Geometric Quantization via integrable systems

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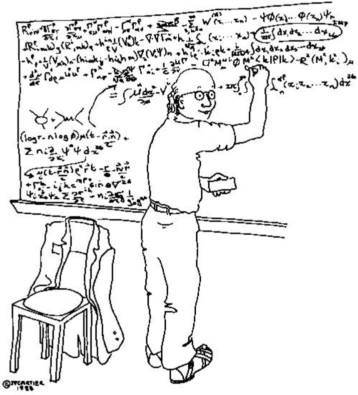
GEOQUANT 2021, Day 1, Introduction to symplectic geometry and integrable systems

Classical vs. Quantum: A love story.

1. Classical systems
2. Observables $C^{\infty}(M)$
3. Bracket $\{f, g\}$

1. Quantum System
2. Operators in $\mathcal{H}$ (Hilbert)
3. Commutator $[A, B]_h = \frac{2\pi i}{h}(AB - BA)$



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

"I still don't understand quantum theory."

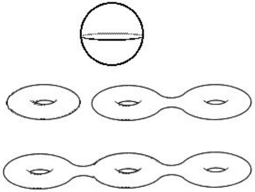
Geometric Quantization in a nutshell

- (M^{2n}, ω) symplectic manifold with integral $[\omega]$.
- $(\mathbb{L}, \nabla)$ a complex line bundle with a connection ∇ such that $curv(\nabla) = -i\omega$ (prequantum line bundle).
- A real polarization $\mathcal{P}$ is a Lagrangian foliation. Integrable systems provide natural examples of real polarizations.
- Flat sections equation: $\nabla_X s = 0$, $\forall X$ tangent to $\mathcal{P}$.

Topological classification of compact surfaces

Any connected closed surface is homeomorphic to:

1. the sphere
2. the connected sum of g tori, for $g \geq 1$
3. the connected sum of k real projective planes, for $k \geq 1$.



Classification with additional structures may depend on new invariants.
Example: Riemannian structure $\rightsquigarrow$ curvature is an invariant.

The antisymmetric case

oriented surface $\leftrightsquigarrow$ area form ω antisymmetric.

It is a closed 2-form and $\omega \neq 0$ (symplectic structure).

Theorem (Moser)

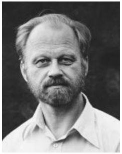
Two area forms on a surface ω_0 and ω_1 $[\omega_0] = [\omega_1]$ are equivalent.

Idea behind: Moser's path method

The linear path $\omega_t = (1-t)\omega_0 + t\omega_1$ is a path of symplectic structures $\rightsquigarrow$ (Moser's trick) integration of the flow of X_t satisfying $\iota_{X_t}\omega_t = -\beta$ for $\omega_0 - \omega_1 = d\beta$, ($X_t(\phi_t) = \frac{d\phi_t}{dt}$) given by the path method yields the diffeomorphism.

Symplectic structures

- A symplectic structure is a non-degenerate closed 2-form ω.
- Non-degeneracy gives a natural isomorphism between $T^*(M)$ and $T(M)$.
- For every f, there is a unique vector field X_f (Hamiltonian vector field), $\iota_{X_f}\omega = -df$

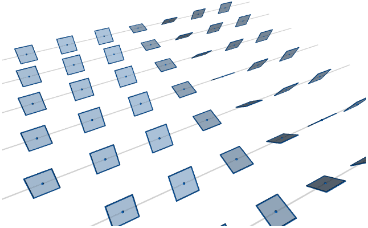


$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Figure: Sir William Rowan Hamilton, Jürgen Moser and Hamilton's equations.

Hamilton's equations are the equations of the flow of a Hamiltonian vector field in Darboux coordinates.



Symplectic	Contact
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$ Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$

Symplectic manifolds

- Locally, any symplectic form ω on a $2n$-dimensional manifold can be written as, $\omega = \sum_{i=1}^{2n} dx_i \wedge dy_i$, Darboux theorem.
- Any orientable surface is a symplectic manifold.
- Cotangent bundles $T^*(M)$ with symplectic form $\omega = -d\lambda$ (λ is a Liouville one form).
- Classification problems for symplectic geometry in dimension > 2 are HARD.
- Moser's path method is still the most famous trick to construct symplectomorphisms.

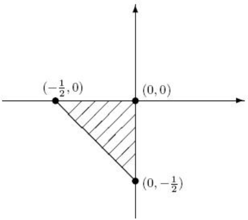
Higher dimensions?

Some classification schemes are still possible when additional data is considered (toric manifolds). Toric manifolds are a particular example of integrable system.

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$



Moment map for the $\mathbb{T}^2$-action on $\mathbb{C}P^2$ given by $(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2] := [z_0 : e^{i\theta_1} z_1 : e^{i\theta_1} z_2]$

Special submanifolds

Definition (Lagrangian submanifold)

Given a symplectic manifold (M, ω), a submanifold $L \subset M$ is called Lagrangian if $i^*(\omega) = 0$ with $i : L \hookrightarrow M$ the inclusion. Lagrangian submanifolds satisfy: $T_pS^\omega = T_pS$

Examples:

- A curve on an orientable surface.
- A fiber of the moment map on a toric manifold.
- The zero section of the cotangent bundle T^*M.
- The fibers of an integrable system.

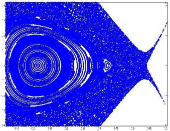
Other important submanifolds are:

- coisotropic when $T_pS^\omega \subset T_pS$.
- isotropic when $T_pS \subset T_pS^\omega$.

Integrability and dynamical systems

Importance of integrability of a Hamiltonian system and dynamical properties of its solutions such as stability.

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$



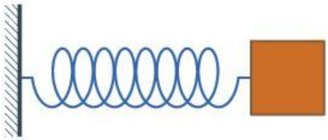
Integrable system

A set of functions $f_1, \ldots, f_n$ on (M^{2n}, ω) such that

- $f_1, \ldots, f_n$ Poisson commute, i.e., $\{f_i, f_j\} = 0, \forall i, j$.
- $df_1 \wedge \cdots \wedge df_n \neq 0$ on an open dense set.

The mapping $F : M^{2n} \longrightarrow R^n$ given by $F = (f_1, \ldots, f_n)$ is called moment map.

Coupling two simple harmonic oscillators



- Phase space: $(T^*(\mathbb{R}^2), \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$.
- Total energy:

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

.

- $H = h$ is a sphere S^3.

We have rotational symmetry on this sphere $\rightsquigarrow$ the angular momentum is a constant of motion, $L = x_1 y_2 - x_2 y_1$, $X_L = (-x_2, x_1, -y_2, y_1)$ and

$$X_L(H) = \{L, H\} = 0.$$

Classical integrable systems

The compact regular level sets of an integrable system $F = (f_1, \ldots, f_n)$ on a symplectic manifold are tori (**Liouville tori**).

Theorem (Liouville-Mineur-Arnold)

Semilocally around a Liouville torus:

- *There exist coordinates (action-angle)* $(p_1, \ldots, p_n, \theta_1, \ldots, \theta_n)$ *with values in* $B^n \times \mathbf{T}^n$ *such that* $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$.
- *The level sets of the coordinates* $p_1, \ldots, p_n$ *correspond to the Liouville tori of* F.
- *The flow of the Hamiltonian vector field on each Liouville torus is linear.*

The problem of global existence of action-angle variables is related to monodromy and the Chern class of the fibration given by the moment map.

The Liouville-Mineur-Arnold theorem

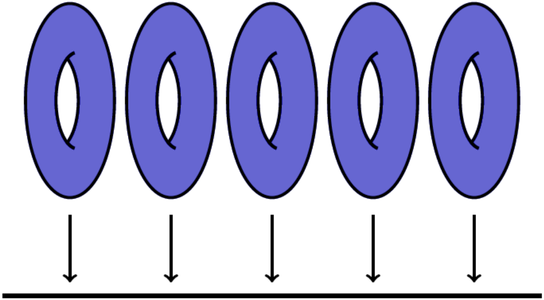
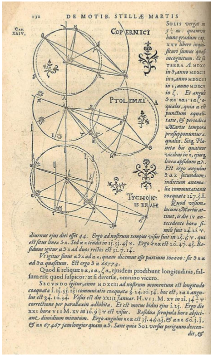


Figure: Liouville tori

- In action-angle coordinates (p_i, θ_i) the fibers of F are the tori $\{p_i = c_i\}$ and the symplectic structure is *simple* (Darboux) $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$.
- This theorem can be reformulated as a cotangent lift (see this afternoon's problem session).

Action coordinates and Liouville 1-form



It was an astronomer and mathematician (Henri Mineur) who gave an explicit formula of action coordinates:

$$p_i = \int_{\gamma_i} \alpha$$

where γ_i is a cycle of a Liouville torus and $(\omega = d\alpha)$.

Definition (Hamiltonian action)

Let G be a compact Lie group acting symplectically on (M, ω).
The action is **Hamiltonian** if there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$- d\mu^X = \iota_{X^\#}\omega, \quad (1)$$

with $\mu^X = <\mu, X>$.
The map μ is called the **moment map**.

Moment maps and reduction provide an effective tool to study symmetries in geometrical models in mechanics.

Torus actions in the proof

1. **Topology of the foliation. The fibration in a neighbourhood of a compact connected fiber is a trivial fibration by compact fibers**
2. **These compact fibers are tori:** We recover a $\mathbb{T}^n$-action tangent to the leaves of the foliation This implies a process of **uniformization of periods**.

$$\begin{aligned} \Phi \quad : \quad \mathbf{R}^r \times (\mathbf{T}^r \times B^s) \quad &\to \quad \mathbf{T}^r \times B^s \\ ((t_1, \ldots, t_r), m) \quad &\mapsto \quad \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(r)}_{t_r}(m). \end{aligned} \tag{2}$$

3. We prove that **this action is symplectic** (we use the fact that if Y is a complete vector field of period 1 and ω is a symplectic form for which $\mathcal{L}^2_Y \omega = 0$, then $\mathcal{L}_Y \omega = 0$).
4. As ω is exact in a neighbourhood of the Liouville torus **the action is Hamiltonian**.
5. To construct action-angle coordinates we use Darboux-Carathéodory theorem and the constructed Hamiltonian action of $\mathbb{T}^n$ to **drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber**.

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Toric symplectic manifolds

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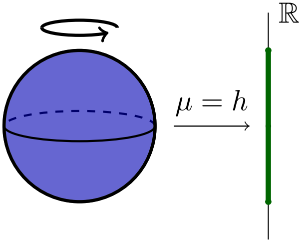
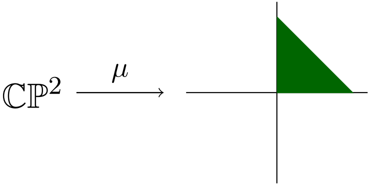




$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

Non-degenerate singularities

A point p is called singular point of an integrable system if $d_pF < n$. As in Morse theory we can define a non-degenerate singular point.

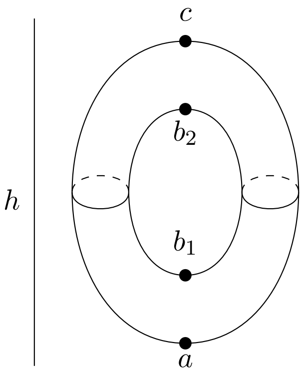


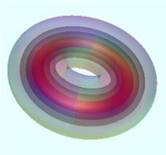
Figure: Critical points of the height function in the torus

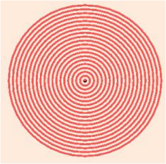
Action-angle coordinates with singularities

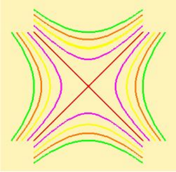
The theorem of Marle-Guillemin-Sternberg for fixed points of toric actions can be generalized to non-degenerate singularities of integrable systems.

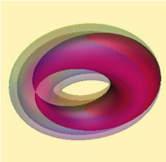
Theorem (Eliasson-M.-Zung)

There exists symplectic Morse normal forms for integrable systems with non-degenerate singularities.









Liouville torus k_e comp. elliptic k_h hyperbolic k_f focus-focus These models are valid in a neighbourhood of an orbit.

Description of singularities

The local model is given in a covering by $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ and $\omega = \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$. and the components of the moment map are:

1. Regular $f_i = p_i$ for $i = 1, ..., k$;
2. Elliptic $f_i = x_i^2 + y_i^2$ for $i = k+1, ..., k_e$;
3. Hyperbolic $f_i = x_i y_i$ for $i = k_e + 1, ..., k_e + k_h$;
4. focus-focus $f_i = x_i y_{i+1} - x_{i+1} y_i$, $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1$, $j = 1, ..., k_f$.

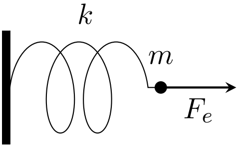
We say the system is semitoric if there are no hyperbolic components.

Cotangent models

This afternoon we will see they can all be seen as cotangent models.

Singularities in physical examples

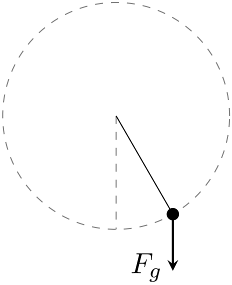
Harmonic oscillator



Elliptic singularity

$$f = x^2 + y^2$$

Simple pendulum

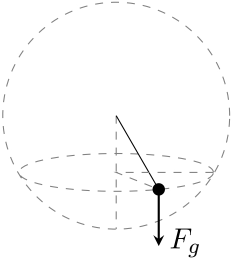


Hyperbolic singularity

$$f = xy$$

(or $f = x^2 - y^2$)

Spherical pendulum



Focus-focus singularity

$$f_1 = x_1 y_2 - x_2 y_1$$
$$f_2 = x_1 y_1 + x_2 y_2$$

The solar system

``We revolve around the Sun like any other planet.''

1514, Nicolaus Copernicus.

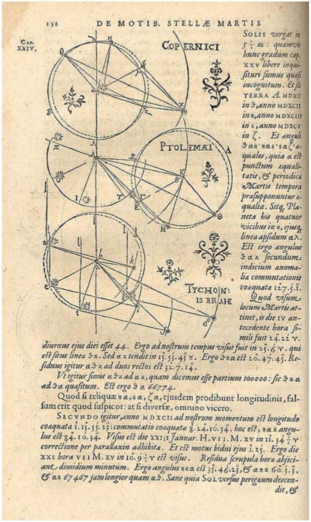

NICOLAI COPERNICI

net, in quo terram cum orbe lunari tanquam epicyclo contineri
diximus. Quinto loco Venus nono mense reducitur. Sextum
denique locum Mercurius tenet, octuaginta dierum spacio circũ
currens. In medio uero omnium residet Sol. Quis enim in hoc

pulcherrimo templo lampadem hanc in alio uel meliori loco po
neret, quàm unde totum simul possit illuminare? Siquidem non
inepte quidam lucernam mundi, alij mentem, alij rectorem uo-
cant. Trimegistus uisibilem Deum, Sophoclis Electra intuentẽ
omnia. Ita profecto tanquam in solio regali Sol residens circum
agentem gubernat Astrorum familiam. Tellus quoque minime
fraudatur lunari ministerio, sed ut Aristoteles de animalibus ait,

The solar system

Kepler: Planets spin in elliptical orbits.



132 DE MOTIB. STELLÆ MARTIS

CAP. XXIV.

COPERNICI

PTOLEMÆI

TYCHONIS BRAHE

SOLIS vergat in 5 ½ ♒: quamvis hunc gradum cap. XXV libere inquisituri sumus quasi incognitum. Et sit TERRA A. MDXC in ϑ, anno MDXCII in η, anno MDXCIII in ε, anno MDXCV in ζ. Et anguli ϑακ· ηακ· εακ· ζακ· æquales, quia α est punctum æqualitatis, & periodica Martis tempora præsupponuntur æqualia. Sitque Planeta his quatuor vicibus in κ, ejusque linea apsidum αλ. Est ergo angulus ϑακ secundum indicium anomaliæ commutationis coæquatæ 127.5.1.

Quod visum locum Martis attinet, is die IV antecedente hora simili fuit 24.22 ♈. diurnus ejus diei esset 44. Ergo ad nostrum tempus visus fuit in 25.6 ♈. qui est situs lineæ ϑκ. Sed ακ tendit in 15.53.45 ♉. Ergo ϑκα est 20.47.45. Residuus igitur αϑκ ad duos rectos est 32.7.14.

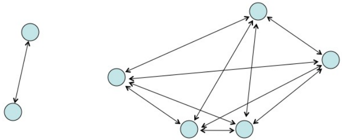
Vt igitur sinus αϑκ ad ακ, quam dicemus esse partium 100000: sic ϑκα ad ϑα quæsitum. Est ergo ϑα 66774.

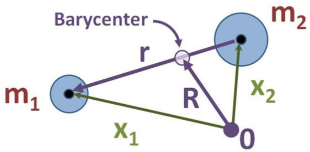
Quod si reliquæ ηα, εα, ζα, ejusdem prodibunt longitudinis, falsum erit quod suspicor: at si diversæ, omnino vicero.

SECVNDO igitur, anno MDXCII ad nostrum momentum est longitudo coæquata 1.15.55.23: commutatio coæquata 8.24.10.34. hoc est, ηακ angulus est 84.10.34. Visus est die XXIII Januar. H. VII. M. XV in 11.34 ½ ♈ correctione per parallaxin adhibita. Et est motus bidui ejus 1.25. Ergo die XXI hora VII M. XV in 10.9 ½ ♈ est visus. Residua scrupula horæ abjiciant dimidium minutum. Ergo angulus ηκα est 35.46.23, & αηκ 60.3.3, & αη 67467 jam longior quam αϑ. Sane quia SOL versus perigæum descendit, &

The n-body problem

The n-body problem describes the movement of n bodies under mutual attraction.





Integrable for $n = 2$: The Hamiltonian function is

$$H(x_1, x_2, p_1, p_2) := E_{kin} - U = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} - U.$$

where $U := \mathcal{G} m_1 m_2 \frac{1}{\|x_2 - x_1\|}$ is the gravitational potential. Integrals:

1. Total linear momentum: The problem reduces to determining the relative position $r = x_2 - x_1$.
2. Total angular momentum: makes the problem *planar*.

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

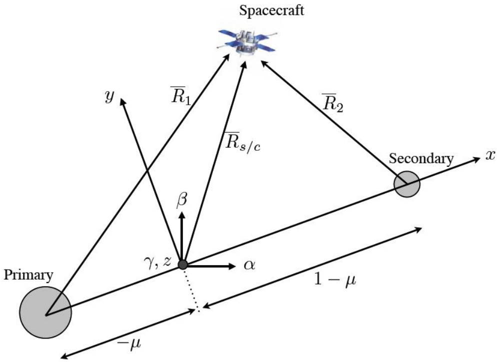


Figure: Circular 3-body problem

Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$, with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbf{R}^2 \times \mathbf{R}^2,$ where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+,$ can be then extended to infinity ($x = 0$).
- The symplectic structure becomes a singular object $\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$ for $x > 0$
- The integrable 2-body problem for $\mu = 0$ is integrable with respect to the singular ω.

Model of these systems

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}} \mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

Close to $x_1 = 0$, the systems behave like,

and not like,