1}{2}(p_\xi^2 + p_\eta^2) - \frac{\kappa}{2}(\xi p_\xi + \eta p_\eta)^2 + \frac{1}{\kappa}(\sqrt{1 - \kappa\rho^2} - 1)\right), \tag{102}$$

where $\rho^2 = \xi^2 + \eta^2$, $\nu^2 = g/l$ and $1/\kappa = ml^2\nu = mgl/\nu$ and they are all constants.

Finally, a last symplectic transformation reveals that the Williamson normal form at the unstable equilibrium of the spherical pendulum corresponds to the focus-focus singularity. It is the following:

$$\sqrt{2}\xi = q_1 - p_1, \quad \sqrt{2}p_\xi = q_1 + p_1, \qquad \sqrt{2}\eta = q_2 - p_2, \quad \sqrt{2}p_\eta = q_2 + p_2\,. \tag{103}$$

In these coordinates, where $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, the Hamiltonian is:

$$H = \nu\left(p_1q_1 + p_2q_2 - \kappa\frac{1}{8}(q^2 - p^2)^2 + \frac{1}{\kappa}\sqrt{1 - \kappa\rho^2} + \frac{\rho^2}{2} - \frac{1}{\kappa}\right), \tag{104}$$

where $q^2 = q_1^2 + q_2^2$, $p^2 = p_1^2 + p_2^2$ and $\rho^2 = p^2/2 + q^2/2 - (p_1q_1 + p_2q_2)$.

Observe that the quadratic part of the potential has been absorbed in the terms $H' = \nu(p_1q_2 + p_2q_2)$ and that the remaining terms of the potential are of order 4 and higher. The quadratic part of H is simply H' and the angular momentum in the p, q variables is $L = q_1p_2 - q_2p_1$. So, the system $F = (H', L)$ has a singularity of focus-focus type.

3 Session 3. Basics on Geometric Quantization. Polarizations. Integrable systems as polarizations. Bohr–Sommerfeld leaves. Some computations

3.1 Introduction to sheaf cohomology

Quantization is a mathematical procedure which seeks to associate a quantum system to a classical Hamiltonian system by replacing functions by operators and Poisson brackets of functions by brackets of operators. Several paths have been traced for this passionate journey from geometry and analysis into Physics: geometric quantization, formal quantization, BRST quantization and semi-classical quantization, to cite a few. All of them supply Taylor-made master formulas to the day-dreamer mathematicians who are looking into the quantum world.

In [MM21] we present a new geometric quantization model which corrects former models and brings us closer to the role of quantization as a mathematical tamer of quantum physics.

One of the virtues of our model is that it takes the cotangent bundle as a general set-up for our systems. The connection between a Hamiltonian system and the cotangent bundle is given by the cotangent lift and provides a unified approach to former attempts in the literature. On the other hand, one of the downfalls of our model is that, unlike other quantization models like Kähler quantization, it depends on choices (in our case, on the choice of a real polarization given by an integrable system) as it usually happens in the standard geometric quantization. One of the advantages of our method is that geometric quantization of integrable systems can be computed even if global classification of integrable systems with non-degenerate singularities is unknown in general (not even semi-local classification), as its recipe is based on gluing local models. So the *"from local to global"* principle prevails here.

Geometric quantization and integrable systems are common mathematical objects on the interface of Geometry and Physics. Integrable systems represent a class of Hamiltonian systems which can be associated to an extra set of functions called first integrals, and are ubiquitous in Physics. Many known systems, such as any two dimensional system, or more complicated systems, such as the coupled harmonic oscillators or the spherical pendulum, are integrable. Other classical systems defined by attracting or repelling particles, such as Toda systems, are integrable. The geometric quantization procedure meets integrable systems when these are used as data attached to the geometric quantization process, in particular as providers of (real) polarizations.

In [MM21], we contemplate the quantization problem considering precisely the real polarization associated to an integrable system. The geometric quantization procedure starts with a prequantum complex line bundle $\mathbb{L}$, which is naturally associated to a symplectic manifold (M^{2n}, ω) of integral class, and an attached connection ∇ with curvature ω. A *flat section* s of the line bundle is a solution to the equation $\nabla_X s = 0$, where the derivation takes place along the direction X of a polarization, which is considered here to be real and given by the integrable system. Flat sections form a sheaf, from which one constructs a cohomology that eventually gives the quantization.

Because of the maximum principle, an integrable system defined via smooth functions on a compact phase space must have singularities. Then, polarizations given by these systems are singular too and one has to analyze the contribution of singularities to the geometric quantization of such singular integrable systems.

In former works by Mark Hamilton [Ham10], Hamilton-Miranda [HM10] and Miranda-Presas-Solha [MPS20], the authors analyze the contributions of non-degenerate singularities of integrable systems to quantization. They find no contribution from elliptic points and infinite dimensional contributions for hyperbolic and focus-focus type singularities. Those infinite dimensional models clash with the initial expectations of obtaining a finite dimensional representation space as the quantization space (and thus, representation space) of a system defined on a compact manifold.

In [MM21], we work out "cotangent models" for integrable systems with non-degenerate singularities which can be of elliptic, hyperbolic and focus-focus type. This is a first step towards understanding the cotangent models of the pairs given by the polarization associated to such integrable systems and the prequantum line bundle. Those singularities naturally appear in polarizations on compact manifolds given by integrable systems of Morse-Bott type. In particular, any semitoric system (such as the ones studied in [PVuN09, PVuN11]) gives rise to singularities of this type). These structures also show up naturally in algebraic geometry, for instance in the study of the K3 surface[1], which can be viewed as a semitoric system. When it comes to considering their quantization, several models have been proposed. However, none of them can compete with the model of Kähler quantization in terms of independence of the polarization and in terms of the principle *quantization commutes with reduction, or simply* $[Q, R] = 0$ (see [GS83], [GS82a], [GS82b]). Notwithstanding, Kähler quantization cannot always be applied since the conditions to have a polarization of Kähler type are not always fulfilled.

For regular integrable systems (without singularities) action-angle coordinates (the classical Arnold-Liouville-Mineur theorem) provide cotangent models as a neighbourhood of the Liouville torus can be symplectically interpreted as its cotangent bundle, $T^*(\mathbb{T}^n)$. This canonical identification gives a way to relate the choice of the Liouville 1-form of the cotangent bundle with the connection 1-form of the prequantum line bundle. In other words, the Liouville 1-form of the cotangent bundle yields a canonical choice of the connection 1-form. This connects the cotangent model to quantization in the regular case (see for instance [MP15] and [Ś77]). With the ambition of extending these ideas to the singular set-up, we analyze the cotangent lift technique for different types of non-degenerate singularities (in the sense of Eliasson-Williamson) and provide brand-new cotangent models for the pair given by the polarization and the connection one-form.

Additionally, the existence of a local model of cotangent type allows to capture symmetry and is compatible with the $[Q, R] = 0$ principle. The cotangent models used to define the new proposal for geometric quantization for non-degenerate singularities also allows to obtain a unique universal cotan-

[1] A K3 surface is an example of a hyperkähler manifold with three compatible complex structures i, j, k. The denomination K3 comes from Kummer, Kähler and Kodaira and, according to André Weil, it is a reminiscence of the beautiful mountain K2 in Kashmir.

gent model. In contrast to the former models of geometric quantization for real polarizations endowed with non-degenerate singularities in [Ham10] and [MPS20], our new models provide finite dimensional representations for systems on compact manifolds which match the physical models.

One interesting advantage of our models is that they fit well with the sheaf-theoretical geometric quantization kit provided in [MP15]. In particular, the Künneth formula and Mayer-Vietoris recipe which were established there can be used to patch the cotangent models to provide global quantization on a compact manifold even if the global symplectic classification of non-degenerate integrable systems is still unknown in some cases. In other words, our cotangent models can be seen as building pieces of the geometric quantization puzzle as partitions of unity in Differential geometry allow us to invoke a local-to-global principle.

To relate the Bohr-Sommerfeld leaves of a quantizable surface with the integer points in the interior of the image of the moment map we need a result from Guillemin and Sternberg [GS83], which is valid for any $2n$-dimensional symplectic manifold.

Theorem 3.1 (Guillemin-Sternberg, Theorem 2.4 in page 4 of [GS83])**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold and $\pi : M \to B$ a moment map whose fibers are compact connected Lagrangian submanifolds of M. Suppose that p and q are two distinct points of B. Let B_0 be an open simply connected subset of B containing p and q. There exists a globally defined system of action coordinates $f_1, \dots, f_n$ on B_0 and we can assume that $f_1(p) = \dots = f_n(p) = 0$. Then, if p is in the Bohr-Sommerfeld set, q is in the Bohr-Sommerfeld set if and only if $f_1(q), \dots, f_n(q)$ are integers.*

Proof. Let $b(t)$, with $0 \le t \le 1$, be a smooth simple curve in B joining p to q. For each $i \in \{1, \dots, n\}$ let $\Gamma_i = \cup_{0 \le t \le 1} \gamma_i(b(t))$, where γ_i is such that

$$f_i(p) = \int_{\gamma_i(p)} \alpha$$

and $d\alpha = \omega$, with α and ω the Liouville 1-form and the symplectic forms, respectively. Γ_i is a smooth two-dimensional submanifold of M with boundary $\partial \Gamma_i = \gamma_i(p) \cup \gamma_i(q)$. The parallel transport around $\gamma_i(p)$ of a nonzero section of the complex line bundle $\mathbb{L}$ gives a monodromy constant $M_i(p)$, and the parallel transport around $\gamma_i(q)$ gives a similar monodromy constant $M_i(q)$.

Now, we apply the following theorem by Kostant to conclude that

$$M_i(q) = M_i(p) e^{2\pi i \int_{\gamma_i} \omega}.$$

Theorem 3.2 (Kostant, Theorem 1.8.1 in page 108 of [Kos70].)**.** *Let $(\mathbb{L}, \alpha)$ be a line bundle with connection over a manifold M. Let γ be a closed piece-wise smooth curve in M which is homotopic to a point and let σ be any surface of deformation of γ. Let $\omega = curv(\mathbb{L}, \alpha)$. Then, the scalar multiplication $Q(\gamma)$, induuced by parallel transport around γ, is given by the surface integral*

$$Q(\gamma) = e^{-2\pi i \int_\sigma \omega}.$$

On the other hand, by Stoke's Theorem we know that

$$\int_{\gamma_i} \omega = f_i(q) - f_i(p),$$

which implies that the monodromy constants $M_i(p)$ and $M_i(q)$ are related by

$$M_i(q) e^{-2\pi i f_i(q)} = M_i(p) e^{-2\pi i f_i(p)}.$$

We assumed that $f_i(p) = 0$ and, since p is a Bohr-Sommerfeld point, $M_i(p) = 1$. Therefore, $M_i(q)$ is a Bohr-Sommerfeld leaf (and $M_i(q)$ is 1 for all i) if and only if all the $f_1(q), \dots, f_n(q)$ are integers. □

3.2 Prequantization and sheaf cohomology

Definition 3.3. Let (M, ω) be a symplectic manifold. It is *quantizable* if there exists a hermitian complex line bundle $\mathbb{L}$ over M with a compatible connection ∇ whose curvature is ω. $\mathbb{L}$ is called the *prequantization line bundle.*

Remark 3.4. (M, ω) is quantizable if ω is exact, which occurs for cotangent bundles with the canonical symplectic structure and for compact manifolds if and ony if $[\omega] \in H^2(M, \mathbb{Z})$[Kos70] or, equivalently, if for every compact oriented two-dimensional submanifold N of M

$$\int_N \omega \in \mathbb{Z}.$$

The reason of this integrality condition, often called Weil integrality condition, is that the de Rham cohomology class $[\alpha]$ of the curvature form α of a connection on a complex line bundle $\mathbb{L} \to M$ is in $H^2(M, \mathbb{Z})$, that is, for every compact oriented two-dimensional submanifold N of M

$$\int_N \alpha \in \mathbb{Z}.$$

Moreover, for every form α on M with $[\alpha] \in H^2(M, \mathbb{Z})$, there exists a complex line bundle $\mathbb{L} \to M$ with connection ∇ such that α is the curvature of ∇.

Remark 3.5. Given a symplectic form $\omega \in H^2(M, \mathbb{R})$ with integer class $[\omega]$, the lift to $H^2(M, \mathbb{Z})$ is not unique in general, we have an exact sequence $Tor(H^2(M, \mathbb{Z})) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$, for manifolds with non-vanishing $Tor(H^2(M, \mathbb{Z}))$ this line bundle is not unique.

One additional piece of structure is required, a *polarization*, to restrict which sections of $\mathbb{L}$ are considered, because the space of all sections is normally "too big".

Definition 3.6. A *polarization* $\mathcal{P}$ on a symplectic manifold (M, ω) is an integrable Lagrangian distribution in $TM \otimes \mathbb{C}$. There are two polarizations in which we are mainly interested:

- A *real polarization*, in which $\bar{\mathcal{P}} = \mathcal{P}$.
- A *Kähler polarization* (or *holomorphic polarization*), in which $\bar{\mathcal{P}} \cap \mathcal{P} = \{0\}$ and the hermitian form

  $$i\omega(\cdot, \cdot) : \bar{\mathcal{P}} \times \mathcal{P} \longrightarrow C^\infty_{\mathbb{C}}(M)$$

  is positive definite.

Remark 3.7. A *real polarization* on a symplectic manifold (M, ω) is basically a foliation of M into Lagrangian submanifolds.

Definition 3.8. A section σ of $\mathbb{L}$ is *flat along the leaves* or *leaf-wise flat* if it is covariant constant along the fibres of F, with respect to the prequantization connection ∇. Namely, if $\nabla_X \sigma = 0$ for any vector field X tangent to fibres of F. The set of sections which are flat along the leaves is a sheaf and it is denoted by $\mathcal{J}$.

We now recall the construction of the cohomology of sheaves or *sheaf cohomology*, which is used to define the geometric quantization. We start defining presehaves and sheaves.

Definition 3.9. Let X be a topological space. A *presheaf* $\mathcal{F}$ on X assigns to every open set U of X an abelian group $\mathcal{F}(U)$, usually called the set of *sections* of $\mathcal{F}$ over U. It also assigns, to any $V \subset U$, a *restriction map* $\mathcal{F}(U) \to \mathcal{F}(V)$, such that if $W \subset V \subset U$ and $\sigma \in \mathcal{F}(U)$, then

$$\sigma|_W = (\sigma|_V)|_W,$$

and if $V = U$ then the restriction is just the identity map.

Definition 3.10. A presheaf $\mathcal{J}$ is a *sheaf* if the following properties hold:

1. For any pair of open sets U, V, and sections $\sigma \in \mathcal{J}(U)$ and $\tau \in \mathcal{J}(V)$ which agree on the intersection $U \cap V$, there exists a section $\rho \in \mathcal{J}(U \cup V)$ which restricts to σ on U and τ on V.
2. If σ and τ in $\mathcal{J}(U \cup V)$ have equal restrictions to U and V, then they are equal on $U \cup V$.

Now, we construct the cochains and the coboundary operator of the cohomology. Fix an open cover $\mathcal{A} = \{A_\alpha\}$ of the manifold M. A *Čech k-cochain* assigns, to each $(k+1)$-fold intersection of elements from the cover $\mathcal{A}$, a section of the sheaf $\mathcal{J}$, and we will denote this kind of intersection $A_{\alpha_0} \cap \cdots \cap A_{\alpha_k}$, where the α_j are distinct, simply by $A_{\alpha_0 \cdots \alpha_k}$. Then, a k-cochain is an assignment $f_{\alpha_0 \cdots \alpha_k} \in \mathcal{J}(A_{\alpha_0 \cdots \alpha_k})$ for each $(k+1)$-fold intersection in the cover $\mathcal{A}$. The set of k-cochains is denoted by $C^k_{\mathcal{A}}(M; \mathcal{J})$, or just by $C^k_{\mathcal{A}}$.

The coboundary operator δ that makes $C^*_{\mathcal{A}}$ into a cochain complex is defined in the following way:

$$(\delta f)_{\alpha_0 \cdots \alpha_k} = \sum_{j=0}^{k} (-1)^j f_{\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_k}|_{A_{\alpha_0 \cdots \cdots \alpha_k}}, \tag{105}$$

where the $\hat{}$ denotes that the index is omitted. Then, it is clear that if $f = \{f_{\alpha_0 \cdots \alpha_{k-1}}\}$ is a $(k-1)$-cochain, δf is a k-cochain. With this definition, for instance, $(\delta f)_{123} = f_{23} - f_{13} + f_{12}$ and $(\delta \circ \delta f)_{123} = \delta(f_{23} - f_{13} + f_{12}) = f_3 - f_2 - f_3 + f_1 + f_2 - f_1 = 0$. In general, $\delta \circ \delta = 0$ and $C^*_{\mathcal{A}}$ is a well-defined cochain complex.

Definition 3.11. With the above definitions, the sheaf cohomology with respect to the cover $\mathcal{A}$ is the cohomology of this complex:

$$H^k_{\mathcal{A}}(M; \mathcal{J}) = \frac{\ker \delta^k}{\operatorname{im} \delta^{k-1}},$$

where by δ^k denotes the map δ on $C^k_{\mathcal{A}}$.

The sheaf cohomology that is useful to work with is defined independently of the cover, the way of doing it is to take a limit over cover refinements. A cover $\mathcal{B}$ is a *refinement* of a cover $\mathcal{A}$ if every element of $\mathcal{B}$ is a subset of some element of $\mathcal{A}$. A refinement provides a map $\rho\colon \mathcal{B} \to \mathcal{A}$, where $B \subset \rho(B)$ for all $B \in \mathcal{B}$, and gives a map $\phi\colon C^k_{\mathcal{A}}(U, \mathcal{J}) \to C^k_{\mathcal{B}}(U, \mathcal{J})$ induced by the restriction maps in the sheaf. Then, if $\eta \in C^k_{\mathcal{A}}$ is a cochain, $\phi\eta$ is defined by

$$(\phi\eta)_{B_0 B_1 \cdots B_k} = (\eta)_{(\rho B_0)(\rho B_1) \cdots (\rho B_k)}|_{B_0 B_1 \cdots B_k}.$$

This map commutes with δ and induces a map on cohomology $H^*_{\mathcal{A}} \to H^*_{\mathcal{B}}$. All the possible choices of maps ρ turn the collection of $H^*_{\mathcal{A}}$ for all open covers of M into a directed system and the sheaf cohomology can be defined as the limit of this system, which can be proved to exist.

Definition 3.12. The *sheaf cohomology* of M is defined as the limit of the directed system:

$$H^*(M; \mathcal{J}) = \varinjlim H^*_{\mathcal{A}}(M; \mathcal{J}).$$

There is a last result on sheaf cohomology that we will use, which is presented as Theorem 3.4 in [MP15]. It gives a classical Künneth formula also holds for the sheaf cohomology in a generalized form. Let $(M_1, \mathcal{J}_1)$ and $(M_2, \mathcal{J}_2)$ be a pair of symplectic manifolds endowed with Lagrangian foliations. The induced sheaf of flat sections associated to the product is denoted $\mathcal{J}_{12}$ and we call it *product sheaf.* Then, there is a natural morphism of cohomology groups,

$$\Psi : H^*(M_1, \mathcal{J}_1) \otimes H^*(M_2, \mathcal{J}_2) \to H^*(M_1 \times M_2, \mathcal{J}_{12}) \tag{106}$$

induced by pull-back of the classes through the natural projections, and there is an isomorphism of cohomology groups

$$H^n(M_1 \times M_2, \mathcal{J}_{12}) \cong \bigoplus_{p+q=n} H^p(M_1, \mathcal{J}_1) \otimes H^q(M_2, \mathcal{J}_2),$$

if the geometric quantization associated to $(M_1, \mathcal{J}_1)$ has finite dimension, M_1 is compact and M_2 admits a good covering or is compact.

3.3 Geometric quantization of regular systems

Definition 3.13. Let (M, ω, F) and $\mathbb{L}$ be as above and let $\mathcal{J}$ be the sheaf of flat sections. The *quantization* of M is

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M; \mathcal{J}).$$

Definition 3.14. A leaf ℓ of the (singular) foliation is a *Bohr-Sommerfeld leaf* if there is a leaf-wise flat section σ defined over all of ℓ.

Although leaf-wise flat sections always exist locally (because by construction the curvature of ∇ is ω, which is zero when restricted to a leaf), we are requiring global existence, which is a strong condition. The set of Bohr-Sommerfeld leaves is discrete in the leaf space and a leaf is Bohr-Sommerfeld if and only if its holonomy is trivial around all the loops contained in the leaf.

The main result about quantization using real polarizations is a theorem of Śniatycki [Ś77].

Theorem 3.15 (Śniatycki, [Ś77])**.** *Let (M^{2n}, ω) be a symplectic manifold with a prequantization line bundle $\mathbb{L}$. Take a real polarization P such that the projection map $\pi\colon M \to B$ is a fibration with compact fibres. Then, $H^q(M; \mathcal{J}) = 0$ for all $q \neq n$. Furthermore, H^n can be expressed in terms of the Bohr-Sommerfeld leaves and the dimension of H^n is exactly the number of Bohr-Sommerfeld leaves.*

This result implies that, for a toric manifold foliated by fibres of the moment map, the Bohr-Sommerfeld leaves correspond to the integer lattice points in the interior of moment polytope, excluding the ones on the boundary.

In [Ham10], Mark Hamilton calculated the sheaf cohomology of a toric manifold, a manifold equipped with a Lagrangian fibration with elliptic singularities. He obtained the explicit expression for the group that is non-zero in Theorem 3.15, which can be computed by counting over all non-singular Bohr-Sommerfeld fibres.

Theorem 3.16 (Hamilton, [Ham10])**.** *Let (M, ω) be a compact symplectic $2n$-manifold and suppose it is equipped with a locally toric singular Lagrangian fibration, with prequantization line bundle $\mathbb{L}$ and connection ∇. Let $\mathcal{J}$ be the sheaf of leaf-wise flat sections of $\mathbb{L}$. Then, the cohomology groups $H^k(M; \mathcal{J})$ are zero for all $k \neq n$, and*

$$H^n(M; \mathcal{J}) \cong \bigoplus_{b \in BS} \mathbb{C}$$

where the sum is taken over all non-singular Bohr-Sommerfeld fibres.

Example 3.17. Consider the case of the torus with irrational slope. Let $(\mathbb{T}^2, \omega)$ be the 2-torus with a symplectic structure ω of integer class and consider the foliation $\mathcal{P}_\eta$ given by $X_\eta = \eta\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, with $\eta \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem 3.18 (Presas-Miranda)**.** *With the previous notation, the following holds:*

- *$Q(T^2, \mathcal{J})$ is always infinite dimensional.*
- *For the limit case of foliated cohomology $\omega = 0$ $Q(\mathbb{T}^2, \mathcal{J}) = \mathbb{C} \bigoplus \mathbb{C}$ if the irrationality measure of η is finite and $Q(\mathbb{T}^2, \mathcal{J})$ is infinite dimensional if the irrationality measure of η is infinite.*

The "Quantization Computation kit" for regular foliations is the following:

1. Künneth formula: Let $(M_1, \mathcal{P}_1)$ and $(M_2, \mathcal{P}_2)$ be two symplectic manifolds endowed with Lagrangian foliations and let $\mathcal{J}_{12}$ be the induced sheaf of basic sections, then under some mild conditions: $H^n(M_1 \times M_2, \mathcal{J}_{12}) = \bigoplus_{p+q=n} H^p(M_1, \mathcal{J}_1) \otimes H^q(M_2, \mathcal{J}_2)$.

2. Mayer-Vietoris: Consider $M \leftarrow U \sqcup V \leftleftarrows U \cap V$, then the following sequence is exact,

   $0 \to \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(M) \xrightarrow{r} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U) \oplus \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(V) \xrightarrow{r_0 - r_1} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U \cap V) \to 0.$

4 Session 4. Sheaf cohomology computations. (Problem session)

4.1 The sheaf cohomology of the cylinder

Exercise 4.1. Compute the sheaf cohomology of the cylinder explicitly and, from it, obtain its geometric quantization.

To illustrate the sheaf cohomology in a simple manifold, let us consider the cylinder $\mathbb{R} \times S^1$ and compute its sheaf cohomology. Set $M = \mathbb{R} \times S^1$, with polar coordinates (t, θ) and symplectic form $\omega = dt \wedge d\theta$.

We consider the real polarization of the symplectic manifold M given by vectors tangent to the S^1 directions, i.e., the polarization P is spanned by $\frac{\partial}{\partial \theta}$ at each point of M.

Define the complex line bundle as $\mathbb{L} = M \times \mathbb{C}$. A section σ of $\mathbb{L}$ can be seen as a complex-valued function on M. It is clear that $\omega = dt \wedge d\theta = d(t\, d\theta)$, so the connection defined by

$$\nabla_X \sigma = X(\sigma) - \sigma it\, d\theta(X),$$

with potential 1-form $t\, d\theta$, will have curvature ω and makes $\mathbb{L}$ into a prequantization line bundle.

The leaf-wise flat sections have to satisfy σ of $\mathbb{L}$ satisfy

$$\nabla_X \sigma = X(\sigma) - \sigma\, it\, d\theta(X) = 0$$

for any $X \in P$ or, equivalently,

$$\frac{\partial \sigma}{\partial \theta} - \sigma it = 0.$$

This partial differential is solved by

$$\sigma(t, \theta) = a(t) e^{it\theta},$$

for any smooth function $a(t)$. These are the leaf-wise flat sections from which we can compute the sheaf cohomology $\mathcal{J}$ of $\mathbb{L}$.

We are looking for the set of Bohr-Sommerfeld leaves, which are the leaves of the polarization P that possess a non-trivial global leaf-wise flat section. The leaves of P are of the form $\{t_0\} \times S^1$. Then, we want to see for which values of t_0 and θ the section $\sigma(t_0, \theta) = a(t_0) e^{it_0\theta}$, for t_0 fixed, is a well defined global section of the polarization.

Since t_0 is fixed along a leaf, $a(t_0) = a$ and $\sigma(t_0, \theta) = a e^{it_0\theta}$. The section $\sigma(t_0, \theta)$ has to be well defined in all the section, implying that $\sigma(t_0, \theta) = \sigma(t_0, \theta + 2\pi)$. Therefore, t_0 has to satisfy $1 = e^{2\pi i t_0}$ and we get the condition $t_0 \in \mathbb{Z}$. Then, the Bohr-Sommerfeld leaves are the leaves of P of the form $\{k\} \times S^1$ with $k \in \mathbb{Z}$.

The space of global covariant constant sections over one leaf is one-dimensional, i.e., there is freedom in the choice of the value of $a \in \mathbb{C}$ in $\sigma = a e^{it_0\theta}$.

To compute the sheaf cohomology of M we can determine the cohomology groups H^k applying directly Śniatycki Theorem. In this case, we obtain that, for any open interval $I \subset \mathbb{R}$ and $U = I \times S^1 \subset M$,

$$H^1(U, \mathcal{J}) \cong \bigoplus_{m \in \mathbb{Z} \cap I} \mathbb{C}, \qquad H^k(U, \mathcal{J}) = 0, \; k \neq 1,$$

where $\mathcal{J}$ is the sheaf of flat sections σ.

We start by calculating the cohomology on a finite subset of the cylinder $M = \mathbb{R} \times S^1$. Let U be a *band* of the cylinder, i.e., a subset of M of the form $I \times S^1$, with $I \subset \mathbb{R}$ open and bounded. Assume that U contains at most one Bohr-Sommerfeld leaf and construct a cover $\mathcal{A}$ of U by three rectangles A, B, C that slightly overlap as in Figure 6.

We want to identify the Čech 0-cochains and 1-cochains with respect to the cover $\mathcal{A}$. A 0-cochain α is an assignment of a flat section over A, B and C to that same subsets. Then, α assigns A to the section $a_A(t)e^{it\theta}$, B to $a_B(t)e^{it\theta}$ and C to $a_C(t)e^{it\theta}$.

The angular coordinate θ can not be defined on all of S^1 so a branch of θ has to be fixed on each rectangle. We choose the branches so that $\theta_A = \theta_B$ on $A \cap B$, $\theta_B = \theta_C$ on $B \cap C$, and $\theta_C = \theta_A + 2\pi$ on $A \cap C$. The coboundary operator δ acts on α as

$$(\delta\alpha)_{ij} = \eta_j - \eta_i = a_j(t)\, e^{it\theta_j} - a_i(t)\, e^{it\theta_i},$$

for $i, j \in \{A, B, C\}$. We impose that, at the three intersections, $\delta\alpha$ is 0, obtaining the following three equations:

$$0 = a_B(t)\, e^{it\theta_B} - a_A(t)\, e^{it\theta_A} \qquad \text{on } A \cap B \tag{107}$$

$$0 = a_C(t)\, e^{it\theta_C} - a_B(t)\, e^{it\theta_B} \qquad \text{on } B \cap C \tag{108}$$

$$0 = a_A(t)\, e^{it\theta_A} - a_C(t)\, e^{it\theta_C} \qquad \text{on } C \cap A \tag{109}$$

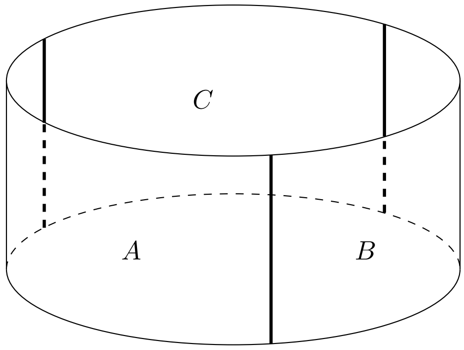

Figure 6: A band U of the cylinder $M = \mathbb{R} \times S^1$ covered by three rectangles A, B, C. The rectangles overlap slightly in the thick joints.

Then, α is a cocycle if and only if the following three equations simultaneously:

$$a_B(t) = a_A(t) \qquad \text{on } A \cap B \tag{110}$$
$$a_C(t) = a_B(t) \qquad \text{on } B \cap C \tag{111}$$
$$a_A(t) = a_C(t)\, e^{2\pi i t} \qquad \text{on } C \cap A \tag{112}$$

which is not possible since $e^{2\pi i t}$ can not equal 1 in an entire interval of values of t. We conclude that are no 0-cocycles and that $H^0 = 0$.

Now, a 1-cochain β is an assignment of a flat section over $A \cap B$, $B \cap C$ and $C \cap A$ to that same subsets. Then, β assigns $A \cap B$ to the section $b_{AB}(t)e^{it\theta}$, $B \cap C$ to $b_{BC}(t)e^{it\theta}$ and $C \cap A$ to $b_{CA}(t)e^{it\theta}$. There only possible triple intersection in the cover $\mathcal{A}$ is empty and 2-cochains do not exist, implying that every 1-cochain is a cocycle. Since the 1-cochain is determined essentially by the three smooth functions $b_{ij}(t)$ on I, the space of 1-cocycles is isomorphic to $C^\infty(I)^3$.

A 1-cochain β is a coboundary if there exists a 0-cochain $\alpha = \{a_A(t)e^{it\theta_A}, a_B(t)e^{it\theta_B}, a_C(t)e^{it\theta_C}\}$ with $\delta\alpha = \beta$. Or, equivalently, if these three equations are satisfied:

$$\beta_{AB} = \alpha_B - \alpha_A \quad \text{on } A \cap B \tag{113}$$
$$\beta_{BC} = \alpha_C - \alpha_B \quad \text{on } B \cap C \tag{114}$$
$$\beta_{CA} = \alpha_A - \alpha_C \quad \text{on } C \cap A \tag{115}$$

Giving the sections explicitly, these equations transform to:

$$b_{AB}(t)e^{it\theta_A} = a_B(t)e^{it\theta_B} - a_A(t)e^{it\theta_A} \qquad \text{on } A \cap B \tag{116}$$
$$b_{BC}(t)e^{it\theta_B} = a_C(t)e^{it\theta_C} - a_B(t)e^{it\theta_B} \qquad \text{on } B \cap C \tag{117}$$
$$b_{CA}(t)e^{it\theta_C} = a_A(t)e^{it\theta_A} - a_C(t)e^{it\theta_C} \qquad \text{on } C \cap A \tag{118}$$

Notice that, on each ordered intersection of two sets $E \cap F$, we use the θ coordinate from E. In each equation all the θ coordinates coincide except in Equation 118, where they differ by a factor of 2π. Then, we obtain a system of equations in the three unknown functions a_A, a_B, and a_C which has to be true for each value of t in I and which has, as a matrix of coefficients, the following:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ e^{-2\pi i t} & 0 & -1 \end{pmatrix} \tag{119}$$

We observe that the matrix has rank 3 (and therefore the system has a unique solution) when $e^{-2\pi i t} \neq 1$. Then if $e^{-2\pi i t}$ is never 1 on U, every cocycle is a coboundary, and U has the zero cohomology. Otherwise, if $e^{-2\pi i t} = 1$ somewhere in I, which happens if and only if I contains an integer m, the system only has a solution if the matrix

$$\begin{pmatrix} -1 & 1 & 0 & b_{AB}(m) \\ 0 & -1 & 1 & b_{BC}(m) \\ e^{-2\pi i t} & 0 & -1 & b_{CA}(m) \end{pmatrix} \tag{120}$$

has rank 2, i.e., if β satisfies the condition

$$b_{AB}(m) + b_{BC}(m) + b_{CA}(m) = 0. \tag{121}$$

Then, we have the following result:

Proposition 4.2. *the cohomology group H^1 of U is precisely*

$$H^1 = C^\infty(I)^3/\{b_{AB}(m) + b_{BC}(m) + b_{CA}(m) = 0\}, \tag{122}$$

which is isomorphic to $\mathbb{C}$.

Observe that for $k > 1$ there are no $(k+1)$-fold intersections in the cover $\mathcal{A}$. Therefore, all the cohomology groups $H^k_{\mathcal{A}}$ are zero for $k > 1$.

The condition $e^{2\pi i t} = 1$ is satisfied exactly at the Bohr-Sommerfeld leaves, so we conclude that if U is a band on the cylinder, the sheaf cohomology of U with respect to the cover $\mathcal{A}$ by the three rectangles is trivial if U does not contain a Bohr-Sommerfeld leaf, and it is:

$$H^k_{\mathcal{A}}(U; \mathcal{J}) \cong \begin{cases} \mathbb{C} & k = 1 \\ 0 & k \neq 1 \end{cases}$$

One can see that if another cover $\mathcal{B}$ of U is made by k rectangles instead of 3, the cohomology calculated with respect to $\mathcal{B}$ is the same as that calculated with respect to $\mathcal{A}$ [Ham10].

4.2 The sheaf cohomology of the b-sphere

Exercise 4.3. Compute the sheaf cohomology with sign of the b-sphere and, from it, obtain its geometric quantization with sign.

Consider the b-sphere (S^2, Z), with Z the equator, and take coordinates (h, θ). The b-symplectic form on the b-sphere is $\omega = \frac{1}{h} dh \wedge d\theta$ or $\omega = d \log \|h\| \wedge d\theta$.

To start with the prequantization procedure, take the real polarization P given by vectors tangent to the directions parallel to Z, i.e., the polarization spanned by $\frac{\partial}{\partial\theta}$ at each point of S^2.

Define the complex line bundle as $\mathbb{L} = S^2 \times \mathbb{C}$. A section σ of $\mathbb{L}$ can be seen as a complex-valued function on (S^2, Z). The connection defined by

$$\nabla_X \sigma = X(\sigma) - \sigma i \log \|h\| \, d\theta(X),$$

with potential 1-form $\log \|h\| \, d\theta$, has curvature ω and makes $\mathbb{L}$ into a prequantization line bundle.

The leaf-wise flat sections are computed as in the cyinder case and are of the form

$$\sigma = a e^{i \log \|h_0\| \theta},$$

with $a \in \mathbb{C}$.

We are looking for the set of Bohr-Sommerfeld leaves, which are the leaves of the polarization P that possess a non-trivial global leaf-wise flat section. The leaves of P are the circles of the form $\{h_0\} \times S^1$, with $-1 < h_0 < 1$, and the two poles. Then, we want to see for which values of h_0 the section $\sigma(h_0, \theta) = a(h_0) e^{i \log \|h_0\| \theta}$ is a well defined global section of the polarization.

We do not consider possible Bohr-Sommerfeld leaves at the critical set Z (or $h_0 = 0$) because they should be viewed as being at infinity and, hence, they do not contribute to the quantization. In other words, since there is an S^1-action and the modular weight generates the circle S^1 itself, any element of the quantization coming from Z is at an infinite value of the moment map and therefore does not contribute to any finite dimensional representation of the circle.

Since h_0 is fixed along a leaf, $a(h_0) = a$ is constant and $\sigma(h_0, \theta) = a e^{i \log \|h_0\| \theta}$. The section $\sigma(h_0, \theta)$ has to be well defined in all the leaf, implying that $\sigma(h_0, \theta) = \sigma(h_0, \theta + 2\pi)$. Therefore, h_0 has to satisfy $1 = e^{2\pi i \log \|h_0\|}$ and we get the condition $\log \|h_0\| \in \mathbb{Z}$. Since $0 < \|h_0\| < 1$, $\log \|h_0\| < 0$ and so $log\|h_0\| = -m$, with $m \in \mathbb{N}$. Then, the Bohr-Sommerfeld leaves are the leaves of P of the form $\{h_0 = \pm e^{-m}\} \times S^1$, with $m \in \mathbb{N}$.

The space of global covariant constant sections over one leaf is one-dimensional, i.e., there is freedom in the choice of the value of $a \in \mathbb{C}$ in $\sigma = a e^{i \log \|h_0\| \theta}$.

Now, if we use the classic definition of geometric quantization and we apply directly Śniatycki's Theorem to compute the standard sheaf cohomology of (S^2, Z), we obtain that, for any open interval $I \subset (-1, 1)$ and $U = I \times S^1 \subset (S^2, Z)$, the cohomology groups are:

$$H^1(U, \mathcal{J}) \cong \bigoplus_{\#\{\pm e^{-\mathbb{N}} \cap I\}} \mathbb{C}, \qquad H^k(U, \mathcal{J}) = 0,\ k \neq 1,$$

where $\mathcal{J}$ is the sheaf of flat sections.

The count $\#\left(\pm e^{-\mathbb{N}} \cap I\right)$ is infinite if $0 \in I$, meaning that the quantization of any open susbet $U = I \times S^1$ of the sphere is an infinite-dimensional group if it contains Z. This is the reason why we have to define a *geometric quantization with sign* in order to obtain a finite quantization for the sphere.

We can compute the *geometric quantization with sign* of the b-sphere explicitly. We will start by calculating the sheaf cohomology on a particular subset of the (S^2, Z) and then we will extend it to any subset of (S^2, Z) and to the whole b-sphere.

Let U be a *band* of the sphere, i.e., a subset of (S^2, Z) of the form $I \times S^1$, with $I \subset (-1, 1)$ open and bounded. Assume that U contains at most one Bohr-Sommerfeld leaf. Notice that this condition can only be met if $0 \notin I$, since any open interval containing 0 has infinite points of the form e^{-m} and $-e^{-m}$ with $m \in \mathbb{N}$. Then, at this moment we are considering only bands U of the sphere that contained entirely on one of the hemispheres.

Construct a cover $\mathcal{A}$ of U by rectangles as in the cylinder case to obtain the sheaf cohomology. This cohomology is trivial if U does not contain a Bohr-Sommerfeld leaf and, if it does, it is:

$$H^k_{\mathcal{A}}(U; \mathcal{J}) \cong \begin{cases} \mathbb{C} & k = 1 \\ 0 & k \neq 1 \end{cases}$$

One can prove that, for any other cover, the sheaf cohomology is the same as the sheaf calculated with respect to $\mathcal{A}$ [Ham08].

Now, we want to consider bands U of the b-sphere (S^2, Z) that do contain Z, i.e., that are of the form $I \times S^1$, with $I \subset (-1, 1)$ open and bounded and containing 0. We can write I as (a, b) with $a \in (-1, 0)$ and $b \in (0, 1)$.

With the classical geometric quantizaiton, we obtain an infinite-dimensional group, but will define a *geometric quantization with sign* which takes into account the sign of the hemisphere of each Bohr-Sommerfeld leaf in such a way that there happens a "cancellation of infinites" and the obtained quantization is finite.

To illustrate this cancellation, start taking a horizontal band U of the b-sphere (S^2, Z) which is symmetric with respect to the equator, i.e., $U = (-a, a) \times S^1$ with $a \in (0, 1)$. We know that in this band there are infinite Bohr-Sommerfeld leaves, because there are infinite values of the form e^{-m} with $m \in \mathbb{N}$ in $(0, a)$ (in the northern hemisphere) and there are also infinite values of the form $-e^{-m}$ with $m \in \mathbb{N}$ in $(-a, 0)$ (in the southern hemisphere). We also know that, for each Bohr-Sommerfeld leaf ℓ_+ in U placed at $h_0 > 0$ (in the northern hemisphere), there is a symmetric Bohr-Sommerfeld leaf ℓ_- in U placed at $-h_0$ (in the southern hemisphere), and vice-versa.

From now on, consider the sign of the Bohr-Sommerfeld leaves, i.e., associate to each Bohr-Sommerfeld leaf a $+$ sign if they belong to the northern hemisphere and a $-$ sign if they belong to the southern hemisphere. Taking these signs into account, we can define the following *geometric quantization with sign* of a general band U cointaining Z.

Definition 4.4 (Geometric quantization with sign)**.** Let $U = I \times S^1$, with $I = (-a, b)$ and $a, b \in (0, 1)$, be a band of the b-sphere (S^2, Z) containing the critical set Z. The geometric quantization of U is given by

$$H^1(U, \mathcal{J}) \cong \bigoplus_{\#\{\sigma_I e^{-\mathbb{N}} \cap I_{ab}\}} \mathbb{C}, \qquad H^k(U, \mathcal{J}) = 0,\ k \neq 1,$$

where $I = (a, b)$ and $\sigma_I = +1$ if $a \leq b$, and $I = (-a, -b)$ if $a > b$ and $\sigma_I = -1$.

Remark 4.5. The quantization of a horizontal band $U = (-a, b) \times S^1$ of the b-sphere (S^2, Z) is finite, since the set $\{e^{-\mathbb{N}} \cap (a, b)\}$ with $0 < a \leq b < 1$ is finite. See Figure 7 for an illustration.

Remark 4.6. In this quantization with sign, the contribution to the quantization of a symmetric band $U = (-a, a) \times S^1$ of each *positive* Bohr-Sommerfeld leaf ℓ_+ is cancelled with the contribution of its symetric *negative* Bohr-Sommerfeld leaf ℓ_-, yielding a total quantization of the symmetric band U of 0.

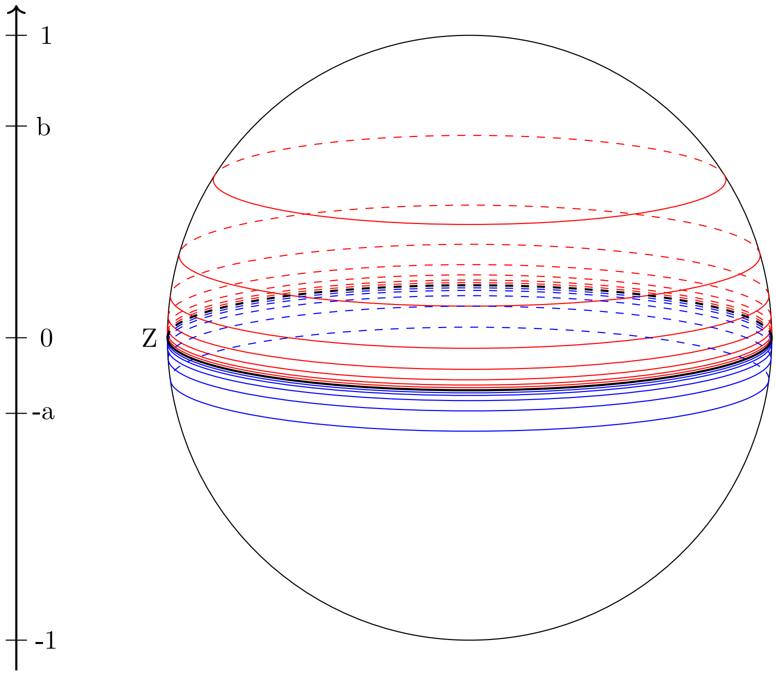

Figure 7: A band $U = (-a, b) \times S^1$ of the b-sphere which cointains 2 more Bohr-Sommerfeld leaves on the northen hemisphere (in red) than on the southern hemisphere (in blue). Its quantization is $H^1(U, \mathcal{J}) \cong \mathbb{C}^2$.

With this definition, for a general horizontal band U on the sphere containing Z, a Bohr-Sommerfeld leaf in U does not contribute to the geometric quantization whenever its symmetric leaf in the other hemisphere is also contained in U. This way, the contribution of an infinite number of Bohr-Sommerfeld leaves in the northern hemisphere cancels with the the contribution of an infinite number of Bohr-Sommerfeld leaves in the southern hemisphere, leaving only a finite contribution from the hemisphere that contains more Bohr-Sommerfeld.

Remark 4.7. The only horizontal bands of the b-sphere whose geometric quantization is infinite are only the bands of the form $(-a, 0) \times S^1$ and $(0, b) \times S^1$, since they contain infinite Bohr-Sommerfeld leaves only in one hemisphere which do not cancel.

Notice that, with this definition, the quantization of the standard b-sphere (S^2, Z) is exactly 0 due to its symmetry. It would not be necessarily so if the critical set Z was not placed at the equator, but it would still be finite.

5 Session 5. Singularities everywhere! How to handle with singularities of integrable systems. How to handle singularities of the symplectic structures

In [HM10], Hamilton and Miranda proved the following key theorem for compact two-dimensional integrable systems with non-degenerate singularities.

Theorem 5.1 (Hamilton-Miranda, [HM10]). *Let (M, ω, F) be a two-dimensional, compact, completely integrable system, whose moment map has only non-degenerate singularities. Suppose M has a prequantum line bundle $\mathbb{L}$, and let $\mathcal{J}$ be the sheaf of sections of $\mathbb{L}$ which are flat along the leaves. The cohomology $H^1(M, \mathcal{J})$ has two contributions of the form $\mathbb{C}^{\mathbb{N}}$ for each hyperbolic singularity, each one corresponding to a space of Taylor series in one complex variable. It also has one $\mathbb{C}$ term for each non-singular Bohr-Sommerfeld leaf. That is,*

$$H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{b \in BS} \mathbb{C}_b. \tag{123}$$

The cohomology in other degrees is zero. Thus, the quantization of M is given by (123).

Two more important theorems were proved by Miranda, Presas and Solha in [MPS20], concerning the geometric quantization of focus-focus fibers.

Theorem 5.2 (Miranda-Presas-Solha, [MPS20]). *The geometric quantization of a saturated neighborhood of a focus-focus fiber with n nodes is:*

- 0 *if the singular fiber is not Bohr–Sommerfeld.*
- *isomorphic to*
$$(C^\infty(\mathbb{R};\mathbb{C}))^{n_f},$$
if the singular fiber is Bohr–Sommerfeld, where $n_f = n$ (for compact fibers) and $n_f = n-1$ otherwise.

Theorem 5.3 (Miranda-Presas-Solha, [MPS20]). *For a 4-dimensional closed almost toric manifold M, with n_r regular Bohr–Sommerfeld fibers and n_f focus-focus Bohr–Sommerfeld fibers:*

$$\mathcal{Q}(M) \cong \mathbb{C}^{n_r} \oplus \left(\bigoplus_{j\in\{1,\ldots,n_f\}} (C^\infty(\mathbb{R};\mathbb{C}))^{n(j)} \right),$$

with $n(j)$ the number of nodes on the j-th focus-focus Bohr–Sommerfeld fiber.

And the expression of the flat sections on a focus-focus singular leaf is known (see [Sol15]). These results allow to compute the quantization of some particular systems such as $K3$ surfaces [MPS20].

In [MM21] Mir and Miranda give a universal model for geometric quantization associated to a real polarization given by an integrable system with non-degenerate singularities. This universal model goes one step further than the cotangent models in [KM17] by both considering singular orbits and adding to the cotangent models a model for the prequantum line bundle. These singularities are generic in the sense that are given by Morse-type functions and include elliptic, hyperbolic and focus-focus singularities. Examples of systems admitting such singularities are toric, semitoric and almost toric manifolds, as well as physical systems such as the coupling of harmonic oscillators, the spherical pendulum or the reduction of the Euler's equations of the rigid body on $T^*(SO(3))$ to a sphere. The new geometric quantization formulation coincides with the models given in [HM10] and [MPS20] away from the singularities and corrects former models for hyperbolic and focus-focus singularities cancelling out the infinite dimensional contributions obtained by former approaches. These cotangent models obey a *local-to-global principle* and can be glued to determine the geometric quantization of the global systems even if the global symplectic classification of the systems is not known in general.

5.1 New model of geometric quantization

Jacques Vey proved in [Vey78] that there was a unique complex model for the linearization of analytical systems. Indeed, he proved the following theorem in the holomorphic set-up.

Theorem 5.4 (Vey, [Vey78]). *Let (M^{2n}, ω) be an analytic complex symplectic manifold. Let A be a Liouville algebra of critical function germs at a point $p \in M$. Then, there exist holomorphic coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ in a neighbourhood of p on M such that $\omega = \sum_i dp_i \wedge dq_i$ and such that A is the analytic algebra generated by the n functions $h_i = p_i q_i$.*

From his idea of a unique model for a non-degenerate singularity in the complexes, we present a new model for geometric quantization. This model unifies the computation for the regular case and the three types of (real) non-degenerate singularities in the Williamson sense, which are equivalent in the complexes.

5.2 Sheaf surgery for non-degenerate singularities

Theorem 3.16 already unifies the geometric quantization of elliptic singularities with the regular case. We introduce some *sheaf surgery* that will allow us to redefine the quantization of the hyperbolic and the focus-focus singularity to also bring together their geometric quantization.

To set up for the hyperbolic case, recall from [HM10] the following result

Proposition 5.5 (Hamilton-Miranda, [HM10]). *Let Z be the neighbourhood of a hyperbolic singular point. If $\sigma\colon Z \to \mathbb{L}$ is a smooth leaf-wise flat section defined over Z, then σ is Taylor flat at the singular point. That is,*

$$\left.\frac{\partial^{j+k}\sigma}{\partial^j x\,\partial^k y}\right|_{(0,0)} = 0 \quad \textit{for all } j,k$$

Corollary 5.6. *The only leaf-wise flat analytic section in a semi-local neighbourhood of a hyperbolic singularity is the zero section.*

Proof. By Theorem 5.5, leaf-wise flat analytic sections in a neighbourhood of a singularity are the zero sections. Since the analytic extension of the zero section is zero, the only leaf-wise flat analytic section in a semi-local neighbourhood of a hyperbolic singularity is the zero section. □

Now, we construct a special sheaf by changing the sheaf of smooth flat sections in a neighbourhood on a hyperbolic singularity.

Lemma 5.7. *Let $p \in (M, \omega, F)$ be a non-degenerate singular point of hyperbolic type on an integrable system defined on a symplectic manifold. Suppose that it is equipped with a complex line bundle $\mathbb{L}$ and a connection ∇ whose curvature is ω. Then, the sheaf of leaf-wise smooth flat sections is still a sheaf when a in a neighbourhood of p smooth sections are required to be analytic. We denote this sheaf by $\mathcal{J}_h$.*

Proof. Analytic sections are a subclass of smooth sections. Observe that the process consists of two steps. Fisrt, thanks to the system linearization around a singularity of Eliasson [Eli90], the flat sections equation $\nabla_X \sigma = 0$ is formed by analytic data (the field $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ in dimension 2) and the solutions sections are analytic. Requiring smooth sections to be analytic in a small neighbourhood does not change the fact that the intersection works well. This is because we are extending a piece where the sections are required to be zero and by Corollary 5.6) this is compatible, since analyticity is a local condition. Therefore, conditions for being a sheaf (recall Definitions 3.9 and 3.10) are satisfied. □

To set up for the focus-focus case, we need another kind of sheaf manipulation. In this case, we define a new sheaf by replacing a section in a neighbourhood.

Lemma 5.8. *Let $p \in (M, \omega, F)$ be a non-degenerate singular point of focus-focus type on an integrable system defined on a symplectic manifold. Suppose that it is equipped with a complex line bundle $\mathbb{L}$ and a connection ∇ whose curvature is ω. Then, the sheaf cohomology of leaf-wise flat smooth sections is still a sheaf cohomology when the cohomology of a neighbourhood of p is replaced by the sheaf cohomology of a cylinder in that neighbourhood. We denote this sheaf by $\mathcal{J}_f$.*

Proof. Consider a semi-local neighbourhood U of the singularity of focus-focus type $p \in (M, \omega, F)$. Apart from the singular focus-focus leaf ℓ containing p, the rest of the leaves of U are regular tori. In the intersection of ℓ and a local neighbourhood V of p change the cohomology of the focus-focus leaf presented in [MPS20] by the sheaf cohomology of the torus.

Explicitly, take $\omega = dt \wedge d\theta$ and take as flat sections the complex functions of the form $\sigma = a(t)e^{it\theta}$. The definition of these flat sections in the singular fiber automatically sets the cohomology of all the fibers in the entire U into the torus cohomology. By continuity, this is a well-defined cohomology in U.

The cohomology glues back well because the focus-focus singularity is isolated and the topology of the complementary is glued back to a cylinder. So the pieces of the puzzle will glue back normally as in the computation of the sheaf cohomology in a neighbourhood of the torus. This is done using the Mayer-Vietoris formula in Section 4 of [MP15]. If this torus lies over an integer point of the lattice this would add a Bohr-Sommerfeld leaf to the computation. □

Remark 5.9. Observe that this argument works whenever there are several pinched nodes in the singular focus-focus fiber. In this case, the total count does not depend on the number of nodes of the multiple pinched torus. At each node of the pinched torus we redo the puzzle argument above to conclude.

In practice, we do an interchange of a single cohomology piece, a focus-focus cohomology piece by a torus cohomology piece which is so close that the substitution can be done smoothly (see Figure 8). This allows to define the geometric quantization of the focus-focus singularity as the quantization of the regular case.

We can think of the sheaf-theoretical approach as a puzzle construction where the manifold comes endowed with an adapted covering admitting local data (sections and polarization) and the invariants associated to the singularity, in this case the focus-focus singularity.

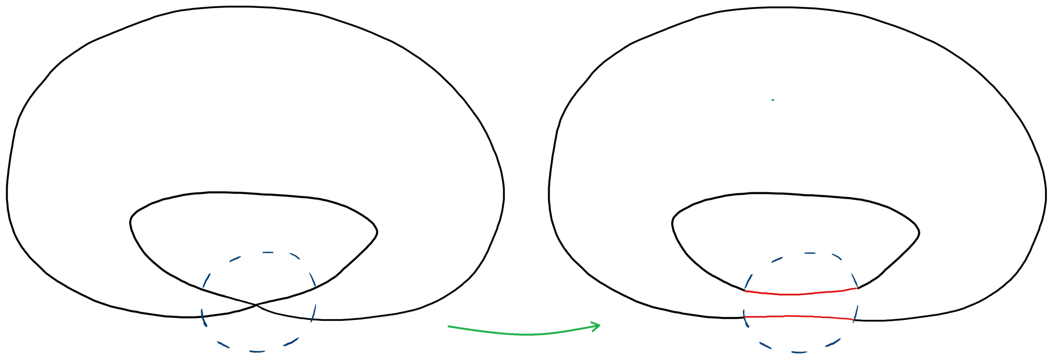

Figure 8: At practice, the model replaces the cohomology in a neighbourhood of the focus-focus singular point by the cohomology of the torus, like pieces of a puzzle.

The attachment of the new piece keeps the global cohomology of the neighbouring fibers well-defined. Understanding the sheaf cohomology computation in [MPS20] under the computation kit provided in [MP15], we can simply reconstruct the sheaf computation in the focus-focus case replacing the "puzzle piece" containing the focus-focus singularities by the torus piece. In practice, this will behave as replacing the puzzle by a red piece (regular cylinder) and gluing it back to the sheaf.

5.3 Geometric quantization for non-degenerate singularities

We propose a new model for geometric quantization which has which is convenient for non-degenerate singularities of both elliptic and hyperbolic type of certain characteristics. First, we introduce, the simple version for manifolds of dimension 2 and later we generalize it to bigger dimensions. In dimension 2, our model can be stated as follows.

Theorem 5.10 (Geometric quantization of non-degenerate singularities in dimension 2)**.** *Let (M^2,ω) be a symplectic 2-manifold and suppose it is equipped with a non-degenerate singular Lagrangian fibration, with singularities of only elliptic and hyperbolic type, with a prequantization line bundle $\mathbb{L}$ and with connection ∇. Let $\mathcal{J}$ be the sheaf of leaf-wise flat sections of $\mathbb{L}$. Then, the cohomology groups $H^k(M;\mathcal{J})$ are zero for all $k \neq 1$, and*

$$H^1(M;\mathcal{J}) \cong \bigoplus_{b\in BS} \mathbb{C},$$

where the sum is taken over all non-singular Bohr-Sommerfeld fibres.

Proof. At the neighbourhood of the singular fibers, take the corresponding sheaf $\mathcal{J}$ of leaf-wise flat sections. Namely, take the sheaf of the cylinder and the corresponding cohomology if it is a singular elliptic fiber, and take the sheaf $\mathcal{J} = \mathcal{J}_h$ of Lemma 5.7 and the corresponding cohomology if it is a singular hyperbolic fiber. In the regular components, just take the usual toric sheaf cohomology.

The quantization is computed as the sum of the cohomology groups, which are 0 for $k \neq n$ and $H^n(M;\mathcal{J}) \cong \bigoplus_{b\in BS} \mathbb{C}$ where the sum is taken over all non-singular Bohr-Sommerfeld fibres. □

For systems in higher-dimensional manifolds, we can still quantize both elliptic and hyperbolic singularities and even a merging of them. Nevertheless, we need that for each singularity of rank k, its Williamson type is $(k_e, k_h, 0)$ with $k_h \leq 1$. We also have to require compactness of the manifold to apply a Künneth Formula for the product sheaf.

Theorem 5.11 (Geometric quantization of non-degenerate systems without focus-focus singularities)**.** *Let (M^{2n},ω) be a compact symplectic $2n$-manifold and suppose it is equipped with a non-degenerate singular Lagrangian fibration, with singularities of rank k and Williamson type $(k_e, k_h, 0)$ with $k_h \leq 1$. Suppose it is also equipped with a prequantization line bundle $\mathbb{L}$ and with connection ∇. Let $\mathcal{J}$ be the product sheaf of leaf-wise flat sections of $\mathbb{L}$. Then, the cohomology groups $H^k(M;\mathcal{J})$ are zero for all $k \neq n$, and*

$$H^n(M;\mathcal{J}) \cong \bigoplus_{b\in BS} \mathbb{C},$$

where the sum is taken over all non-singular Bohr-Sommerfeld fibres.

Proof. Suppose that a singularity of rank k has Williamson type $(k_e, 0, 0)$. Then, the result of quantization for toric systems (see Theorem 3.16) applies and the quantization is given by $H^n(M;\mathcal{J}) \cong \bigoplus_{b\in BS}\mathbb{C}$.

Now, suppose that a singularity of rank k has Williamson type $(k_e, 1, 0)$. At the neighbourhood of the singular fiber, consider the decomposition in elementary blocks given by Theorem 1.42, which in this case is $\overbrace{M^{\text{reg}} \times \cdots \times M^{\text{reg}}}^{k} \times \overbrace{M^{\text{ell}} \times \cdots \times M^{\text{ell}}}^{k_e} \times M^{\text{hyp}}$. Because of this decomposition, the sheaf $\mathcal{J}$ of leaf-wise flat sections of the prequantum line bundle is of product type. In other words, it is the product of the following components:

- the elliptic components which are modelled as in Theorem 3.16;
- at the hyperbolic component where the sheaf $\mathcal{J}_h$ is given in Lemma 5.7 and the corresponding cohomology, recall that this sheaf is an extension of the zero section in a neighbourhood of the singularity;
- and in the regular components, just take the usual toric sheaf cohomology given by 3.15.

Because M is compact and the sheaf can be seen as a product, we can apply the Künneth formula for geometric quantization (see Theorem 3.4 in [MP15]).

Therefore, in both cases the total cohomology is the sum of the cohomology of each block, and coincides with the quantization of the toric case because the hyperbolic component does not add any contribution. Then, the quantization is computed as the sum of the cohomology groups, which are 0 for $k \neq n$ and

$$H^n(M;\mathcal{J}) \cong \bigoplus_{b\in BS}\mathbb{C}$$

where the sum is taken over all non-singular Bohr-Sommerfeld fibres. □

In short, the theorem states that geometric quantization of an integrable system with only non-degenerate singular points without focus-focus singularities is locally given by the number of regular Bohr-Sommerfeld leaves in the neighbourhood of any singular point.

Applying this mantra to the set-up of [MP15] (where Śniatycki's Theorem is restated in a cotangent version, see Theorem 4.2 there) and [MPS20] we obtain a simplification of the quantization formulas. Concerning the focus-focus case in dimension 4, at each focus-focus component we can apply Lemma 5.8 and Theorem 5.2 now becomes:

Theorem 5.12. *The new geometric quantization of a saturated neighborhood of a focus-focus fiber with n nodes is the same as the quantization of a regular fiber. It is* 0 *if it is not Bohr-Sommerfeld and contributes with the addition of* $\mathbb{C}$ *if it is Bohr-Sommerfeld.*

This computation can be made global using again the Mayer-Vietoris formula presented in [MP15] (see Theorem 3.1. and Corollary 3.2 there) to obtain a new quantization model for almost toric manifolds. Theorem 5.3 becomes:

Theorem 5.13. *For a* 4*-dimensional closed almost toric manifold* M*, with* n_r *regular Bohr–Sommerfeld fibers and* n_f *focus-focus Bohr–Sommerfeld fibers:*

$$\mathcal{Q}(M) \cong \mathbb{C}^{n_r+n_f}.$$

The new definition of the geometric quantization of a system with non-degenerate singularities of rank k and Williamson type (k_e, k_h, k_f) has advantages with respect to other definitions. From the physical point of view, the most important advantage is that it associates a discrete set to a model which is semi-locally compact. This is much more natural than to associate to it a continuous set. On the other hand, since the complexifications of the real, elliptic and hyperbolic singularities are equivalent, it is reasonable that the quantization of the focus-focus and hyperbolic cases essentially coincide with the quantization of the elliptic and the regular cases. The nodal trade, which can be thought as a smooth operator, allows the compatibility of this definition with the properties of the Delzant polytope.

Our definition, also, can be used to explicitly compute the quantization in particular examples, such as the $K3$ surface. In [MPS20] the authors apply Kostant's model to singularities of focus-focus type to compute the cohomology groups associated to the real geometric quantization of a neighborhood of a focus-focus fiber of a 4-dimensional semitoric integrable system. They conclude that the first cohomology group is trivial but the second is not, since it is infinite dimensional if the singular fiber is Bohr–Sommerfeld (Theorem 5.2). As an application of Theorem 5.3 in a particular example, they analyze the effect of nodal trades [LS10] in a K3 surface, which is an almost toric manifold.

In their example, a K3 surface is built from two copies of a symplectic and toric blow-up of the complex projective plane at 9 different points, applying nodal trades to all of their elliptic-elliptic singular fibers and taking their symplectic sum along the symplectic tori corresponding to the preimage of the boundary of their respective bases. This construction provides a K3 surface with 24 Bohr–Sommerfeld focus-focus fibers (see Figure 9). For this surface, they obtain that its quantization, with the previous model, is $\mathcal{Q}(K3) \cong \mathbb{C}^{26} \oplus (C^\infty(\mathbb{R};\mathbb{C}))^{24}$.

Then, the real geometric quantization of the K3 surface with the previous model from [MPS20] is essentially different from the Kähler case, which is always finite dimensional. In the particular example we are considering, the dimension of its Kähler quantization is $1/2 \cdot c_1(L)^2 + 2 = 1/2 \cdot (2 \cdot 24 + 2 \cdot 24) + 2 = 50$, which is the same dimension given by our proposed model.

However, both models (Kähler quantization and the one provided in [MPS20]) have something in common: The total number of Bohr-Sommerfeld leaves (26 regular +24 singular in [MPS20]) coincides with the dimension 50 in the Kähler quantization.

In our model we exchange each of the infinity contributions from the focus-focus Bohr-Sommerfeld leaves by a finite dimensional contribution. Therefore, the representation space is $\mathbb{C}^{50}$ and it coincides with the dimension provided by the Kähler quantization count.

Thus, a remarkable point of the K3 example is that our model for geometric quantization does coincide with the Kähler quantization, validating our new model and correcting the infinite dimensional contributions in [MPS20].

Other models for the quantization of K3 have been obtained from the perspective of Berezin-Toeplitz Quantization [CD16]. In those cases, the representation space is the same as in the Kähler case, since one uses one of the compatible Kähler structures associated to the hyperkähler structure of K3. The Berezin-Toeplitz operators for the hyperkähler K3 in [CD16] are built up from former work on the Kähler case in [Sch10].

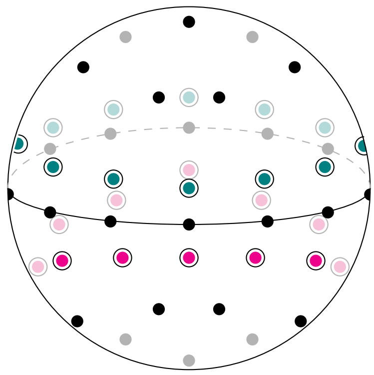

Figure 9: K3 surface as a singular fiber bundle over the sphere. The preimage of the equator of the sphere contains 12 regular Bohr-Sommerfeld fibers, obtained from the 12 elliptic singular Bohr-Sommerfeld fibers which become regular after the symplectic sum.

With this new model (see Figure 8) we replaced the sheaf cohomology computation by the computation of the Bohr-Sommerfeld leaves, killing the infinite contributions of the focus-focus singularities and obtaining a quantization that coincides with the Kähler quantization of the nodal trade.

In the new model for the focus-focus singularity, the singular focus-focus fiber does not add any contribution to the quantization when compared with respect to the elliptic model [MPS20].

5.4 Poisson geometry

Definition 5.14 (Poisson structure)**.** Let M be a manifold. A Poisson structure on M is an $\mathbb{R}$-bilinear operation

$$\{\cdot,\cdot\} : \mathcal{F}(M) \to \mathcal{F}(M) \tag{124}$$
$$(f,g) \mapsto \{f,g\}, \tag{125}$$

called the Poisson bracket, which satisfies the following properties:

- Antisymmetry: $\{f, g\} = -\{g, f\}$ for all $f, g \in \mathcal{F}(M)$.
- Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ for all $f, g, h \in \mathcal{F}(M)$.
- Leibniz rule: $\{f, gh\} = h\{f, g\} + g\{f, h\}$ for all $f, g, h \in \mathcal{F}(M)$.

Remark 5.15. We note that any manifold admits a trivial Poisson structure which is given by $\{f, g\} = 0$ for all $f, g \in \mathcal{F}(M)$. This is an important point, as it shows that there are no topological obstructions on a manifold to admit a Poisson structure, as opposed to the case of symplectic forms.

The Poisson bracket can be expressed in local coordinates as follows. Given a set of coordinates $(x_1, \ldots, x_n)$ on a Poisson manifold M, we denote

$$\omega_{ij} = \{x_i, x_j\}. \tag{126}$$

Proposition 5.16. *The following properties hold:*

1. $\omega_{ij} = -\omega_{ji}$ *for all* $i, j = 1, \ldots, n$.
2. $\{\omega_{ij}, x_k\} + \{\omega_{jk}, x_i\} + \{\omega_{ki}, x_j\} = 0$ *for all* $i, j, k = 1, \ldots, n$.
3. *For all* $i, j, k = 1, \ldots, n$,

$$\sum_{l=1}^{n} \left(\omega_{li} \frac{\partial \omega_{jk}}{\partial x_l} + \omega_{lj} \frac{\partial \omega_{ki}}{\partial x_l} + \omega_{lk} \frac{\partial \omega_{ij}}{\partial x_l} \right) = 0. \tag{127}$$

Proof. Exercise (it follows directly from the axioms applied to the coordinate functions). □

Proposition 5.17. *Given* $f, g \in \mathcal{F}(M)$*, the Poisson bracket can be computed in coordinates as*

$$\{f, g\} = (\mathrm{d}f)^\top \cdot \Omega \cdot (\mathrm{d}g), \tag{128}$$

where $(\Omega)_{ij} = \omega_{ij}$.

Proof. Exercise. □

Note that the fact that the Poisson bracket satisfies Leibniz rule implies that for each $f \in \mathcal{F}(M)$, the map

$$\{f, \cdot\} : \mathcal{F}(M) \to \mathcal{F}(M) \tag{129}$$

$$g \mapsto \{f, g\} \tag{130}$$

is a derivation on $\mathcal{F}(M)$. Hence, as in the symplectic case, we have:

Definition 5.18 (Hamiltonian vector field)**.** Let $f \in \mathcal{F}(M)$ be a function on a Poisson manifold M. The Hamiltonian vector field associated with f is the vector field X_f defined by $\{f, \cdot\}$.

This fact together with antisymmetry suggests that Poisson structures are associated with a contravariant analogue of differential forms. These are precisely the k-vectors.

Definition 5.19 (k-vectors)**.** A k-vector field is a smooth section of the k-th exterior power of the tangent bundle. That is, a map $\Pi : M \to \Lambda^k(TM)$ such that $\pi \circ \Pi = \mathrm{id}_M$, where $\pi : TM \to M$ is the canonical projection.

As in the case of differential forms, k-vector fields can be locally expressed as linear combinations of wedge products of the basis vectors. That is,

$$\Pi = \sum_{i_1 < \cdots < i_k} \Pi_{i_1, \ldots, i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}. \tag{131}$$

A 2-vector field is called a bivector field.

Proposition 5.20. *Let M be a Poisson manifold. There exists a bivector field Π such that for all $f, g \in \mathcal{F}(M)$*

$$\{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}g). \tag{132}$$

Conversely, a bivector field defines a Poisson structure on M if, and only if, it satisfies the Jacobi identity.

Proof. We define a bivector field Π in local coordinates by setting

$$\Pi_{ij} = \omega_{ij} = \{x_i, x_j\}. \tag{133}$$

The equality follows by construction and by the bilinearity of the Poisson bracket.

Now suppose that we have a bivector field Π defined on M. From the definition, we know that the map defiend by

$$\{f, g\}_\Pi = \Pi(\mathrm{d}f, \mathrm{d}g), \quad f, g \in \mathcal{F}(M) \tag{134}$$

is alternating and $\mathbb{R}$-bilinear. It is immediate to check that it also satisfies Leibniz's rule. □

As a consequence, we will often refer to the Poisson structure indifferently by the Poisson bracket or by its associated bivector field.

As always, we will be interested in maps that preserve the additional structure that we have in our space. In this case:

Definition 5.21 (Poisson morphism)**.** Let $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$ be two Poisson manifolds. A Poisson morphism is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*(\{f, g\}_2) = \{\varphi^* f, \varphi^* g\}_1$ for all $f, g \in \mathcal{F}(M_2)$.

Definition 5.22. Let (M, Π) be a Poisson manifold. A vector field $X \in \mathfrak{X}(M)$ is called a Poisson vector field if its flow is a Poisson morphism. Equivalently, if $\mathcal{L}_X \Pi = 0$.

Proposition 5.23. *A vector field $X \in \mathfrak{X}(M)$ is a Poisson vector field if, and only if,*

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}. \tag{135}$$

Proof. We have, for all $f, g \in \mathcal{F}(M)$,

$$(\mathcal{L}_X \Pi(\mathrm{d}f, \mathrm{d}g)) = (\mathcal{L}_X \Pi)(\mathrm{d}f, \mathrm{d}g) + \Pi(\mathrm{d}(X(f)), g) + \Pi(f, \mathrm{d}(X(g))), \tag{136}$$

or, in terms of the Poisson bracket,

$$X(\{f, g\}) = (\mathcal{L}_X \Pi)(\mathrm{d}f, \mathrm{d}g) + \{X(f), g\} + \{f, X(g)\}. \tag{137}$$

Hence the identity is equivalent to $\mathcal{L}_X \Pi = 0$. □

Properties analogous to those seen in the constext of symplectic geometry hold for the Poisson case:

Proposition 5.24. *Let X_f be a Hamiltonian vector field associated to $f \in \mathcal{F}(M)$. Then,*

1. *X_f is a Poisson vector field.*
2. *f is constant along the flowlines of X_f.*
3. *Let X_g be another Hamiltonian vector field. Then,*

$$[X_f, X_g] = X_{\{f,g\}}. \tag{138}$$

5.5 b-Poisson structures

Definition 5.25. Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a **b-Poisson structure** on (M, Z).

Theorem 5.26. *For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \ldots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and*

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

The following are examples of b-Poisson manifolds:

- A Radko surface.
- The product of a Radko compact surface (R, π_R) with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- A corank 1 Poisson manifold(N, π) and a Poisson vector field $X \Rightarrow (S^1 \times N, f(\theta)\frac{\partial}{\partial \theta} \wedge X + \pi)$ is a b-Poisson manifold if:
    1. f vanishes linearly.
    2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

At this point, it is necessary to mention that in [GMP14] it was proved that b-Poisson equals b-symplectic.

Proposition 5.27. *A two-form ω on a b-manifold (M, Z) is b-symplectic if and only if its dual bivector field Π is a b-Poisson structure.*

Remark 5.28. Because ω is of maximal rank in $\Lambda^2({}^bT^*M)$ and Π is of maximal rank in $\Lambda^2({}^bT^*M)$, it makes sense to say that they are dual to each other. Similarly, this it makes sense to say that a volume form (of maximal rank in $\Lambda^{2n}(T^*M)$) has a dual $2n$-vector field (of maximal rank in $\Lambda^{2n}(TM)$).

First, recall Weinstein's splitting theorem, which will be applied to the particular case of b-Poisson manifolds:

Theorem 5.29 (Weinstein)**.** *Let (M^m, Π) be a Poisson manifold of rank $2k$ at a point $p \in M$. Then there exists a neighborhood and a local coordinate system $(x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_{m-2k})$ centered at p for which the Poisson structure can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{m-2k} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{139}$$

where f_{ij} are functions which depend only on the variables $(z_1, \ldots, z_{m-2k})$ and which vanish at the origin.

Let Π be a b-Poisson structure on (M, Z), and Ω and Ξ a volume form on M and its dual $2n$-vector field, respectively. Then, $\Pi^n = f\Xi$ for some $f : M \to \mathbb{R}$ which vanishes on Z. Since the $2n$-vector field Π^n doesn't vanish identically, the generic rank of the Poisson structure is $2n$, and it is less than $2n$ on Z. This implies that the two-form ω_Π dual to Π is a smooth symplectic form on $M \setminus Z$. Because Π^n intersects the zero section of $\bigwedge^n(TM)$ transversally, 0 is a regular value of f and so $Z = f^{-1}(0)$ must be a codimension-one submanifold of M, a union of hypersurfaces. Furthermore, we can assume that in a neighborhood of a point in Z, the function f is simply the coordinate function z_1, with $z_1 = 0$ locally defining the hypersurface. When restricted to Z, the Poisson structure defines a symplectic foliation of codimension one. Observe that Π^n vanishing transversally at Z implies that the transverse Poisson manifolds at points of Z must be of dimension two, so Z is the union of symplectic leaves of corank 2 in M. This defines a codimension-one foliation of Z by symplectic leaves.

Summarizing, a b-Poisson manifold is a Poisson manifold (M^{2n}, Π) for which the $2n$-vector field Π^n vanishes linearly along a disjoint union of smooth hypersurfaces Z and such that the Poisson structure Π defines a symplectic structure ω_Π on $M \setminus Z$ and when restricted to Z gives a symplectic foliation of codimension one in Z. In particular, the rank maximality of a b-symplectic form implies that its dual bivector field must be a b-Poisson structure; this proves one of the directions of Proposition 5.27.

Consider now the particular case of a 2-dimensional b-Poisson manifold (M, Z). The Poisson bivector field Π vanishes linearly at Z, and the dual two-form will be given locally by $\omega_\Pi = \frac{1}{z_1 f(z_1, z_2)} dz_1 \wedge dz_2$, with $f(z_1, z_2)$ nonvanishing on Z, which is given locally by $z_1 = 0$. The diffeomorphism ϕ given by the change of coordinates $z = z_1$ and $t = \int f(z_1, z_2) dz_2$ satisfies $\phi^*(\omega_\Pi) = \frac{1}{z} dz \wedge dt$ (here we give the explicit diffeomorphism, but the existence of such a diffeomorphism derives simply from the fact that Π^n intersects the zero section $\bigwedge^n(TM)$ transversally and uses the regular value theorem).

Proposition 5.30. *Let (M, Z) be a b-Poisson manifold, with Poisson bivector field Π and dual two-form ω_Π. Then, on a neighborhood of a point $p \in Z$, there exist coordinates $(x_1, y_1, \ldots x_{n-1}, y_{n-1}, z, t)$ centered at p such that*

$$\omega_\Pi = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z} dz \wedge dt.$$

Remark 5.31. In other words, we can find a splitting such that

$$\omega_\Pi = \omega_L + (\Pi^T)^\sharp$$

where ω_L is the symplectic form on the symplectic leaf through the point $p \in Z$ and $(\Pi^T)^\sharp$ is the dual to a b-Poisson structure on a 2-dimensional manifold. In particular, we have obtained a linearization result for bivector fields associated to b-manifolds.

Because being symplectic is a local property, Proposition 5.30 implies that a b-Poisson manifold is b-symplectic, the other direction of Proposition 5.27. From now on, we will refer to these manifolds as b-symplectic manifolds.

5.6 b-symplectic geometry

b-Symplectic geometry is a tool that extends the symplectic structure to manifolds with boundary by considering the boundary as a hypersurface of the double of the manifold and considering vector fields which are tangent to this hypersurface along it. It is then possible to associate a vector bundle (the b-tangent bundle) to model this situation and work with forms as sections of the dual bundle. This setting (see [GMPS15], [GMP11] and [GMP14] for the complete overview) provides a singular model for integrable systems which is useful for families of physical problems for which symplectic manifolds are not enough to describe them properly.

When dealing with physical models it is usual to encounter singularities in the phase space. A natural formulation for these type of singularities at the level of the manifold is the b-geometry and, in the symplectic context, the b-symplectic geometry. We give some definitions and results on b-symplectic manifolds that will be necessary later. The proofs of these results are contained in [GMP11], [GMP14] and [GMPS15].

Definition 5.32. A pair (M, Z), where M is a manifold and Z a (not necessarily connected) hypersurface in M is called a *b-manifold*.

Definition 5.33. A map $f : (M_1, Z_1) \longrightarrow (M_2, Z_2)$ between b-manifolds is called a *b-map* if f is transverse to Z_2 and $Z_1 = f^{-1}(Z_2)$.

Definition 5.34. We call an action ρ of a Lie group G on a b-manifold (M, Z) a *b-action* if for every $g \in G$, the induced diffeomorphism ρ_g is a b-map on (M, Z).

Differential forms with singularities can be introduced formally for b-Poisson manifolds. The idea is that it is possible to extend the symplectic structure from $M \backslash Z$ to the whole manifold M. This singular form will be called a *b-symplectic* form on M.

Definition 5.35. A *b-vector field* on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.

If x is a local defining function for Z on an open set $U \subset M$ and $(x, y_1, \ldots, y_{n-1})$ is a chart on U, then the set of b-vector fields on U is a free $C^\infty(M)$-module with basis

$$(x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}).$$

There exists a vector bundle associated to this module. This vector bundle is called the *b-tangent bundle* and denote it ${}^bTM$. The *b-cotangent bundle* ${}^bT^*M$ of M is defined to be the vector bundle dual to ${}^bTM$.

For each $k > 0$, let ${}^b\Omega^k(M)$ denote the space of *b-de Rham k-forms*, i.e., sections of the vector bundle $\Lambda^k({}^bT^*M)$. The usual space of de Rham k-forms sits inside this space in a natural way; for f a defining function of Z every b-de Rham k-form can be written as

$$\omega = \alpha \wedge \frac{df}{f} + \beta, \text{ with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M). \tag{140}$$

The decomposition given by (140) enables us to extend the exterior d operator to ${}^b\Omega(M)$ by setting

$$d\omega = d\alpha \wedge \frac{df}{f} + d\beta.$$

The right hand side is well defined and agrees with the usual exterior d operator on $M \setminus Z$ and also extends smoothly over M as a section of $\Lambda^{k+1}({}^bT^*M)$. Note that $d^2 = 0$, which allows us to define the complex of b-forms, the b-de Rham complex. The cohomology associated to this complex is called *b-cohomology* and it is denoted by ${}^bH^*(M)$.

A special class of closed 2-forms of this complex are b-symplectic forms as defined in [GMP14].

Definition 5.36. Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is *b-symplectic* if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$. We call (M, Z, ω) a *b-symplectic manifold.*

Definition 5.37. The set of *b-functions* ${}^bC^\infty(M)$ consists of functions with values in $\mathbb{R} \cup \{\infty\}$ of the form

$$c \log|f| + g,$$

where $c \in \mathbb{R}$, where f is a defining function for Z and g is a smooth function. The differential operator d is defined as: $d(c\log|f| + g) := \frac{c\,df}{f} + dg \in {}^b\Omega^1(M)$, where dg is the standard de Rham derivative.

The Lie derivative of b-forms is defined via the Cartan formula:

$$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega) \in^b \Omega^k(M), \tag{141}$$

where $\omega \in^b \Omega^k(M)$ and X is a b-vector field.

Finally, the following theorem shows how the b-cohomology is related to De Rham cohomology:

Theorem 5.38 (Mazzeo-Melrose)**.** *The b-cohomology groups of M^{2n} satisfy*

$${}^bH^*(M) \cong H^*(M) \oplus H^{*-1}(Z).$$

5.7 Formal geometric quantization

Formal geometric quantization was introduced by Weitsman in [Wei01] and later reviewed by Paradan in [Par09], who also followed [ADPW91].

Let (M, ω) be a compact symplectic manifold and let $(\mathbb{L}, \nabla)$ be a line bundle with connection of curvature ω. Choose an almost complex structure J compatible with the symplectic structure. Then this almost complex structure gives $\mathbb{L}$ the structure of a complex line bundle, and by twisting the spin-$\mathbb{C}$ Dirac operator on M by $\mathbb{L}$ we obtain an elliptic operator $\bar{\partial}_{\mathbb{L}}$. Since M is compact, $\bar{\partial}_{\mathbb{L}}$ is Fredholm, and we define the geometric quantization $Q(M)$ by

$$Q(M) = \mathrm{ind}(\bar{\partial}_{\mathbb{L}})$$

as a virtual vector space.

If M is equipped with a Hamiltonian action of a torus T, the action lifts to $\mathbb{L}$, and one can choose the almost complex structure to be T-invariant. Then the quantization $Q(M)$ is a finite-dimensional virtual T-module, and it satisfies the following principle.

For $\xi \in \mathfrak{t}^*$, we denote by $M//_\xi T$ the reduced space of M at ξ. Also, for α a weight of T, and V a virtual T-module, denote by V^α the submodule of V of weight α.

Theorem 5.39 (Meinrenken, [Mei96])**.** *Let α be a weight of T. Then*

$$Q(M)^\alpha = Q(M//_\alpha T). \tag{142}$$

In other words,

$$Q(M) = \bigoplus_\alpha Q(M//_\alpha T)\alpha \tag{143}$$

as virtual T-modules.

Remark 5.40. Both Theorem 5.39 and equation (143) are strictly speaking valid only for regular values of the moment map. In the case where α is a singular value of the moment map, the singular quotient must be replaced by a slightly different construction using a shift of α. For details, we refer the interested reader to [Mei96]. A similar caution applies in the case of noncompact Hamiltonian T-spaces and of b-symplectic manifolds below.

Remark 5.41. If (M, ω) is a compact, integral symplectic manifold, one can always find a line bundle $\mathbb{L}$ with connection ∇ of curvature ω, and the quantization $Q(M)$ is independent of this choice. We therefore suppress the line bundle and connection and simply write $Q(M)$ for the quantization.

5.8 FGQ in noncompact Hamiltonian T-spaces

If we now consider the case where M is not compact, the analysis above cannot be carried out, since the operator $\bar{\partial}_{\mathbb{L}}$ is elliptic, but no longer Fredholm. Instead, in [Wei01], equation (142) is used to *define* the quantization of such Hamiltonian T-spaces, where the moment map is proper, so that the reduced spaces are compact and the right hand side of equation (142) makes sense.

Definition 5.42 (Weitsman, [Wei01])**.** Let M be a Hamiltonian T-space with integral symplectic form. Suppose the moment map for the T-action is proper. Let V be an infinite-dimensional virtual T-module with finite multipliticies. We say

$$V = Q(M)$$

if for any compact Hamiltonian T-space N with integral symplectic form, we have

$$(V \otimes Q(N))^T = Q((M \times N)//_0 T). \tag{144}$$

In other words, as in (143),

$$Q(M) = \bigoplus_{\alpha} Q(M//_{\alpha} T)\alpha,$$

where the sum is taken over all weights α of T.

Note that the fact that the moment map is proper implies that the reduced space $(M \times N)//_0 T$ is compact for any compact Hamiltonian T-space N, so that the right hand side of equation (144) is well-defined.

In other words, we have used Theorem 5.39 to give us enough functoriality to force a definition of the quantization in this case, despite the fact that the elliptic operator $\bar{\partial}_{\mathbb{L}}$ is not Fredholm.

5.9 FGQ in b-symplectic manifolds

Suppose now that M is a compact $b-$symplectic manifold, with integral $b-$symplectic form as above. Suppose that it is equipped with a Hamiltonian action of a torus T with *nonzero modular weight.* Then, in analogy with Definition 5.42, we define

Definition 5.43. Let V be a virtual T-module with finite multipliticies. We say

$$V = Q(M)$$

if for any compact Hamiltonian T-space N with integral symplectic form, we have

$$(V \otimes Q(N))^T = Q((M \times N)//_0 T). \tag{145}$$

In other words,

$$Q(M) = \bigoplus_{\alpha} Q(M//_{\alpha} T)\alpha,$$

where the sum is taken over all weights α of T. In this b-symplectic case the condition that the modular weight be nonzero guarantees that the reduced space $(M \times N)//_0 T$ is a compact and symplectic (and in the generic case, a manifold) for any compact Hamiltonian T-space N; so that as in the case of noncompact Hamiltonian T-spaces, the right hand side of equation (145) is well-defined.

Another way to say this is to note that

$$Q(M) = Q(M - Z)$$

where $Q(M-Z)$ is the quantization of the noncompact Hamiltonian T-space $M-Z$. The fact that the modular weights on M are nonzero insures that the moment map on $M-Z$ is proper.

A b-symplectic manifold is prequantizable if:

- $M \setminus Z$ is prequantizable
- The cohomology classes given under the Mazzeo-Melrose isomorphism applied to class given by the b-symplectic form $[\omega]$ are integral.

Theorem 5.44 (Guillemin-M.-Weitsman, [GMW18]). *Assume (M, ω) is prequantizable then:*

- *The formal geometric quantization $Q(M)$ exists.*
- *$Q(M)$ is finite-dimensional.*

Idea of proof: We use that

$$Q(M) = Q(M_+) \bigoplus Q(M_-)$$

and the fact that an ϵ-neighborhood of Z does not contribute to quantization.

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