Geometric Quantization via integrable systems

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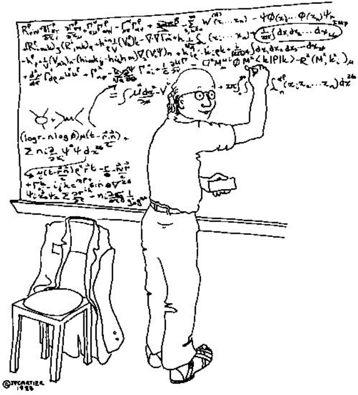
GEOQUANT 2021, Day 2, Geometric Quantization via action-angle coordinates

Classical vs. Quantum: A love story.

1. Classical systems
2. Observables $C^\infty(M)$
3. Bracket $\{f, g\}$

1. Quantum System
2. Operators in $\mathcal{H}$ (Hilbert)
3. Commutator $[A, B]_h = \frac{2\pi i}{h}(AB - BA)$



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

"I still don't understand quantum theory."

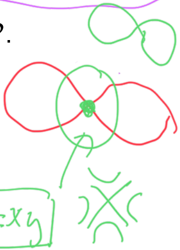
$[\frac{\Omega}{2\pi i}] = c_1(L)$

- (M^{2n}, ω) symplectic manifold with integral $[\omega]$.
- $(\mathbb{L}, \nabla)$ a complex (and hermitian) line bundle with a connection ∇ such that $curv(\nabla) = -i\omega$ (prequantum line bundle).
- A real polarization $\mathcal{P}$ is a Lagrangian foliation. Integrable systems provide natural examples of real polarizations.
- Flat sections equation: $\nabla_X s = 0$, $\forall X$ tangent to $\mathcal{P}$.

$L \downarrow \uparrow s$ M $\quad \nabla_X s = 0$ (Flat section equation)

in the directions of the polarization

X_f = Ham's v.f.s. $\quad f = xy$



A connection on a vector bundle V is a map $\nabla : \Gamma(V) \to \Omega^1(M) \otimes \Gamma(V)$ satisfying:

1. $\nabla(\sigma_1 + \sigma_2) = \nabla\sigma_1 + \nabla\sigma_2$
2. $\nabla(f\sigma_1) = (df) \otimes \sigma_1 + f\nabla\sigma_1$

for all sections σ_1 and σ_2 and functions f.
We write $\nabla_X\sigma$ for $\nabla\sigma$ applied to the vector field X (the *covariant derivative* of σ in the direction X.)

Basics of Quantization

- Let $\mathbb{L}$ be a complex line bundle and s the unit section in some local trivialization. Fix a connection ∇ on $\mathbb{L}$. Define the potential one-form Θ of ∇, by
  $\nabla_X s = -i\Theta(X)s.$
- Changing s by another section $s' = fs$
  $\nabla_X s' = df(X)s - fi\Theta(X)s.$
  and $\Theta' = \Theta - i\frac{1}{\psi}\,d\psi.$
- Locally as $\psi = e^{if}$ for some real-valued function f, and $d\psi = e^{if}idf$.
  thus $i\frac{1}{\psi}\,d\psi = -df$ is real-valued.
- So as $curv\nabla = i\omega$ we can take locally a given Θ connection one-form with $d\Theta = \omega$.

Space for proofs

$[\omega] \in H^2(M,\mathbb{Z}) \Longleftrightarrow \exists$ complex line bundle

(Prop — Hatton's lecture today)

Recall $[a] \in H^2_{DR}(M) \longrightarrow [c] \in H^2_{\check{C}ech}(M,\mathbb{R})$

$\{U_i\}$ good cover (contractible intersections) $a_i = db_i$ on U_i (P. Lemma), $a_j = db_j$ on U_j

on $U_i \cap U_j \to db_i = db_j \Rightarrow \exists\, c_{ij}$ (function) on $U_i \cap U_j$

$b_i - b_j = dc_{ij}$

on $U_i \cap U_j \cap U_k$: $dc_{ij} + dc_{jk} + dc_{ki} = 0 \to c_{ijk} = c_{ij} + c_{jk} + c_{ki}$ is constants

$\to$ defines a cohomology class in $H^2_{\check{C}ech}(M,\mathbb{R})$

Let's play this game for $[\omega]$: $\omega = d\theta_i$ on $U_i \cap U_j \to$

$\theta_i - \theta_j = d(\log(g_{ij}))$ $\quad c_1(L) = [\frac{1}{2\pi}\omega] \longrightarrow [c] \in H^2_{\check{C}ech}(M,\mathbb{R})$

$c_{ijk} = \frac{1}{2\pi i}(\log(g_{ij}) + \log g_{jk} + \log(g_{ki}))$

g_{ij} transition functions $\longrightarrow$ cocycle condition $g_{ij}\, g_{jk}\, g_{ki} = 1$

$\Rightarrow e^{2\pi i c_{ijk}} = 1 \Rightarrow c_{ijk} \in \mathbb{Z} \longrightarrow [c_{ijk}] \in H^2(M,\mathbb{Z})$

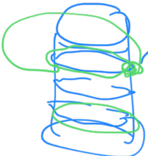
Bohr-Sommerfeld leaves

Definition

A Bohr-Sommerfeld leaf is a leaf of a polarization admitting global flat sections.

Example: Take $M = S^1 \times \mathbb{R}$ with $\omega = dt \wedge d\theta$, $\mathcal{P} = < \frac{\partial}{\partial \theta} >$, $\mathbb{L}$ the trivial bundle with connection 1-form $\Theta = t d\theta \rightsquigarrow \nabla_X \sigma = X(\sigma) - i < \Theta, X > \sigma$ $\rightsquigarrow$Flat sections: $\sigma(t, \theta) = a(t).e^{it\theta} \rightsquigarrow$ Bohr-Sommerfeld leaves are given by the condition $t = 2\pi k, k \in \mathbb{Z}$.

Liouville-Mineur-Arnold $\leftrightsquigarrow$ this example is the canonical one.



Bohr-Sommerfeld leaves: continued...

Theorem (Guillemin-Sternberg)

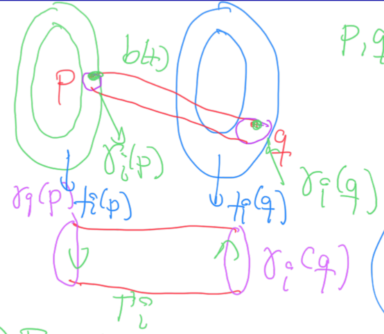
If the polarization is a regular fibration with compact leaves over a simply connected base B, then the Bohr-Sommerfeld is given by,

$$BS = \{p \in M, (f_1(p), \dots, f_n(p)) \in \mathbb{Z}^n\}$$

where $f_1, \dots, f_n$ are global action coordinates on B.

- In a semilocal cotangent model for the connection given by Liouville-Mineur-Arnold, Bohr-Sommerfeld leaves coincide with integral points.
- For toric manifolds the base B may be identified with the image of the moment map.

Space for proofs



$p, q \in BS$ ← Bohr-Sommerfeld set

(G.-Sternberg → Gelfand-Zeitlin syst. JFA 83)

$\partial F_i = \gamma_i(q) - \gamma_i(p)$

$$\int_{\Gamma_i} \omega = \int_{\gamma_i(q)-\gamma_i(p)} \alpha = \int_{\gamma_i(q)} \alpha - \int_{\gamma_i(p)} \alpha = f_i(q) - f_i(p)$$

Stokes theorem, $\omega = d\alpha$

$$M_i(q) = M_i(p)\, e^{2\pi i \int_{\Gamma_i} \omega}$$

monodromy (transport along γ_i)

[Kostant]

$$M_i(q)\, e^{2\pi i f_i(q)} = M_i(p)\, e^{2\pi i f_i(p)}$$

p → Bohr-Sommerfeld set: $M_i(p) = 1$, $f_i(p) = 0$

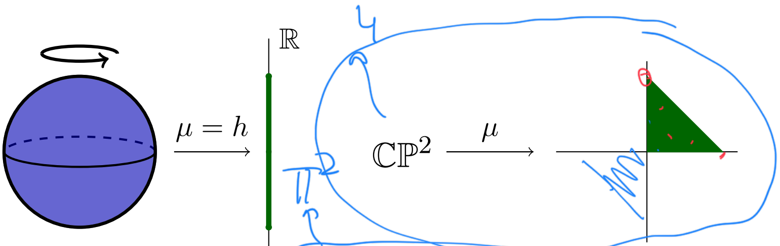
$M_i(q) = 1$

$e^{2\pi i f_i(q)} = 1$ → $f_i(q) \in \mathbb{Z}$

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes and the bijective correspondence is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$



$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{it_1} z_1 : e^{it_2} z_2]$$

The case of fibrations

Kostant $Q(M) = \bigoplus_i H^i(M, \mathcal{J})$, $\mathcal{J} \to$ sheaf of flat section

- "Quantize" these systems counting Bohr-Sommerfeld leaves.
- For real polarization given by integrable systems Bohr-Sommerfeld leaves are just "integral" Liouville tori.

Theorem (Sniatycki)

If the leaf space B^n is Hausdorff and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibers, then quantization is given by the count of Bohr-Sommerfeld leaves.

But how exactly?

$Q(M) = H^n(M, \mathcal{J}) \stackrel{?}{=} \#$ Bohr-Sommerfeld (Sniatycki)

Quantization: The cohomological approach

- Following the idea of Kostant when there are no global sections we define the quantization of $(M^{2n}, \omega, \mathbb{L}, \nabla, P)$ as

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M, \mathcal{J}).$$

- $\mathcal{J}$ is the sheaf of flat sections.

Then quantization is given by:

Theorem (Sniatycki)

$\mathcal{Q}(M^{2n}) = H^n(M^{2n}, \mathcal{J})$, with dimension the number of Bohr-Sommerfeld leaves.

What is this cohomology?

1. Define the sheaf: $\Omega^i_{\mathcal{P}}(U) = \Gamma(U, \wedge^i \mathcal{P})$..
2. Define $\mathcal{C}$ as the sheaf of complex-valued functions that are locally constant along $\mathcal{P}$. Consider the natural (fine) resolution

$$0 \to \mathcal{C} \stackrel{i}{\to} \Omega^0_{\mathcal{P}} \stackrel{d_{\mathcal{P}}}{\to} \Omega^1_{\mathcal{P}} \stackrel{d_{\mathcal{P}}}{\to} \Omega^1_{\mathcal{P}} \stackrel{d_{\mathcal{P}}}{\to} \Omega^2_{\mathcal{P}} \stackrel{d_{\mathcal{P}}}{\to} \cdots$$

   The differential operator $d_{\mathcal{P}}$ is the one of foliated cohomology.
3. Use this resolution to obtain a fine resolution of $\mathcal{J}$ by twisting the previous resolution with the sheaf $\mathcal{J}$.

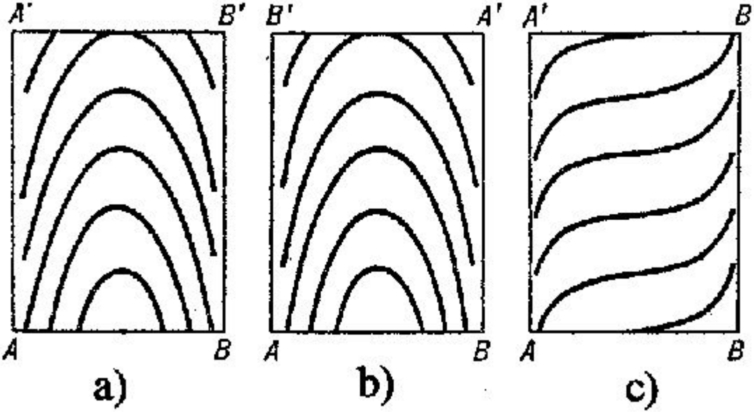
$$0 \to \mathcal{J} \stackrel{i}{\to} \mathcal{S} \stackrel{\nabla_{\mathcal{P}}}{\to} \mathcal{S} \otimes \Omega^1_{\mathcal{P}} \stackrel{\nabla_{\mathcal{P}}}{\to} \mathcal{S} \otimes \Omega^2_{\mathcal{P}} \to \cdots$$

   with $\mathcal{S}$ the sheaf of sections of the line bundle $\mathbb{L}(\otimes N^{1/2})$.

4. Computation kit: Mayer-Vietoris, Künneth formula, Remarkable fact: S^1-actions help prove semilocal Poincaré lemma (toric, almost toric, semitoric case).

Applications to the general case of Lagrangian foliations

This fine resolution approach can be useful for polarizations given by general Lagrangian foliations.
Classification of foliations on the torus (Kneser-Denjoy-Schwartz theorem).



The case of the torus: irrational slope.

Consider $X_\eta = \eta \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, with $\eta \in \mathbb{R} \setminus \mathbb{Q}$. This vector field descends to the quotient torus denote by $\mathcal{P}_\eta$ the associated foliation in $\mathbb{T}^2$. Let $(\mathbb{T}^2, \omega)$ be the 2-torus with a symplectic structure ω of integer class, then,

Theorem (Presas-Miranda)

- $\mathcal{Q}(T^2, \mathcal{J})$ *is always infinite dimensional.*
- *For the limit case of foliated cohomology* $\omega = 0$ $\mathcal{Q}(\mathbb{T}^2, \mathcal{J}) = \mathbb{C} \bigoplus \mathbb{C}$ *if the irrationality measure of* η *is finite and* $\mathcal{Q}(\mathbb{T}^2, \mathcal{J})$ *is infinite dimensional if the irrationality measure of* η *is infinite.*

This generalizes a result El Kacimi for foliated cohomology.

"Quantization Computation kit" for regular foliations

Most computations rely on proving Künneth and Mayer-Vietoris (joint with Presas)

1. Künneth formula: Let $(M_1, \mathcal{P}_1)$ and $(M_2, \mathcal{P}_2)$ be symplectic manifolds endowed with Lagrangian foliations and let $\mathcal{J}_{12}$ be the induced sheaf of basic sections, then:
   $H^n(M_1 \times M_2, \mathcal{J}_{12}) = \bigoplus_{p+q=n} H^p(M_1, \mathcal{J}_1) \otimes H^q(M_2, \mathcal{J}_2)$.
2. Mayer-Vietoris: Consider $M \leftarrow U \sqcup V \overset{\leftarrow}{\leftarrow} U \cap V$, then the following sequence is exact,
   $0 \to \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(M) \xrightarrow{r} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U) \oplus \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(V) \xrightarrow{r_0 - r_1} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U \cap V) \to 0.$

Application II: Regular integrable system

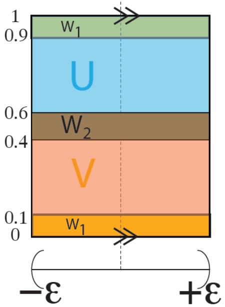
$I_j = (-\varepsilon, \varepsilon), j = 1, 2.$

Computation 1: $\mathcal{Q}(I_1 \times I_2, \omega = dx_1 \wedge dx_2; \mathcal{P} = \frac{\partial}{\partial x_2})$.

- $H^0(I_1 \times I_2; \mathcal{J}) = C^\infty(I_1, \mathbb{C})$,
- $H^1(I_1 \times I_2; \mathcal{J}) = 0$.

Computation 2: $\mathcal{Q}(I_1 \times \mathbb{S}^1_2, \omega = dx_1 \wedge d\theta_2; \mathcal{P} = \frac{\partial}{\partial \theta_1})$.

- $H^0(I_1 \times \mathbb{S}^1_2; \mathcal{J}) = 0$ since BS leaves are isolated.
- Consider $I_1 \times \mathbb{S}^1_2 = U \cup V = (I_1 \times (0.4, 1.1)) \cup (I_1 \times (-0.1, 0.6))$.



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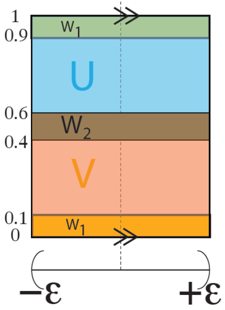

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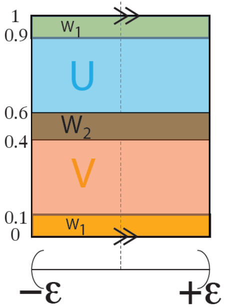

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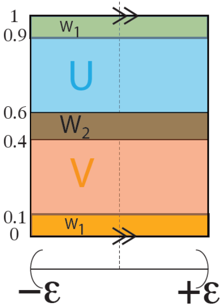

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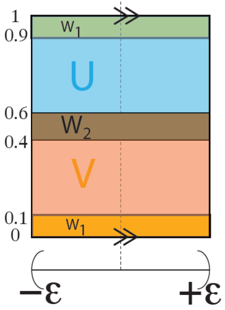

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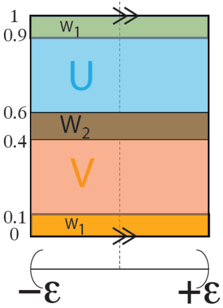

Regular integrable system

Apply Mayer-Vietoris and computation 1 to obtain

$$H^0(V) \oplus H^0(U) \hookrightarrow H^0(W_1) \oplus H^0(W_2) \twoheadrightarrow H^1(I_1 \times \mathbb{S}^1_2).$$

$H^0(V) = H^0(U) = H^0(W_1) = C^\infty(I_1 \times \{0\}; \mathbb{C})$ and $H^0(W_2) = C^\infty(I_1 \times \{0.5\}; \mathbb{C})$.Take $f_0 \in H^0(V)$ and $f_1 \in H^0(U) = C^\infty(I_1 \times \{0\}; \mathbb{C})$. The first map of the sequence is given by

$$\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ e^{i\theta x} & e^{-i\theta x} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

Thus

$$H^1(I_1 \times \mathbb{S}^1_2) = \begin{cases} 0 & \text{if non BS,} \\ \mathbb{C} & \text{if there is one BS.} \end{cases}$$

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Regular integrable system

Computation 3: $\mathcal{Q}(I^k \times \mathbb{T}^k; \mathbb{T}^k)$.

By Künneth $H^j(I^k \times \mathbb{T}^k; \mathcal{J}) = 0$, if $j \neq k$, and

$$H^k(I^k \times \mathbb{T}^k; \mathcal{J}) = \begin{cases} 0 & \text{if non BS,} \\ \mathbb{C} & \text{if there is one BS.} \end{cases}$$

Computation 4:

$$\mathcal{Q}(M^{2n}_{Tor,Reg}; \mathcal{P}(Torus)) = \bigoplus_{j=1}^{n} H^j(M; \mathcal{J}) = \mathbb{C}^b, \ b = \#\text{BS}.$$

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Regular integrable system

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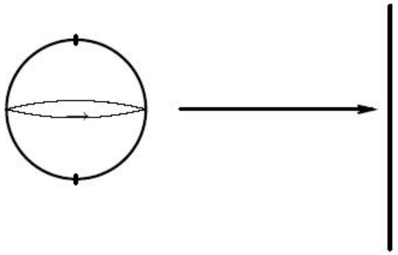
Toric Manifolds

What happens if we go to the edges and vertexes of Delzant's polytope?



Rotations of S^2 and moment map

There are two leaves of the polarization which are singular and correspond to fixed points of the action.

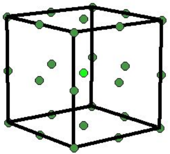
Quantization of toric manifolds

Theorem (Hamilton)

For a $2n$-dimensional compact toric manifold

$$\mathcal{Q}(M) = H^n(M; \mathcal{J}) \cong \bigoplus_{l \in BS_r} \mathbb{C}$$

with a BS_r the set of regular Bohr-Sommerfeld leaves.



In the example of the sphere Bohr-Sommerfeld leaves are given by integer values of height (or, equivalently) leaves which divide out the manifold in integer areas.

Key computation in a neighbourhood of an elliptic point

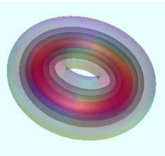
- The coordinates we use on $\mathbb{C}$ are (s, ϕ), where (r, ϕ) are standard polar coordinates and $s = \frac{1}{2}r^2$.
- Then $\omega = ds \wedge d\phi = d(s\, d\phi)$. and the polarization is $P = \text{span}\{\frac{\partial}{\partial \phi}\}$,
- The sections which are flat along the leaves are of the form $a(s)e^{is\phi}$, for arbitrary smooth functions a.

Action-angle coordinates with singularities

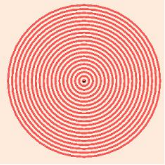
The theorem of Marle-Guillemin-Sternberg for fixed points of toric actions can be generalized to non-degenerate singularities of integrable systems.

Theorem (Eliasson, M-Zung)

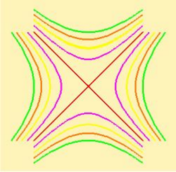
There exists symplectic Morse normal forms for integrable systems with non-degenerate singularities.



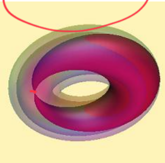
Liouville torus



k_e comp. elliptic



k_h hyperbolic



k_f focus-focus

Description of singularities

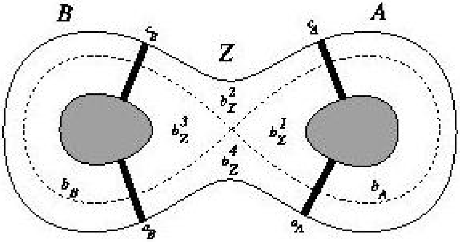
The local model is given in a covering by $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ and $\omega = \sum_{i=1}^k dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$. and the components of the moment map are:

1. Regular $f_i = p_i$ for $i = 1, ..., k$;
2. Elliptic $f_i = x_i^2 + y_i^2$ for $i = k+1, ..., k_e$;
3. Hyperbolic $f_i = x_i y_i$ for $i = k_e + 1, ..., k_e + k_h$;
4. focus-focus $f_i = x_i y_{i+1} - x_{i+1} y_i$, $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1$, $j = 1, ..., k_f$.

We say the system is semitoric if there are no hyperbolic components.

Hyperbolic singularities

We consider the following covering



Key point in the computation

We may choose a trivializing section of such that the potential one-form of the prequantum connection is $\Theta_0 = (xdy - ydx)$.

Theorem

Leafwise flat sections in a neighborhood of the singular point in the first quadrant are given by

$$a(xy)e^{\frac{i}{2}xy\ln\left|\frac{x}{y}\right|}$$

where a is a smooth complex function of one variable which is flat at the origin.

The case of surfaces

We can use Čech cohomology computation and a Mayer-Vietoris argument to prove:

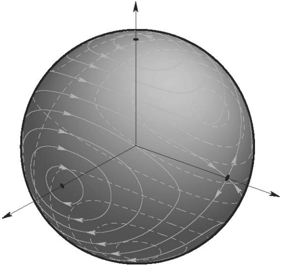
Theorem (Hamilton-M.)

The quantization of a compact surface endowed with an integrable system with non-degenerate singularities is given by,

$$\mathcal{Q}(M) = H^1(M;\mathcal{J}) \cong \bigoplus_{p\in\mathcal{H}}(\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{l\in BS_r} \mathbb{C} \ ,$$

where $\mathcal{H}$ is the set of hyperbolic singularities.

The rigid body



Using this recipe and the quantization of this system is

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}_p^{\mathbb{N}})^2 \oplus \bigoplus_{b \in BS} \mathbb{C}_b.$$

Comparing this system with the one of rotations on the sphere $\rightsquigarrow$ This quantization depends strongly on the polarization.